Meier HIRSCH'S

COLLECTION OF

EXAMPLES, FORMULÆ, & CALCULATIONS,

ON THE

LITERAL CALCULUS

AND

ALGEBRA.

TRANSLATED FROM THE GERMAN,

BY THE

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TRANSLATOR OF HIRSCH'S INTEGRAL TABLES.

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T. C. HANSARD, Paternoster-Row Press.

PREFACE.

THAT I have discovered the general solution of equations, my readers may very probably know already, from the announcement which I have ordered to be put in the public papers, for such a discovery deserves the utmost publicity. A complete history of the unsuccessful endeavours of my predecessors for the same purpose, would appear like a panegyric on myself, and therefore I shall only relate what exactly belongs to the matter.

In the tenth part of the "*Memorie di Matematica e di Fisica della Societa Italiana della Scientia*," p. 1 (1803), a celebrated Analyst, M. Ruffini, gave a proof of the impossibility of such a solution: no very elaborate discussions are required to show its insufficiency. Read his proof, and compare it with my solution, and it will be found that M. Ruffini, in recounting the possible cases, has never thought of this mode of solution. M. Ruffini tries to depress the equation for the assumed function, by taking as many equal formulæ as are proper for his purpose. I do just the contrary: with me the assumed functions, with the

exception of a single condition, are always arbitrary: all their forms may be different, and the depression will be produced by making them dependent upon resolvable equations, the coefficients of which depend on other resolvable equations, the coefficients of which depend again on other resolvable equations, &c. As, for instance, with me the assumed function for the equation of the fifth degree originally depends on an equation of the 120th degree: this I reduce, first to a double equation of the fifth degree; its coefficient, which is still depending on an equation of the 24th degree, I make depending on an equation of the fourth degree, the coefficients of which only still depend on equations of the sixth degree. I reduce these equations again to equations of the third degree, the coefficients of which, lastly, depend on equations of the second degree. Of all this process, there is, with the exception of the reduction to the double equation, not the slightest indication in the proof of M. Ruffini. Moreover, this Analyst has only shown that none of the methods he was acquainted with could succeed, and in this respect his proof is, no doubt, very masterly. His error can be no reflection on his well-deserved reputation, for he has shown to his successors the paths they must avoid, and has thus put them on the right course of investigation.

M. Lagrange gave, in the third volume of the new Memoirs of the Berlin Academy of Sciences, an incomparable analysis of the methods of Tschirnhausen, Euler, and Bezout, which I have adopted in the sixth chapter,

with a few alterations suitable to my purpose. He showed, that when μ is a prime number,* all these methods lead at last to a reduced equation, the coefficients of which are those functions of the roots $x', x'', x''' \ldots\ldots x^{(\mu)}$, which change when you change only the $\mu-2$ last roots among each other, but leave both the first in their places, and that, therefore, the coefficients depend on equations of the $1 . 2 . 3 \ldots\ldots \mu$ second degree; and consequently an equation of the fifth degree, on an equation of the sixth degree. For the explanation of his method, he takes the equation of the fifth degree as an example, and shows how to begin to form the reduced equation. He denotes the roots of the reduced equations by $z', z'', z''', z^{IV}, z^{V}, z^{VI}$, and finds the value of them in* $x', x'', x''', x^{IV}, x^{V}$. From this he calculates the first coefficients, and says that you can find the other coefficients in a similar way. He concludes by saying "but we will not enter into the execution of this calculation, which, with all its immense labour, would not afford any clue to the resolution of equations of the fifth degree; for as the reduced equation for z is of the sixth degree, it is not resolvable, unless it is to be brought to a lower degree than the fifth. But this seems to me to be almost impossible, according to the form of the roots z', z'', &c."

But from these very forms, I affirm that the solution of the reduced equation is possible. For the functions z'', z'''

* μ is with him, what n is with me.

* Pages 432 and 433 of Euler's Introduction, translated by Michelsen.

z^{IV}, z^{V}, z^{VI} are derived from z, as M. Lagrange observed himself, when you change the roots x''', x^{IV}, x^{V}, among each other. According to my notation, therefore, the roots of the reduced equation admit of being presented by $f:(12345)$, $f:(12453)$, $f:(12534)$, $f:(12435)$, $f:(12354)$, and $f:(12543)$, and with these notations the functions z', z^{IV}, z^{VI}, z''' z'', z^{V}, correspond according to the order in which they are here put. The three former formulæ $f:(12345)$, $f:(12453)$, $f:(12534)$, form evidently a cyclical period of the three last roots, as well as the formulæ $f:(12435)$, $f:(12354)$, $f:(12543)$. If we therefore combine the three functions z', z^{IV}, z^{VI}, in one equation of the third degree, the coefficients of them can only have besides their value another one, namely, that which the change of x''' with x^{IV} gives. These coefficients, therefore, have no more than two unequal values, and consequently they depend only on equations of the second degree, or, what is the same, the reduced equation of the sixth degree can be divided into two equations of the third degree, the one of which has the roots z', z^{IV}, z^{VI}, the other the roots z''', z'', z^{V}. That this simple observation escaped the keen penetration of a Lagrange, looks indeed like a miracle. I am not the inventor, it is he; but he did not know it. Whether I should have found the solution without him, may be doubted.

I come now to my method of solution: it is very simple, uniform for all degrees, and as general as could be desired. It gives, not one solution, but as many as we

please; for the functions I mark with ϕ are quite arbitrary. However, the actual calculation is very troublesome, and even in the sixth degree is scarcely practicable without resorting to particular artifices. We cannot escape the difficulties of calculation, when the degree of an equation is a prime number. When, however, it is a compound number, we have, no doubt, a method, which leads more rapidly to the result: it will be reserved for the third volume, in which I shall give also the solution of the equations of the 5th, 6th, and 7th degree. The Combinatorial Analysis is here of great service; and with its help I shall perhaps be able at once to exhibit the reduced equation with little more trouble than the mere combinatorial operations.

A brief sketch of the contents of this volume will not be here improper. I begin with the Symmetrical Functions; they are the foundation of all others. The two first chapters treat of them; the first gives the recurred solution, the second the independent one. Generality was the object I aimed at. The third chapter treats of the Non-symmetrical Functions. They are derived altogether from certain equations, which I call transformed equations. It is shown how to find the equal formulæ of these functions, when their nature is given by certain properties; and how to form a transformed equation of the unequal formulæ. The numberless references to it require particular observation. The utility of some of these propositions will appear in the sequel. The fourth chapter treats of Elimination. I was not obliged to be

so minute as Bezout in his *Theorie générale des Equations Algebraiques*, who confines himself exclusively to this subject. My work would have become too voluminous. Should my readers wish for further details, this may be done in an appendix. The fifth chapter treats of the properties of the roots of the equation $x^n-1=0$. Waring and Euler were my conductors. The labours of Lagrange, in the Memoirs of the Berlin Academy, gave me the materials for the sixth chapter. I desire my readers to bestow particular attention on the seventh chapter; its value will be shown in the third volume of this work. The eighth chapter treats of the General Solution of equations, but must be regarded only as a sketch.

My reader is no longer the same whom I thought of in the Collection, the continuation of which I now give him: he has gone much further in the sciences. The Combinatorial Analysis is no longer strange to him: he has also made already considerable progress in the Differential Calculus. Provided with this knowledge, he will, I trust, find my book not entirely useless. He will not remain where I have remained: he will look further. I do not lead him through an unfertile, but, for want of labour, an uncultivated field. For since the Differential and Integral Calculation employed the Analysts, Algebra has been little thought of.

The next part will contain, besides the deeper researches about the general solution of equations, a great many

other subjects, and amongst them the important, almost inexhaustible, one, of the Analysis of Equations. I shall constantly, as far as my leisure hours permit, labour on the Third Part to hasten its appearance as much as possible. But if on all these subjects the same pains be bestowed, some time must elapse before its appearance. However, not to let my readers wait for what belongs to equations in particular too long, I am inclined to prepare for the next *fair* a *supplement* of about four or five sheets on this subject, and in it to communicate the complete solution of the general equations of the fifth, sixth, and seventh degree.

Berlin, October, 1808.

TRANSLATOR'S

PREFACE.

SINCE the publication of Waring's *Meditationes Algebraicæ* there has not appeared, either in this country or on the continent, so elaborate and able a treatise on the theory of Equations as that of MEYER HIRSCH. Its merit has long been recognized, and the work held in the highest esteem by those who were able to read it; a small number, undoubtedly in this country, where the German language is so rarely understood by those who devote themselves to these studies.

As a treatise on equations of the higher orders, it is not less admirable for the clear and simple manner in which the theories are laid down, than for the numerous and apt examples by which they are elucidated.

It is necessary to remind the reader, however, that the Solution of the General Equation on the imagined discovery of which Meyer Hirsch so warmly congratulates himself in his preface, turns out, on examination, like all the other attempts of the same kind, to be a failure. What he has written on the subject, I have, nevertheless, permitted to remain; because, in the first place, these speculations occupy only a small space; next, because they are highly curious and interesting; and finally,

because I did not think myself at liberty to mutilate a work which I undertook only to translate.

J. A. ROSS.

Westwell-Vicarage, Kent,
January, 1827.

Extract from Meyer Hirsch's Preface to his Integral Tables.

"At the end of this preface I must observe, that I was mistaken when I maintained, in my collection of Problems on the Theory of Equations, that the general solution of equations was not only practicable, but even thought I had found it. The eighth chapter of the above-mentioned work must, therefore, be read with mistrust. It is true that I have found the solution of a number of very remarkable equations, that do not admit of being analysed, but by no means their general solution, in the sense in which Euler, Lagrange, and other great Analysts take these words (general solution); for I am now convinced of the impossibility of effecting it. The mistake arose from haste, and is so readily discovered, that every person who reads so far, will easily perceive it."

I—ON

TABLE I.

2

	A^2	B
$[2] =$	1	−
$[1^2] =$		

5

	A^5	A^3B	AB^2	A^2C	BC	AD	E
$5] =$	1	− 5	+ 5	+ 5	− 5	− 5	+ 5
$[14] =$		1	− 3	− 1	+ 5	+ 1	− 5
$[23] =$			1	− 2	− 1	+ 5	− 5
$[1^23] =$				1	− 2	− 1	+ 5
$[12^2] =$					1	− 3	+ 5
$[1^32] =$						1	− 5
$[1^5] =$							1

	A^6	A^4B
$[6] =$	1	− 6
$[13] =$		1
$[24]$		
[		

7

	AC^2	A^3D	ABD	CD	A^2E	BE	AF	G
	+ 7	− 7	+14	− 7	+ 7	− 7	− 7	+ 7
	− 4	+ 1	− 8	+ 7	− 1	+ 7	+ 1	− 7
	− 7	+ 2	− 4	+ 7	− 2	− 3	+ 7	− 7
	+ 5	+ 3	− 2	− 5	− 7	+ 7	+ 7	− 7
	+ 4	− 1	+ 3	− 7	+ 1	− 2	− 1	+ 7
	− 1	− 3	+ 8	− 2	+ 3	− 4	− 8	+14
	− 2	*	− 1	+ 5	+ 4	− 7	− 4	+ 7
=	1	*	− 2	− 1	+ 2	+ 3	− 7	+ 7
$4] =$		1	− 3	+ 3	− 1	+ 2	+ 1	− 7
$[1^223] =$			1	− 3	− 4	+ 6	+ 9	−21
$[12^3] =$				1	*	− 3	+ 5	− 7
$[1^43] =$					1	− 2	− 1	+ 7
$[1^32^2] =$						1	− 5	+ 1
$[1^52] =$							1	− 7
$[1^7] =$								1

[35] = 1|— 2| ·|— 3|+ 6|+ 3|— 7|+ 3|— 9|+ 8|+ 1|— 4|— 3|+ 1|+ 7|+ 8|— 8|— 8|+ 8|
[42] = 1| ·| ·|— 4|+ 2|+ 4| ·|+ 4|— 4|— 8|+ 6|— 4|+ 8|— 4|+ 4| 4|— 4|+ 4|

TABLE II.

$[35]$	=	1	− 2	*	− 3	+ 6	+ 3	− 7	+ 3	− 9	+ 8	+ 1	− 4	− 3	+ 1	+ 7	+ 8	− 8	− 8	+ 8
$[4^2]$	=		1	*	*	− 4	+ 2	+ 4	*	+ 4	− 4	− 8	+ 6	− 4	+ 8	− 4	+ 4	− 4	− 4	+ 4
$[1^2 6]$	=			1	− 5	+ 5	+ 5	− 5	− 1	+ 4	− 2	− 9	+ 4	+ 1	− 3	+ 8	− 1	+ 2	+ 1	− 8
$[125]$	=				1	− 3	− 1	+ 5	− 3	+11	− 4	−10	+ 8	+ 3	− 8	+ 1	− 3	+ 4	+ 9	−16
$[134]$	=					1	− 2	− 1	*	− 1	*	+10	− 8	+ 4	−10	+ 1	− 9	+16	+ 9	−16
$[2^2 4]$	=						1	− 2	*	− 2	+ 4	*	− 4	+ 2	− 4	+ 8	− 2	− 4	+ 8	− 8
$[23^2]$	=							1	*	*	− 2	− 1	+ 4	*	+ 5	− 7	− 5	+ 2	+ 8	− 8
$[1^3 5]$	=								1	− 4	+ 2	+ 4	− 4	− 1	+ 3	− 3	+ 1	− 2	− 1	+ 8
$[1^2 24]$	=									1	− 2	− 1	+ 4	− 4	+11	− 9	+ 4	− 6	−10	+24
$[1^2 3^2]$	=										1	− 2	+ 2	*	− 1	+ 3	+ 5	− 9	− 5	+12
$[12^2 3]$	=											1	− 4	*	− 3	+ 6	+ 5	*	−17	+24
$[2^4]$	=												1	*	*	− 2	*	+ 2	− 2	+ 2
$[1^4 4]$	=													1	− 3	+ 3	− 1	+ 2	+ 1	− 8
$[1^3 23]$	=														1	− 3	− 5	+ 8	+11	−32
$[1^2 2^3]$	=															1	*	− 4	+ 9	−16
$[1^5 3]$	=																1	− 2	− 1	+ 8
$[1^4 2^2]$	=																	1	− 6	+20
$[1^6 2]$	=																		1	− 8
$[1^8]$	=																			1

I.—On the roots of equations, the sums of their powers, and the products of these powers, and symmetrical functions in general.

SECTION I.

In all good elementary books on Algebra it is shewn, that the first part of an equation of the undetermined nth degree......(ψ)......

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + Px + Q = o$$

may always be considered as a product of n simple factors of the form $x-a$, $x-b$, $x-c$, $x-d$, &c., and that then a, b, c, d, &c. are all the values of the unknown magnitude x, which verify the equation (ψ). If we actually multiply these factors, and compare the product resulting from this multiplication with the polynomial in the first part of the equation, then we have the following relations between the coefficients A, B, C, D, &c. of the equation, and its roots a, b, c, d, &c.:

$$-A = a + b + c + d + \&c.$$
$$B = ab + ac + ad + bc + bd + cd + \&c.$$
$$-C = abc + abd + acd + bcd + \&c.$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\pm Q = abcd \;\&c.$$

Thus the first coefficient A, with the sign changed, is always the sum of all the roots; the second coefficient B, with its own sign, is the sum of all the products of every two of these roots; the third coefficient C, with the sign changed, is the sum of all the products of every three of these roots; and generally the undetermined mth coefficient, with its sign changed or not, according as m is an odd or even number, is the sum of all the products which arise from the combination of all the roots, taken m and m together; finally, the last term Q, (which may also be considered as the coefficient of x^0,) with its sign changed or not, according as n is an odd or even number, is merely the product of all the roots.

The coefficients A, B, C, D, &c., are consequently no other than the aggregates of the combinations of the roots a, b, c, d, &c., taken singly (one by one), two and two, three and three, four and four, &c., or, to express myself more precisely according to Hindenburg, the aggregates of the combinations without repetitions of the first, second, third, &c. class. How these combinations may be easily represented, will be shown under the head of combinations, which are here supposed to be known only in their first principles.

SECTION II.

In the following pages, certain notations are frequently used, which, in fact, are already known to the greater part of my readers; the meaning of these notations, however, in order to prevent mistakes and confusion, I shall give in this place.

1. When determinate or indeterminate magnitudes are spoken of, all algebraical expressions, in which these two kinds of magnitudes are in any way involved, are called functions. We then use the formula: "This or that expression is a function of these or those magnitudes"—because we only mention the indeterminate magnitudes, omitting the determinate.

On account of the particular use which we shall make of functions in this Work, I wish it to be remembered, once for all, that here (when the contrary is not expressly mentioned), only such functions are meant, as may immediately be determined by means of the six arithmetical operations of Addition, Subtraction, Multiplication, Division, Involution and Evolution, so soon as the magnitudes contained in these functions are known, and when they do not contain a magnitude considered as indeterminate either as the exponent of powers, or the index of roots.

2. A rational function is one in which either there are no irrational magnitudes, or one in which at least those magnitudes which are considered as indeterminate are not under the radical sign; in the contrary case the function is an irrational one.

3. An integral function is one in which either there is no denominator, or in which, at least, those magnitudes which are considered as indeterminate, are not found in the denominator; in the contrary case it is called a fractional function.

The coefficients A, B, C, &c. of the equation (ψ) in

§ I., are consequently integral and rational functions of the roots a, b, c, d, &c., so long as these magnitudes are considered as indeterminate. Here no reference is made to the particular properties of the magnitudes a, b, c, d, &c. themselves; consequently these may be rational or irrational, integral or fractional, and, generally, may have every possible form, or they may even be functions of other magnitudes.

4. Those functions which are here called symmetrical, are those in which the indeterminate magnitudes are so combined, that, independently of the particular values of these magnitudes, no change takes place in the value of the function, however these magnitudes are substituted for one another.

The coefficients A, B, C, D, &c. of the equation (ψ) in § I., are ∴ symmetrical functions of the roots a, b, c, d, &c.; thus they remain the same when a is substituted for b, or b for c, or a for c, and b for d, and so in like manner of other substitutions.

From this definition it immediately follows, that the sums, remainders, products, quotients, powers and roots of symmetrical functions, are again symmetrical functions, provided the functions, which are combined together by addition, subtraction, multiplication and division, contain all the same indeterminate magnitudes, and in the same number. Thus the expression

$$\frac{(ab + ac + bc)^m + abc}{\sqrt[n]{(abc - a - b - c)}}$$

is a symmetrical function of a, b, c, because $ab + ac + bc$,

abc, $a+b+c$, are functions of this kind. Generally, every function of one or more symmetrical functions is always again a symmetrical function, when these last contain the same indeterminate magnitudes and in the same number.

Now it may be shown that every rational, integral, or fractional function of the roots of an equation, however constituted, may always be expressed rationally by the coefficients of this equation. This highly important relation between the coefficients and the roots, has thrown more light on the theory of equations than any other; and should human genius ever succeed in fully discovering the secret of its solution, so far as this is possible, it will probably be by such inquiries as are exactly founded on this very property.

SECTION III.

For the sake of brevity and perspicuity, I shall use the following symbols:

Let the sum of all the roots of an equation, their squares, their cubes, their biquadrates, and, in general, the sum of their μth powers, be represented by (1), (2), (3), (4) . . . . . (μ), so that only the exponents, but not the roots, are indicated, because the latter are not considered in the present case. Therefore,

$$(1) = a + b + c + d + e + \text{\&c.}$$
$$(2) = a^2 + b^2 + c^2 + d^2 + e^2 + \text{\&c.}$$
$$(3) = a^3 + b^3 + c^3 + d^3 + e^3 + \text{\&c.}$$
$$\cdots\cdots\cdots\cdots$$
$$(\mu) = a^\mu + b^\mu + c^\mu + d^\mu + e^\mu + \text{\&c.}$$

If the roots a, b, c, d, &c. taken two and two, be combined in all possible ways, and in each such combina-

tion every alternate root raised to the power α, the other to the power β, then the sum of all the products thus obtained may be represented by $(\alpha\beta)$. Consequently, when only four roots are assumed, the equation (ψ), in § I., is of the fourth degree,

$$(\alpha\beta) = a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + a^\alpha d^\beta + a^\beta d^\alpha + b^\alpha c^\beta + b^\beta c^\alpha + b^\alpha d^\beta + b^\beta d^\alpha + c^\alpha d^\beta + c^\beta d^\alpha.$$

In a similar manner $(\alpha\beta\gamma)$ denotes the sum of all the products which arise from the combination of all the roots, taken three and three, and in each such combination raising one root to the power α, another to the power β, and the third to the power γ, and this in as many ways as possible. When again only four roots are assumed, we ∴ have

$$\begin{aligned}(\alpha\beta\gamma) = {} & a^\alpha b^\beta c^\gamma + a^\alpha b^\gamma c^\beta + a^\beta b^\alpha c^\gamma + a^\beta b^\gamma c^\alpha + a^\gamma b^\alpha c^\beta + a^\gamma b^\beta c^\alpha + \\ & a^\alpha b^\beta d^\gamma + a^\alpha b^\gamma d^\beta + a^\beta b^\alpha d^\gamma + a^\beta b^\gamma d^\alpha + a^\gamma b^\alpha d^\beta + a^\gamma b^\beta d^\alpha + \\ & a^\alpha c^\beta d^\gamma + a^\alpha c^\gamma d^\beta + a^\beta c^\alpha d^\gamma + a^\beta c^\gamma d^\alpha + a^\gamma c^\alpha d^\beta + a^\gamma c^\beta d^\alpha + \\ & b^\alpha c^\beta d^\gamma + b^\alpha c^\gamma d^\beta + b^\beta c^\alpha d^\gamma + b^\beta c^\gamma d^\alpha + b^\gamma c^\alpha d^\beta + b^\gamma c^\beta d^\alpha.\end{aligned}$$

In general, $(\alpha\beta\gamma\delta\ldots\ldots\kappa)$, when m is the number of the letters $\alpha, \beta, \gamma, \delta, \ldots\ldots \kappa$, denotes a sum of the products which arise from the combination of all the roots to the mth class, and in each class one root is raised to the power α, another to the power β, a third to the power γ, and so on, in as many ways as possible.

In order ∴ to represent actually the expression $(\alpha\beta\gamma\delta \ldots\ldots\kappa)$, find all the combinations of the roots a, b, c, d, &c. taken m and m together, give the roots in each such combination the exponents $\alpha, \beta, \gamma, \ldots\ldots, \kappa$, and then

permute the exponents in all possible ways. By this method we get

$[\alpha\beta\gamma\delta] =$

$$a^\alpha b^\beta c^\gamma d^\delta + a^\alpha b^\beta c^\delta d^\gamma + \ldots\ldots + a^\delta b^\gamma c^\beta d^\alpha +$$
$$a^\alpha b^\beta c^\gamma e^\delta + a^\alpha b^\beta c^\delta e^\gamma + \ldots\ldots + a^\delta b^\gamma c^\beta e^\alpha +$$
&c.

$[\alpha\alpha\alpha\beta\beta] =$

$$a^\alpha b^\alpha c^\alpha d^\beta e^\beta + a^\alpha b^\alpha c^\beta d^\alpha e^\beta + \ldots + a^\beta b^\beta c^\alpha d^\alpha e^\alpha +$$
$$a^\alpha b^\alpha c^\alpha d^\beta f^\beta + a^\alpha b^\alpha c^\beta d^\alpha f^\beta + \ldots + a^\beta b^\beta c^\alpha d^\alpha f^\alpha +$$
&c.

In order to render this notation more convenient, it would be better to use the repeating exponents in those terms where an exponent occurs more than once; thus, for instance, $[\alpha^3\beta^2]$ instead of $[\alpha\alpha\alpha\beta\beta]$, and $[\alpha^2\beta^2\gamma^2]$ instead of $[\alpha\alpha\beta\beta\gamma\gamma]$.

By § I., the relation between the coefficients and the roots of an equation may be thus represented:
$-A = [1]$, $B = [11] = [1^2]$, $-C = [111] = [1^3]$, $D = [1111], = [1^4]$, $-E = [11111], = [1^5]$, &c.

From the construction of the function $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$, it follows, that it belongs to symmetrical functions, because it undergoes no change when the roots a, b, c, d, &c. are substituted for one another.

The function $[\alpha\beta\gamma\delta\ldots\kappa]$, or more generally $[\alpha^a\beta^b\gamma^c\delta^d \ldots\kappa^k]$, I also sometimes call a numerical expression. The exponents α, β, γ, δ, ... κ, are called radical exponents, in order to distinguish them from the repeating exponents a, b, c, d, ... k.

The radical exponents may also be negative, and then

$$[-1] = a^{-1} + b^{-1} + c^{-1} + \&c. = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \&c.$$

$$[-2] = a^{-2} + b^{-2} + c^{-2} + \&c. = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \&c.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[-\mu] = a^{-\mu} + b^{-\mu} + c^{-\mu} + \&c. = \frac{1}{a^\mu} + \frac{1}{b^\mu} + \frac{1}{c^\mu} + \&c.$$

and in the same manner

$$[\overline{-\alpha}\beta] = a^{-\alpha}b^\beta + a^\beta b^{-\alpha} + a^{-\alpha}c^\beta + a^\beta c^{-\alpha} + \&c.$$

$$= \frac{b^\beta}{a^\alpha} + \frac{a^\beta}{b^\alpha} + \frac{c^\beta}{a^\alpha} + \frac{a^\beta}{c^\alpha} + \&c.$$

$$[\overline{-\alpha}\,\overline{-\beta}] = a^{-\alpha}b^{-\beta} + a^{-\beta}b^{-\alpha} + a^{-\alpha}c^{-\beta} + a^{-\beta}c^{-\alpha} + \&c.$$

$$= \frac{1}{a^\alpha b^\beta} + \frac{1}{a^\beta b^\alpha} + \frac{1}{a^\alpha c^\beta} + \frac{1}{a^\beta c^\alpha} + \&c.$$

and in a similar way it obtains with the numerical expressions, in which more than one negative radical exponent occur.

SECTION IV.

Prob. To find the number of terms of which the numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ consists.

Solution 1. Let the number of the roots a, b, c, d, &c. $= n$, and the number of the radical exponents α, β, γ, δ, $\ldots \kappa = m$.

2. The terms which compose the numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ may be found by combining the n roots a, b, c, d, &c. to the mth class, and by permuting in each such combination the m radical exponents α, β, γ, δ, $\ldots \kappa$ (§ III.) The number of these terms is consequently equal to the product of the number of combinations of n things to the mth class, and the number of permutations of m different things.

3. But the number of combinations of n things, taken m and m, is

$$=\frac{n\,.\,n-1\,.\,n-2\,\ldots\ldots\,n-m+2\,.\,n-m+1}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\,m-1\,.\,m}$$

and the number of permutations of m things is

$$=1\,.\,2\,.\,3\,\ldots\ldots\ldots\,m-1\,.\,m.$$

4. Hence the number of the terms of the numerical expression $[\alpha\beta\gamma\delta\,\ldots\ldots\,\kappa]$

$$=\frac{n\,.\,n-1\,.\,n-2\,\ldots\ldots\,n-m+1}{1\,.\,2\,.\,3\,\ldots\ldots\,m}\times 1\,.\,2\,.\,3\,\ldots\,m$$

$$=n\,.\,n-1\,.\,n-2\,\ldots\ldots\,n-m+2\,.\,n-m+1$$

SECTION V.

Prob. To find the number of the terms of the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\,\ldots\ldots\,\kappa^{\mathfrak{k}}]$.

Solution 1. Let the number of all the roots a, b, c, d, &c. $=n$; let the number of the radical exponents, without reference to their equality or difference, or, which is the same in this case, let the sum of the repeating exponents $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+\,\ldots\ldots\,+\mathfrak{k}$, &c. $=m$.

2. Since each term of $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\ldots\kappa^{\mathfrak{k}}]$ contains m of the roots a, b, c, d, &c., in order to find the number of terms, we must here, as in the preceding section, multiply the number of combinations of n roots taken m and m by the number of permutations of their m radical exponents.

3. But the former

$$=\frac{n\,.\,n-1\,.\,n-2\,\ldots\ldots\,n-m+2\,.\,n-m+1}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,m-1\,.\,m}.$$

The second will be obtained by finding how often the letters $\alpha, \beta, \gamma, \delta, \ldots\ldots, \kappa$, can be transposed in a series of elements $\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\ldots\kappa^{\mathfrak{k}}$, in which one letter occurs $\mathfrak{a}$ times, another $\mathfrak{b}$, a third $\mathfrak{c}$, and so on. The number of these transpositions, however, is (as may be seen from the rule of combinations)

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots m-1 \cdot m}{1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots \times 1 \cdot 2 \ldots \mathfrak{k}}$$

4. If $\therefore$ these two numbers are multiplied together, we obtain the number of the terms of $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\ldots\kappa^{\mathfrak{k}}]$

$$= \frac{n \cdot n-1 \cdot n-2 \ldots\ldots\ldots\ldots\ldots n-m+2 \cdot n-m+1}{1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 \cdot 2 \ldots \mathfrak{k}}$$

Corollary. If the m radical exponents are all different from one another, then, by the preceding section, the number of terms $= n \cdot n-1 \cdot n-2 \ldots n-m+2 \cdot n-m+1$. Hence it follows, that in the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\ldots\kappa^{\mathfrak{k}}]$, the number of terms is less by $1 \cdot 2 \ldots \mathfrak{a} \times 1 \cdot 2 \ldots \mathfrak{b} \times 1 \cdot 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 \cdot 2 \ldots \mathfrak{k}$ than when the radical exponents are different.

EXAMPLE. For the expression $[\alpha^4\beta^3\gamma^2]$, when the equation to which it relates is of the twelfth degree, we have $n = 12$, $\mathfrak{a} = 4$, $\mathfrak{b} = 3$, $\mathfrak{c} = 2$; $\therefore m = 9$. The number of terms, of which this expression consists, is consequently

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \times 1 \cdot 2 \cdot 3 \times 1 \cdot 2} = 277200$$

SECTION VI.

Prob. Let Σ denote the sum of all the combinations without repetitions of n letters a, b, c, d, &c. of the mth class; further, let Σ' be the sum of all the combinations of this class, which do not contain a; Σ'' the sum of all those which do not contain b; Σ''' the sum of all those which do not contain c, and so on; find the relation between $\Sigma' + \Sigma'' + \Sigma''' +$ &c. and Σ.

Solution 1. If in Σ', Σ'', Σ''', &c. all the combinations are completed, or if $\Sigma = \Sigma' = \Sigma'' = \Sigma''' =$ &c. then $\Sigma' + \Sigma'' + \Sigma''' +$ &c. $= n\,\Sigma$. But since in these some of the combinations are wanting, consequently their sum must be less than $n\,\Sigma$.

2. But it is evident, that each distinct combination contained in Σ must be wanting in the sum $\Sigma' + \Sigma'' + \Sigma''' +$ &c. exactly as many times as it contains elements. Then supposing the different combinations in Σ consist of four letters, then, for instance, the combination $a\,b\,c\,d$ fails once in Σ', Σ'', Σ''', Σ^{iv}.

3. If $\therefore$, in general, m is the number of elements contained in each combination, then the sum of all the combinations which fail in $\Sigma' + \Sigma'' + \Sigma''' +$ &c. is $m\,\Sigma$.

4. Hence and from 1. it follows, that

$$\Sigma' + \Sigma'' + \Sigma''' + \text{\&c.} = (n-m)\,\Sigma.$$

Corollary. Therefore $\Sigma' + \Sigma'' + \Sigma''' +$ &c. for the first class $= (n-1)\,\Sigma$; for the second $= (n-2)\,\Sigma$; for the third $= (n-3)\,\Sigma$; and so on.

Example. Let Σ be the sum of all the combinations, taken three and three, of five elements a, b, c, d, e; then $\Sigma = abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$, $\Sigma' = bcd + bce + bde + cde$, $\Sigma'' = acd + ace + ade + cde$, $\Sigma''' = abd + abe + ade + bde$, $\Sigma^{IV} = abc + abe + ace + bce$, $\Sigma^{V} = abc + abd + acd + bcd$, and $\Sigma' + \Sigma'' + \Sigma''' + \Sigma^{IV} + \Sigma^{V} = 2\,\Sigma = (5 - 3)\ \Sigma$, as was required.

SECTION VII.

Prob. There are two equations,

$$x^n \ + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = o$$

$$x^{n-1} + A'x^{n-2} + B'x^{n-3} + C'x^{n-4} + \&c. = o$$

of which the second has the same number of roots as the first, except a: find the relation between the coefficients of these two equations.

Solution. The second equation may be obtained from the first, by dividing the latter by $x - a$. By actually performing this division, we obtain

$$x^{n-1} + (a + A)\ x^{n-1} + (a^2 + aA + B)\ x^{n-2} + (a^3 + a^2A + aB + C)\ x^{n-3} + \&c. = o.$$

Hence now it follows, that $A' = a + A$, $B' = a^2 + aA + B$, $C' = a^3 + a^2A + aB + C$ &c. and, in general,

$$\overset{m}{A'} = a^{-m} + a^{m-1}A + a^{m-2}B + a^{m-3}C + \dots\dots + \overset{m}{A}$$

When $\overset{m}{\mathrm{A}}$ and $\overset{m}{\mathrm{A}'}$ denote the mth coefficients in the first and second equations.

SECTION VIII.

Prob. From a given equation to find the sums of the

squares, cubes, biquadrates, and, generally, the sum of any power of its roots, without knowing these roots, assuming that the exponent of this power is a whole positive number.

Solution 1. Let

$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots\ldots + Px + Q = o$

be the given equation, whose roots are a, b, c, d, &c. Further, let

$$\begin{aligned} &x^{n-1} + A'x^{n-2} + B'x^{n-3} + C'x^{n-4} + \&c. = o \\ &x^{n-1} + A''x^{n-2} + B''x^{n-3} + C''x^{n-4} + \&c. = o \\ &x^{n-1} + A'''x^{n-2} + B'''x^{n-3} + C'''x^{n-4} + \&c. = o \\ &\qquad\qquad \&c. \end{aligned}$$

be the n equations, which arise from dividing the given equation by $x-a$, $x-b$, $x-c$, &c. successively.

2. Then the coefficients A, B, C, D, &c. are the positive or negative sums of the letters, taken singly, two and two, three and three, four and four, and so on, of n roots a, b, c, d, &c.; the coefficients A', B', C', D', &c., the sums of the letters, taken singly, two and two, three and three, four and four, and so on, of the $n-1$ roots, b, c, d, e, &c.; the coefficients A'', B'', C'', D'', &c.; the positive or negative sums of the $n-1$ roots, a, c, d, e, &c. taken singly, two and two, three and three, four and four. Then (§ VI.)

$$\begin{aligned} A' + A'' + A''' + \&c. &= (n-1)\,A \\ B' + B'' + B''' + \&c. &= (n-2)\,B \\ C' + C'' + C''' + \&c. &= (n-3)\,C \\ &\&c. \end{aligned}$$

3. But from the preceding §

$$A' = a + A,\ A'' = b + A,\ A''' = c + A,\ \&c.$$

If we use the symbols in § III., we consequently have

$$A' + A'' + A''' + \&c. = (1) + nA$$

Since further (foregoing §)

$$B' = a^2 + aA + B,\ B'' = b^2 + bA + B,$$
$$B''' = c^2 + cA + B,\ \&c.$$

then we have

$$B' + B'' + B''' + \&c. = (2) + A\,(1) + nB$$

In like manner we find

$$C' + C'' + C''' + \&c. = (3) + A\,(2) + B\,(1) + nC$$
$$\&c.$$

4. From 2. and 3. we obtain the following equations:

$$(1) + nA = (n-1)\,A$$
$$(2) + A\,(1) + nB = (n-2)\,B$$
$$(3) + A\,(2) + B\,(1) + nC = (n-3)\,C$$
$$\&c.$$

or

$$(1) + A = o$$
$$(2) + A\,(1) + 2B = o$$
$$(3) + A\,(2) + B\,(1) + 3\,C = o$$

and in general

$$(m) + A\,(m-1) + B\,(m-2) + \ldots + \overset{m-1}{A}\,(1) + m\overset{m}{A} = o$$

where $\overset{m-1}{A}$, $\overset{m}{A}$, denote the $(m-1)$th and the mth coefficients when, $m < n$. But if $m =$ or $> n$, then the conclusions which have been drawn no longer obtain, because, in this case, the sixth section, on which they are founded, ceases to be applicable. We can, however, for this case find a similar equation by another method.

5. Thus, if we multiply the given equation by x^{m-n}, we obtain

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots\ldots + Px^{m-n+1} + Qx^{m-n} = 0$$

and if we substitute in this equation a, b, c, d successively for x, we have

$$a^m + Aa^{m-1} + Ba^{m-2} + \ldots\ldots + Pa^{m-n+1} + Qa^{m-n} = 0$$
$$b^m + Ab^{m-1} + Bb^{m-2} + \ldots\ldots + Pb^{m-n+1} + Qb^{m-n} = 0$$
$$c^m + Ac^{m-1} + Bc^{m-2} + \ldots\ldots + Pc^{m-n+1} + Qc^{m-n} = 0$$
&c.

If we add these equations together, we obtain

$$(m) + A\,(m-1) + B\,(m-2) + \ldots\ldots + P\,(m-n+1) + Q\,(m-n) = 0.$$

6. If in this equation we put $m = n$, then, because $(0) = a^0 + b^0 + c^0 + d^0 + \&c. = n$ we have

$$(n) + A\,(n-1) + B\,(n-2) + \ldots\ldots + P\,(1) + nQ = 0$$

7. By means of the equations found in 4. 5. and 6. we are now enabled to express the sum of every higher power by the sums of all the lower; and consequently, when these last are found, we are enabled to find the former. On account of the frequent use which is made of them in the following pages, I shall here arrange them together.

$$(1) + A = 0$$
$$(2) + A\,(1) + 2B = 0$$
$$(3) + A\,(2) + B\,(1) + 3C = 0$$
$$(4) + A\,(3) + B\,(2) + C\,(1) + 4D = 0$$
. . . .

$$(n-1)+A(n-2)+B(n-3)+\ldots+M(1)+(n-1)P=0$$
$$(n)+A(n-1)+B(n-2)+\ldots+P(1)+nQ=0$$
$$(n+1)+A(n)+B(n-1)+\ldots+P(2)+Q(1)=0$$
$$(n+2)+A(n+1)+B(n)+\ldots+P(3)+Q(2)=0$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(m)+A(m-1)+B(m-2)+\ldots\ldots+P(m-n+1)$$
$$+Q(m-n)=0$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

8. From these equations we successively obtain

$$(1)=-A$$
$$(2)=A^2-2B$$
$$(3)=-A^3+3AB-3C$$
$$(4)=A^4-4A^2B+2B^2+4AC-4D$$
$$(5)=-A^5+5A^3B-5AB^2-5A^2C+5BC+5Ad-5E$$
$$(6)=A^6-6A^4B+9A^2B^2-2B^3+6A^3C+12ABC+3C^2-6A^2D+6BD+6AE-6F$$

&c.

and consequently the sum of the powers of the roots are expressed directly by the coefficients of the given equation.

EXAMPLE. When the equation $x^4-x^3-19x^2+49x-30=0$, $A=-1$, $B=-19$, $C=49$, $D=-30$. By substituting these values in the equations in 8. we obtain $(1)=1$, $(2)=39$, $(3)=-89$, $(4)=723$, $(5)=-2849$, $(6)=16419$, &c. Any person may easily convince himself of the truth of these results, by substituting in the first equation 1, 2, 3, -5 for x.

REMARK. The formulæ in 8. are known by the name of the Newtonian Theorem, because Newton is supposed to be the first who has mentioned it. Other proofs of this theorem, and also much information relating to the subject itself, may be found amongst other matter in Kästner's Principles of Finite Analytical Magnitudes, third edition, p. 538, &c. also in Klügel's Mathematical Dictionary, part first, p. 465, &c. Art. Combination.

SECTION IX.

PROB. The sums of the powers of the roots of an equation, or the expressions (1), (2), (3), &c. are given: find the coefficients of this equation.

Solution. From the equations in 7. of the foregoing section, we obtain by transposition

$$A = -(1)$$

$$B = -\frac{A(1) + (2)}{2}$$

$$C = -\frac{B(1) + A(2) + (3)}{3}$$

$$D = -\frac{C(1) + B(2) + A(3) + (4)}{4}$$

&c.

By means of these equations we are enabled to determine successively the coefficients A, B, C, D, &c. when the numerical expressions (1), (2), (3), (4), &c. are given, as they are assumed to be in the problem.

SECTION X.

PROB. From a given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Mx^3 + Nx^2 + Px + Q = 0$$

whose unknown roots are called a, b, c, d, &c. find another, whose roots are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c.

Solution. Substitute $\frac{1}{y}$ for x, and then multiply the whole equation by y^n; we then get

$$Qy^n + Py^{n-1} + Ny^{n-2} + My^{n-3} + \ldots\ldots + Ay + 1 = 0$$

or

$$y^n + \frac{P}{Q}y^{n-1} + \frac{N}{Q}y^{n-2} + \frac{M}{Q}y^{n-3} + \ldots\ldots + \frac{A}{Q} + \frac{1}{Q} = 0$$

and this is the required equation. For since $x = \frac{1}{y}$, then $y = \frac{1}{x}$, and since a, b, c, d, &c. are the values of x, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. are the values of y.

Corollary. The roots $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. in reference to the roots of the given equation, we term reciprocal roots. Therefore, if $x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Mx^3 + Nx^2 + Px + Q = 0$ be any equation, and $x^n + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \ldots\ldots + P'x + Q' = 0$, be the equation for its reciprocal roots; we then have

$$A' = \frac{P}{Q},\ B' = \frac{N}{Q},\ C' = \frac{M}{Q},\ \&c.$$

SECTION XI.

Prob. Find the sum of a power of roots, when the exponent of this power is a whole negative number.

Solution. Let

$$x^n + Ax^{n-1} + \ldots\ldots + Mx^3 + Nx^2 + Px + Q = 0$$

be the given equation, and

$$x^n + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \ldots\ldots + P'x + Q' = 0$$

the equation for its reciprocal roots (foregoing section.) Then according to § VIII, when the numerical expressions (1), (2), (3), &c. are taken in reference to the second equation,

$$(1) + A' = 0$$
$$(2) + A'(1) + 2B' = 0$$
$$(3) + A'(2) + B'(1) + 3C' = 0$$

2. But (1), (2), (3), in reference to the second equation, are precisely what (—1), (—2), (—3), &c. are in reference to the first equation; we have ∴, when for A', B', C', &c. their values $\frac{P}{Q}, \frac{N}{Q}, \frac{M}{Q}$, &c., are substituted (foregoing section)

$$(-1) + \frac{P}{Q} = 0$$
$$(-2) + \frac{P}{Q}(-1) + \frac{2N}{Q} = 0$$
$$(-3) + \frac{P}{Q}(-2) + \frac{N}{Q}(-1) + \frac{3M}{Q} = 0$$

&c.

from which we can determine successively the sums of powers for negative exponents.

Example. When the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, we have $Q = -30$, $P = +49$, $N = -19$, $M = -1$: we have ∴ $(-1) = \frac{49}{30}$, $(-2) = \frac{1261}{900}$,

$(-3) = \frac{31159}{27000}$. Any one may readily be convinced of the accuracy of these results; for the roots of the given equation are 1, 2, 3, -5, consequently $(-1) = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{5} = \frac{49}{30}$, $(-2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{25} = \frac{1261}{900}$, $(-3) = 1 + \frac{1}{8} + \frac{1}{27} - \frac{1}{125} = \frac{31159}{27000}$, &c.

SECTION XII.

PROB. Express the symmetrical function $(\alpha\beta)$ by the sums of powers.

Solution. For the sake of greater perspicuity, I shall assume that there are only four roots, because this does not affect the general principle. Then

$$(\alpha) = a^\alpha + b^\alpha + c^\alpha + d^\alpha$$
$$(\beta) = a^\beta + b^\beta + c^\beta + d^\beta.$$

If we multiply these equations together, we then obtain

$$\begin{aligned}(\alpha)\,(\beta) = {} & a^{\alpha+\beta} + b^{\alpha+\beta} + c^{\alpha+\beta} + d^{\alpha+\beta} + \\ & a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + a^\alpha d^\beta + a^\beta d^\alpha + \\ & b^\alpha c^\beta + b^\beta c^\alpha + b^\alpha d^\beta + b^\beta d^\alpha + c^\alpha d^\beta + c^\beta d^\alpha.\end{aligned}$$

The first row of the second part of this equation $= (\alpha + \beta)$, and the remaining two rows $= (\alpha\beta)$; consequently we have

$$(\chi) \ldots\ldots (\alpha)\,(\beta) = (\alpha+\beta)\,(\alpha\beta)$$

and $\therefore$

$$(\alpha\beta) = (\alpha)\,(\beta) - (\alpha + \beta).$$

It is easily seen, that these conclusions obtain, let the number of the roots be what it may. Since then (α), (β),

$(\alpha + \beta)$, are only the sums of powers, what was required is now done.

The radical exponents α, β, may besides be either positive or negative. For example, if α be negative, then we have

$$\overline{(-\alpha\beta)} = (-\alpha)\ (\beta) = (\beta - \alpha)$$

and when α and β are both negative,

$$(\overline{-\alpha}\ \overline{-\beta}) = (-\alpha)\ (-\beta) - (-\alpha - \beta).$$

Corollary. Since we are always enabled to express the sums of powers, either for positive or negative exponents, by the coefficients of the given equation, in like manner we can always find the values of the expressions of the form $a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + \&c.$, $\frac{b^\beta}{a^\alpha} + \frac{a^\beta}{b^\alpha} + \frac{c^\beta}{a^\alpha} + \frac{a^\beta}{c^\alpha} + \&c.$, $\frac{1}{a^\alpha b^\beta} + \frac{1}{a^\beta b^\alpha} + \frac{1}{a^\alpha c^\beta} + \frac{1}{a^\beta c^\alpha} + \&c.$ from these coefficients, without knowing the roots.

SECTION XIII.

Prob. Reduce the numerical expression $(\alpha\beta\gamma)$, which contains three radical exponents, to a numerical expression containing no more than two radical exponents.

Solution. For the sake of perspicuity, I shall only begin with three roots a, b, c. If we multiply the equation

$$(\alpha\beta) = a^\alpha b^\beta + a^\beta b^\alpha + a^\alpha c^\beta + a^\beta c^\alpha + b^\alpha c^\beta + b^\beta c^\alpha$$

by

$$(\gamma) = a^\gamma + b^\gamma + c^\gamma,$$

we then obtain

$(\gamma)\,(\alpha\beta) =$
$$a^{\alpha+\gamma}b^{\beta} + a^{\beta}b^{\alpha+\gamma} + a^{\alpha+\gamma}c^{\beta} + a^{\beta}c^{\alpha+\gamma} + b^{\alpha+\gamma}c^{\beta} + b^{\beta}c^{\alpha+\gamma} +$$
$$a^{\beta+\gamma}b^{\alpha} + a^{\alpha}b^{\beta+\gamma} + a^{\beta+\gamma}c^{\alpha} + a^{\alpha}c^{\beta+\gamma} + b^{\beta+\gamma}c^{\alpha} + b^{\alpha}c^{\beta+\gamma} +$$
$$a^{\alpha}b^{\beta}c^{\gamma} + a^{\alpha}b^{\gamma}c^{\beta} + a^{\beta}b^{\alpha}c^{\gamma} + a^{\beta}b^{\gamma}c^{\alpha} + a^{\gamma}b^{\alpha}c^{\beta} + a^{\gamma}b^{\beta}c^{\alpha}.$$

2. Now since the first row in the second part of this equation $= (\overline{\alpha+\gamma\beta})$, the second $= (\overline{\beta+\gamma\alpha})$, and the third $= (\alpha\beta\gamma)$, we consequently have

$$(\psi) \ldots\ldots (\gamma)\,(\alpha\beta) = (\overline{\alpha + \gamma\beta}) + (\overline{\beta + \gamma\alpha}) + (\alpha\beta\gamma)$$

and hence further

$$(\alpha\beta\gamma) = (\gamma)\,(\alpha\beta) - (\overline{\alpha + \gamma\beta}) + (\overline{\beta + \gamma\alpha})$$

The functions $(\alpha\beta\gamma)$ is reduced to three others $(\alpha\beta)$, $(\overline{\beta+\gamma\beta})$, $(\overline{\beta+\gamma\alpha})$, each of which contains only two radical exponents, as required.

3. But since in this case only three roots $a, b, c,$ were assumed, there may be a doubt as to the general application of the result so found. The following proof will remove this doubt.

4. In the first place, it is evident, that there are no equal terms in the product $(\gamma)\,(\alpha\beta)$. Take any term, for instance $b^{\beta+\gamma}d^{\alpha}$ of this product. This term can arise in no other way than by the multiplication of b^{γ} in (γ) by $b^{\beta}d^{\alpha}$ in $(\alpha\beta)$. If this term occurred more than once in the evolution of $(\gamma)\,(\alpha\beta)$, then $b^{\beta}d^{\alpha}$ must also occur more than once in $(\alpha\beta)$, which is impossible. But in the aggregate $(\overline{\alpha + \gamma\beta}) + (\overline{\beta + \gamma\alpha}) + (\alpha\beta\gamma)$, which constitutes the second term of the equation (ψ), there are not even two equal terms; this immediately appears from

the construction of the numerical expression, of which this aggregate consists.

5. But when the number of roots $= n$, the numerical expression $(\alpha\beta)$ consists of $n \, . \, n-1$ terms (§ IV), and consequently the product (γ) $(\alpha\beta)$, of $n^2 \, . \, n-1$ terms. The number of terms, which the sums $(\overline{\alpha+\gamma\beta})$, $(\overline{\beta+\gamma\alpha})$, $(\alpha\beta\gamma)$ contain, are for each of the two first $=n \, . \, n-1$, and for the last $= n \, . \, n-1 \, . \, n-2$; consequently for the aggregate $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma) = 2n \, . \, n-1 + n \, . \, n-1 \, . \, n-2 = n^2 \, . \, n-1$. This aggregate $\therefore$ contains exactly as many terms as the product (γ) $(\alpha\beta)$.

6. Further I affirm, that in the aggregate $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma)$ there can be no term, which is not also in (γ) $(\alpha\beta)$: for if, for instance, the terms $c^{\alpha+\gamma} + d^{\beta}$, $b^{\gamma} c^{\beta} e^{\alpha}$ are not in (γ) $(\alpha\beta)$, then likewise the terms $c^{\alpha} d^{\beta}$, $c^{\beta} e^{\alpha}$ are not in $(\alpha\beta)$; which is impossible.

7. From these conclusions it follows: first, that the terms of each of the two functions (γ) $(\alpha\beta)$, $(\overline{\alpha+\gamma\beta}) + (\overline{\beta+\gamma\alpha}) + (\alpha\beta\gamma)$ are different from one another; secondly, that the number of terms in the one is the same as in the other; thirdly, that there can be no term in the second, which is not also contained in the first. Hence it evidently follows, that they must be the same, and that consequently the equation (ψ) is true for every number of roots.

SECTION XIV.

Prob. Reduce the numerical expression $(\alpha\beta\gamma\delta)$, with four radical exponents, to another, in which there at most only three radical exponents.

Solution 1. If we examine the equations (χ) and (ψ) § XII and XIII, we are justified by analogy in making the following hypothesis:

$$(\delta)\ (\alpha\beta\gamma) =$$

$$(\overline{\alpha + \delta}\,\beta\gamma) + (\overline{\beta + \delta}\,\alpha\gamma) + (\overline{\gamma + \delta}\,\alpha\beta) + (\alpha\beta\gamma\delta).$$

In order to prove that this hypothesis is allowable, we can proceed in the same way as in the foregoing Section.

2. In the first place it may be proved, that in the function $(\delta)\ (\alpha\beta\gamma)$, no term can occur more than once. For if, for instance, the term $b^{\alpha+\delta} + c^{\gamma} f^{\beta}$, or $a^{\alpha} c^{\beta} d^{\gamma} e^{\delta}$ is oftener contained in this function, then also must $b^{\alpha} c^{\gamma} f^{\beta}$, or $a^{\alpha} c^{\beta} d^{\gamma}$, be oftener found in $(\alpha\beta\gamma)$, which is impossible. But from the nature of the function $(\overline{\alpha+\delta}\,\beta\gamma) + (\overline{\beta + \delta}\,\alpha\gamma) + (\overline{\gamma + \delta}\,\alpha\beta) + (\alpha\beta\gamma\delta)$ it can contain no term more than once. Consequently the terms of both functions are all different from one another.

3. The number of terms in $(\alpha\beta\gamma) = n \,.\, n-1 \,.\, n-2$ (§ IV), consequently those in the function $(\delta)\ (\alpha\beta\gamma)$ $= n^2 \,.\, n-1 \,.\, n-2$. The number of terms in the function $(\overline{\alpha + \delta}\,\beta\gamma) + (\overline{\beta + \delta}\,\alpha\gamma) + (\overline{\gamma + \delta}\,\alpha\beta) + (\alpha\beta\gamma\delta = 3 \times n \,.\, n-1 \,.\, n-2 + n \,.\, n-1 \,.\, n-2 \,.\, n-3$ $= n^2 \,.\, n-1 \,.\, n-2$. Both functions $\therefore$ have the same number of terms.

4. Further, in the function $[\overline{\alpha+\delta}\beta\gamma] + [\overline{\beta+\delta}\alpha\gamma]$ $+ [\overline{\gamma+\delta}\alpha\beta] + [\alpha\beta\gamma\delta]$, there can be no term which is not contained also in the function $[\delta]\,[\alpha\beta\gamma]$. For if, for instance, the term $b^{\alpha} d^{\gamma} e^{\beta+\delta}$, or $a^{\gamma} b^{\beta} e^{\alpha} f^{\delta}$, in the first function, be not also contained in the second, $\therefore$ also $b^{\alpha} \delta^{\gamma} e^{\beta}$, or $a^{\gamma} b^{\beta} e^{\alpha}$ is not in $[\alpha\beta\gamma]$; which is impossible.

5. From 2, 3, 4, it may be inferred, as in the foregoing section, that the two assumed functions must be equal, and that consequently the assumed equation is correct.

6. But from this equation we obtain

$$[\alpha\beta\gamma\delta] =$$
$$[\delta]\,[\alpha\beta\gamma] - [\overline{\alpha+\delta}\beta\gamma] - [\overline{\beta+\delta}\alpha\gamma] - [\overline{\gamma+\delta}\alpha\beta]$$

which verifies the problem, because in the second part of this equation, besides $[\delta]$, there are only numerical expressions with three radical exponents.

SECTION XV.

PROB. Reduce the general numerical expression $[\alpha\beta\gamma\delta \ldots\ldots \iota\kappa\lambda]$ with m radical exponents, to others, which contain at most $m-1$ radical exponents.

Solution 1. From §§ XII, XIII and XIV, we have sufficient reasons for assuming the following equations:

$$[\lambda]\,[\alpha\beta\gamma\delta \ldots \iota\kappa] =$$
$$[\overline{\alpha+\lambda}\beta\gamma\delta \ldots \iota\kappa] + [\overline{\beta+\lambda}\alpha\gamma\delta\,\iota\kappa] + [\overline{\gamma+\lambda}\alpha\beta\delta \ldots \iota\kappa]$$
$$+ \ldots\ldots + [\overline{\kappa+\lambda}\alpha\beta\gamma\delta \ldots \iota] + [\alpha\beta\gamma\delta \ldots \iota\kappa\lambda];$$

the accuracy of which may be proved.

2. For, in the first place, it may be proved by a similar manner as that used in §§ XIII and XIV, that both the function in the first part, and that in the second part of this equation, contain terms widely different from one another, and that for each term in the second of these two functions, there must be one equal to it in the first.

3. Further, since all numerical expressions in the equation contain $'m-1$ radical exponents, $[\lambda]$ and $[\alpha\beta\gamma\delta \ldots \iota\kappa\lambda]$ excepted, of which the last contains m radical exponents; $\therefore$ by § IV, the number of the terms in the function of the first part of this equation

$$= n \times n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2$$

and in the function of the second part

$$= m-1 \times n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2 +$$
$$n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2 \,.\, n-m+1$$
$$= n \times n \,.\, n-1 \,.\, n-2 \ldots\ldots n-m+2.$$

These two functions consequently consist of the same number of terms.

4. From 2 and 3 we may infer, in the same manner as in §§ XIII and XIV the accuracy of the assumed equation. From this equation, however, we obtain

$$(\Theta) \ldots\ldots [\alpha\beta\gamma\delta \ldots \iota\kappa\lambda] = [\lambda]\,[\alpha\beta\gamma\delta \ldots \iota\kappa]$$
$$- [\overline{\alpha+\lambda}\beta\gamma\delta \ldots \iota\kappa] - [\overline{\beta+\lambda}\alpha\gamma\delta \ldots \iota\kappa]$$
$$- \ldots\ldots\ldots - [\overline{\kappa+\lambda}\alpha\beta\gamma\delta \ldots \iota]$$

which answers the condition of the problem.

Remark. The formula (Θ) obtains both for positive

and negative radical exponents, because the conclusions remain the same when the signs are changed. By means of this formula we are enabled to reduce any numerical expression $[\alpha\beta\gamma\delta\ldots\lambda]$ to others, which contain one radical exponent less; and if we continue the operation with this diminished radical exponent, we at length arrive at sums of powers only, which, whether the exponents be positive or negative, may again be always expressed, by §§ VIII and XI, by the coefficients of the equation, to which the numerical expression refers.

SECTION XVI.

Hitherto it has been assumed, that the radical exponents in the numerical expression $[\alpha\beta\gamma\delta\ldots\lambda]$ are all different from one another. If this be not the case, and the expression is consequently of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}]$, then the preceding formulæ, if we wish to apply them further, must undergo some modifications. It has already been shewn in § V, that if two numerical expressions $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}]$, $[\alpha\beta\gamma\delta\ldots\zeta]$, the first with, and the other without repeating exponents, contain the same number of radical exponents, the number of terms in the first is less than those in the second by $1 . 2 \ldots \mathfrak{a} \times 1 . 2 \ldots \mathfrak{b} \times 1 . 2 \ldots \mathfrak{c} \times \ldots\ldots \times 1 . 2 \ldots \mathfrak{k}$. The reason of this is only to be accounted for in this way, that in the case of equal radical exponents, there are exactly the same number of terms, which are equal to one another in the numerical expression $[\alpha\beta\gamma\delta\ldots\zeta]$ for each combination of the roots a, b, c, d, &c., of which terms, only one is retained in $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}]$, as we already know from the rule of com-

Solution 1. From the equation (Θ), § XV., when we put $\mathfrak{a}$ of the radical exponents $= \alpha$, $\mathfrak{b}$ of them $= \beta$, and the remaining ones $\mathfrak{c} = \gamma$, we get

$$\kappa \,.\, [\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] =$$

$$\kappa' \,.\, [\alpha]\,[\alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] - [\mathfrak{a}-1] \,.\, \kappa'' \,.\, [\overline{2\alpha}\, \alpha^{\mathfrak{a}-2} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}]$$
$$- \mathfrak{b} \,.\, \kappa''' \,.\, [\overline{\alpha + \beta}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}-1} \gamma^{\mathfrak{c}}]$$
$$- \mathfrak{c} \,.\, \kappa^{IV} [\alpha + \gamma\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}-1}]$$

2. But

$$\kappa \;= 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{a} \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa' \;= 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa'' \;= 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-2 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa''' = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 .. \mathfrak{b}-1 \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$$
$$\kappa^{IV} = 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}-1.$$

If we substitute these values, and then divide by $1 \,.\, 2 \,.\, 3 \ldots \mathfrak{a}-1 \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{b} \times 1 \,.\, 2 \,.\, 3 \ldots \mathfrak{c}$ we obtain

$$\alpha\,[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] = [\alpha]\,[\alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}] - [\overline{2\alpha}\, \alpha^{\mathfrak{a}-2} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}}]$$
$$- [\overline{\alpha + \beta}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}-1} \gamma^{\mathfrak{c}}] - [\overline{\alpha + \gamma}\, \alpha^{\mathfrak{a}-1} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}-1}]$$

as required.

SECTION XX.

Prob. Reduce the general numerical expression $[\alpha^{\mathfrak{a}} \beta^{\mathfrak{b}} \gamma^{\mathfrak{c}} \delta^{\mathfrak{d}} \ldots \kappa^{\mathfrak{k}} \lambda^{\mathfrak{l}}]$, with $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} + \ldots + \mathfrak{k} + \mathfrak{l}$ radical exponents, to others, which contain one radical exponent less.

Solution 1. If we compare the operation in §§ XVIII, XIX, XX., we shall, with very little trouble, obtain from it the following general equation :

$$(\mathfrak{C}) \ldots\ldots\ldots \mathfrak{a}\,[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] =$$

$$[\alpha]\,[\alpha^{\mathfrak{a}-1}\alpha^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$
$$-[\overline{\alpha+\beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{\alpha+\gamma}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$
$$-\ \ldots\ldots\ldots\ldots\ldots\ldots - [\overline{\alpha+\kappa}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}-1}\lambda^{\mathfrak{l}}]$$
$$-[\overline{\alpha+\lambda}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}-1}]$$

in which each of the numerical expressions in the second part contains no more than $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d} + \ldots\ldots + \mathfrak{k} + \mathfrak{l} - 1$ radical exponents.

The formula thus found obtains, as also the one in § XV, whether the radical exponents, α, β, γ, &c. be positive or negative, because in this case the conclusions remain the same.

For the particular case, in which $\mathfrak{a} = 1$, this formula ceases to be applicable, because the repeating exponents $\mathfrak{a}-1$, $\mathfrak{a}-2$, which are contained in it, then become 0 and -1, which is impossible. In this case we must make use of the following formula:

$$[\alpha\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] =$$
$$[\alpha]\,[\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - [\overline{\alpha+\beta}\,\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$$
$$-[\overline{\alpha+\gamma}\,\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] - \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots - [\overline{\alpha+\lambda}\,\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}-1}]$$

which may be derived from the same sources as the preceding.

Remark. By means of the equation $(\mathfrak{C})$ we are now enabled to reduce every numerical expression of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$, by a certain reduction of the repeating exponents, to another numerical expression of the form $[\alpha\beta\gamma\delta\ldots\kappa\lambda]$; and since this last, by means of the equation (Θ) in § XV, may always be reduced merely to the sums of powers, and ∴ may at length be expressed

by the coefficients of the given equation; consequently we are always enabled to express any numerical expression of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]$ by the coefficients of the given equation.

Moreover, the equation ($\mathfrak{C}$) is always true, so long as we assign no determinate values to the radical exponents α, β, γ, δ, &c. For determinate values, it may happen that radical exponents are equal to one another, which in the general expression were considered as different: thus, for instance, when in the equation ($\mathfrak{C}$) $2\alpha = \beta$, or $\alpha + \beta = \gamma$. In such cases as these, we shall do well, in order to avoid mistakes, to add the following equation derived from (Θ), § XV:

$$\begin{aligned}\kappa\,.\,[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}] &= \kappa'\,.\,[\alpha]\,[\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]\\ &- [\mathfrak{a}-1]\,.\,\kappa''\,.\,[\overline{2\alpha}\,\alpha^{\mathfrak{a}-2}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]\\ &- \mathfrak{b}\,.\,\kappa'''\,.\,[\overline{\alpha+\beta}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}-1}\gamma^{\mathfrak{c}}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]\\ &- \mathfrak{c}\,.\,\kappa^{IV}\,.\,[\overline{\alpha+\gamma}\,\alpha^{\mathfrak{a}-1}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}-1}\delta^{\mathfrak{d}}\ldots\kappa^{\mathfrak{k}}\lambda^{\mathfrak{l}}]\\ &\qquad\&c.\end{aligned}$$

in which the coefficients, κ, κ', κ'', κ''', &c., have the values given in § XVI.

Every integral or fractional rational symmetrical function of the roots a, b, c, d, &c., however constituted, must necessarily be composed of numerical expressions of the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\lambda^{\mathfrak{l}}]$. Now since these last, as we have already seen, can always be expressed rationally by the coefficients of the equation to which they relate, consequently also the former can always be expressed rationally by these coefficients; to prove which was the aim of the present chapter.

Since, however, the rule of symmetrical functions is of the greatest importance in the theory of equa-

tions, it is often requisite to express these functions by the coefficients of the given equation, I have ∴ subjoined three tables, in which all numerical expressions, in which the sum of the radical exponents does not exceed the number 10, are fully calculated. Thus Table I* contains, in six small tables, the values of all numerical expressions for the Nos. 2, 3, 4, 5, 6, 7; Table II, those for the Nos. 8 and 9; and Table III, those for the No. 10. The arrangement of these tables is evident at first sight. The letters A, B, C, D, &c. are the coefficients of the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \&c. = 0$, which is the basis of the numerical expressions, and these last, for the sake of facilitating the calculation, are assumed with alternate signs. In the horizontal lines, the numerical expressions themselves are found in a combination series; in the first upper horizontal series, are the different terms of their values, and in the vertical columns under them the numerical coefficients belonging to each term, according to the difference of the numerical expressions. Where there are terms wanting, or the numerical coefficients = 0, asterisks are placed. Thus, for instance, in Table III $[1^2 4^2] = B^3 D - 3\ ABCD + 3\ C^2 D + 3\ A^2 D^2 - 3\ BD^2 - AB^2 E + 2\ A^2 CE + BCE - 8\ ADE + 5\ E^2 + A^2 BF - B^2 F - 8\ ACF + 9\ DF - 6\ A^3 G + 17\ ABG - 15\ CG + 6\ A^2 H - 11\ BH - 6\ AI + 15\ K$.

These tables were calculated by means of the equations (Θ) and (ↄ) in § XV. and § XX. For the more easy application of these tables, it is, however, necessary that the calculation be made successively, and

* *Note.*—These tables are to be found at the end. *Translator.*

that, in order to find the numerical expressions for any determinate sum of theadical exponents, we should first know all those for lower sums. Likewise the sums of powers must be previously calculated by means of the equations in 8, § VIII, which equations, on account of the change of the signs in the assumed equation, have the following values:

$$
\begin{aligned}
(1) &= A \\
(2) &= A^2 - 2B \\
(3) &= A^3 - 3AB + 3C \\
(4) &= A^4 - 4A^2B + 4AC + 2B^2 - 4D
\end{aligned}
$$

&c.

Thus, for the successive calculation of the numerical expressions, Table III, we have the following equations:

$$
\begin{aligned}
(\overline{10}) &= (\overline{10}) \\
(19) &= (1)\,(9) - (\overline{10}) \\
(28) &= (2)\,(8) - (\overline{10}) \\
(37) &= (3)\,(7) - (\overline{10}) \\
(46) &= (4)\,(6) - (\overline{10}) \\
2\,(5^2) &= (5)\,(5) - (\overline{10}) \\
2\,(1^28) &= (1)\,(18) - (28) - (19) \\
(127) &= (1)\,(27) - (37) - (28) \\
(136) &= (1)\,(36) - (46) - (37) \\
(145) &= (1)\,(45) - 2\,(5^2) - (46) \\
2\,(2^26) &= (2)\,(26) - (46) - (28)
\end{aligned}
$$

The numerical expressions in the first part of the equations, depend here, as is easily seen, either on the foregoing, or on such numerical expressions as have a less sum of radical exponents, which, when these last are already found, may successively be determined.

II.—Complete solution of the symmetrical functions of the roots of an equation.

Section XXII.

TO solve a symmetrical function, here means no more than to find an expression for it, which contains only sums of powers.

A compound radical exponent implies one, which is compounded of more than one compound one, as $a + \beta + \gamma + \delta +$ &c. in $(a+\beta+\gamma+\delta+\&c.)$, or $(aa+b\beta+c\gamma+d\delta +\&c.)$ in $(aa+b\beta+c\gamma+d\delta+\&c.)$ The terms of the first are a, β, γ, δ, &c.; the terms of the latter are aa, $b\beta$, $c\gamma$, $d\delta$, &c. In the opposite case, a and aa, in (a), (aa), are simple radical exponents.

In order to show, that a numerical expression, viz. (a), can be raised to any power μ, I shall merely write $(a)^\mu$. We must then very carefully transform $(a)^\mu$ into (a^μ); for $(a)^\mu = (a)\ (a)\ (a)\ (a)\ \ldots$; on the other hand, $(a^\mu)=(a\ a\ a\ \ldots)$. So in like manner $(3a+2\beta)^\mu$ denotes the μth power of $(3a+2\beta)$, and $(aa+b\beta+c\gamma+d\delta+\&c.)^\mu$ the μth power of $(aa + b\beta + c\gamma + d\delta + \&c.)$

Section XXIII.

Prob. Represent the numerical expressions $(a\beta)$, $(a\beta\gamma)$, $(a\beta\gamma\delta)$, &c. fully developed.

Solution 1. From § XII we immediately have

$$(\alpha\beta) = (\beta)\,(\alpha) - (\beta + \alpha)$$

2. From § XIII we at first obtain

$$(\alpha\beta\gamma) = (\gamma)\,(\alpha\beta) - (\overline{\gamma + \alpha}\beta) - (\overline{\gamma + \beta}\alpha).$$

But from 1 we have, when first $\gamma+\alpha$ is put for α, and afterwards $\gamma+\beta$ for β,

$$(\overline{\gamma + \alpha}\beta) = (\beta)\,(\gamma + \alpha) - (\gamma + \beta + \alpha)$$
$$(\overline{\gamma + \beta}\alpha) = (\gamma + \beta)\,(\alpha) - (\gamma + \beta + \alpha)$$

If these values, together with the value of $(\alpha\beta)$ from 1, be substituted in the foregoing equation, we then obtain

$$(\alpha\beta\gamma) = (\gamma)\,(\beta)\,(\alpha) - (\gamma)\,(\beta+\alpha) - (\gamma+\beta)\,(\alpha) - (\gamma+\alpha)\,(\beta) + 1\,.\,2\,(\gamma+\beta+\alpha)$$

3. From § XIV, we have

$$(\alpha\beta\gamma\,\delta)=(\delta)\,(\alpha\beta\gamma) - (\overline{\delta + \alpha}\beta\gamma) - (\overline{\delta + \beta}\alpha\gamma) - (\overline{\delta + \gamma}\alpha\beta)$$

In order to find the numerical expressions $(\overline{\delta + \alpha}\beta\gamma)$, $(\overline{\delta + \beta}\alpha\gamma)$, $(\overline{\delta + \gamma}\alpha\beta)$, we need only successively substitute $\delta+\alpha$ for α, afterwards $\delta+\beta$ for β; and lastly $\delta+\gamma$ for γ in the last equation in 2. If, after this, we substitute the values thus obtained, together with the value of $(\alpha\beta\gamma)$, we obtain

$$\begin{aligned}(\alpha\beta\gamma\delta) = {} & (\delta)\,(\gamma)\,(\beta)\,(\alpha) - (\delta)\,(\gamma)\,(\beta+\alpha) - (\delta)\,(\gamma+\beta)\,(\alpha)\\ & -(\delta)\,(\gamma+\alpha)\,(\beta)+1\,.\,2\,(\delta)\,(\gamma+\beta+\alpha) - (\delta+\gamma)\,(\beta)\,(\alpha)\\ & -(\delta+\beta)\,(\gamma)\,(\alpha) - (\delta+\alpha)\,(\gamma)\,(\beta) + (\delta+\gamma)\,(\beta+\alpha)\\ & +1.2\,(\delta+\beta+\alpha)\,(\gamma)+1\,.\,2\,(\delta+\gamma+\beta)\,(\alpha)+(\delta+\alpha)\,(\gamma+\beta)\\ & +1.2\,(\delta+\gamma+\alpha)\,(\beta)+(\delta+\beta)(\gamma+\alpha)-1.2.3(\delta+\gamma+\beta+\alpha).\end{aligned}$$

4. So, in like manner, from § XV, we have

$$(\alpha\beta\gamma\delta\varepsilon) = (\varepsilon)(\alpha\beta\gamma\delta) - (\overline{\varepsilon+\alpha\beta\gamma\delta}) - (\overline{\varepsilon+\beta\alpha\gamma\delta}) - (\overline{\varepsilon+\gamma\alpha\beta\delta}) - (\overline{\varepsilon+\delta\alpha\beta\gamma}).$$

We get the values of $(\overline{\varepsilon+\alpha\beta\gamma\delta})$, $(\overline{\varepsilon+\beta\alpha\gamma\delta})$, $(\overline{\varepsilon+\gamma\alpha\beta\delta})$, $(\overline{\varepsilon+\delta\alpha\beta\gamma})$ completely developed from the last equation in 3, by substituting in it successively $\varepsilon+\alpha$ for α, $\varepsilon+\beta$ for β, $\varepsilon+\gamma$ for γ, and $\varepsilon+\delta$ for δ. The substitution of these values, together with that of $(\alpha\beta\gamma\delta)$ in the foregoing equation, gives the solution required.

5. In this way we could proceed further, since we always go from one solution to another, and thus find the solutions of the numerical expressions, which contain six, seven, eight, &c. radical exponents.

6. Generally, if we have already found the solution of a numerical expression $(\alpha\beta\gamma\delta\ldots\kappa)$, and wish from hence to derive the solution of another $(\alpha\beta\gamma\delta\ldots\kappa\lambda)$, which contains λ more radical exponents, we must, in the first place, multiply merely the solution of $(\alpha\beta\gamma\delta\ldots\kappa)$ by (λ), then in this solution substitute throughout, first $\lambda+\alpha$ for α, then $\lambda+\beta$ for β, $\lambda+\gamma$ for γ, &c. and change the signs of the results and the former product.

SECTION XXIV.

Prob. Find the law, by which the terms in the solutions of $(\alpha\beta)$, $(\alpha\beta\gamma)$, $(\alpha\beta\gamma\delta)$, &c. are formed, when the coefficients and the signs are left out.

Solution 1. If in the solutions of the above numerical

expressions in the foregoing §, we omit the brackets and the signs, and separate the radical exponents, which belong to the different numerical expressions in each term, by a comma, and all the terms by a semicolon, we then, by means of (α), find the following:

I. α

II. $\beta, \alpha;\ \beta+\alpha$

III. $\gamma, \beta, \alpha;\ \gamma, \beta+\alpha;\ \gamma+\beta, \alpha;\ \gamma+\alpha, \beta;\ \gamma+\beta+\alpha$

IV. $\delta, \gamma, \beta, \alpha;\ \delta, \gamma, \beta+\alpha;\ \delta, \gamma+\beta, \alpha;\ \delta, \gamma+\alpha, \beta$
$\delta, \gamma+\beta+\alpha;\ \delta+\gamma, \beta, \alpha;\ \delta+\beta, \gamma, \alpha;\ \delta+\alpha, \gamma, \beta$
$\delta+\gamma, \beta+\alpha;\ \delta+\beta+\alpha, \gamma;\ \delta+\gamma+\beta, \alpha;\ \delta+\alpha, \gamma+\beta$
$\delta+\gamma+\alpha, \beta;\ \delta+\beta, \gamma+\alpha;\ \delta+\gamma+\beta+\alpha$

&c.

2. Hence we may perceive at first sight and from 6 in the foregoing § the law of the successive formation of the terms. Thus, in order to derive the terms of a solution from the terms of the immediately preceding one, we must

(a) put before each of all the terms of the preceding solution, the radical exponent which is now to be added;

(b) we must connect this by the sign + with each radical exponent of every term, while, at the same time, we add the remaining radical exponents of the same term without any change.

Thus, for instance, if we wish to derive IV from III, by the rule (a) we get

$\delta, \gamma, \beta, \alpha;\ \delta, \gamma, \beta+\alpha;\ \delta, \gamma+\beta, \alpha;\ \delta, \gamma+\alpha, \beta;\ \delta, \gamma+\beta+\alpha$

and by the rule (b) from the first term in III

$$\delta + \gamma, \beta, \alpha ; \delta + \beta, \gamma, \alpha ; \delta + \alpha, \gamma, \beta ;$$

from the second term in III

$$\delta + \gamma, \beta + \alpha ; \delta + \beta + \alpha, \gamma ;$$

from the third term in III

$$\delta + \gamma + \beta, \alpha ; \delta + \alpha, \gamma + \beta ;$$

from the fourth term in III

$$\delta + \gamma + \alpha, \beta ; \delta + \beta, \gamma + \alpha ;$$

lastly from the fifth term in III

$$\delta + \gamma + \beta + \alpha .$$

The foundation of this method is so evident from the foregoing §, that it requires no further explanation.

3. But since this mode of representation possesses this disadvantage, that in finding the following solutions, we must first add the foregoing, we can ∴ with great advantage make use of the Hindenburgian method of involution, which is already known to my readers from the first principles of the rule of combination. Here follows this involution, whose construction is immediately deducible from 2 :

$$\begin{array}{l|l|l|l}
\delta & \gamma & \beta & \alpha \\
\delta & \gamma & \beta + \alpha & \\
\delta & \gamma + \beta, & & \alpha \\
\delta & \gamma + \alpha, & & \beta \\
\delta & \gamma + \beta + \alpha & & \\
\hline
\end{array}$$

$$\begin{array}{llll}
\delta + \gamma, & \beta, & & \alpha \\
\delta + \beta, & \gamma, & & \alpha \\
\delta + \alpha, & \gamma, & & \beta \\
\delta + \gamma, & \beta, & + & \alpha \\
\delta + \beta + \alpha, & & & \gamma \\
\delta + \gamma + \beta, & & & \alpha
\end{array}$$

$$\begin{array}{l} \delta + \alpha,\ \gamma + \beta \\ \delta + \gamma + \alpha,\ \beta \\ \delta + \beta,\ \gamma + \alpha \\ \delta + \gamma + \beta + \alpha \\ \&c. \end{array}$$

It is not necessary, in the first place, to remind my readers, that this mode of representation, besides the advantage that it immediately gives what is sought, possesses also this one, that each solution includes all the foregoing, as the brackets denote, and this follows of course from the very nature of involution.

REMARK. The involution which is here given, includes, besides, as may easily be observed, all possible combinations of the radical exponents α, β, γ, δ, &c. both simple and compound, consequently can be used with advantage in many other cases, in which it is required to find all the possible combinations of this kind in a given number of things.

SECTION XXV.

PROB. Find the law of the coefficients and the signs in the solution of the numerical expression $(\alpha\beta\gamma\delta\ldots\lambda)$.

Solution 1. From the method in § XXIII, by which the solutions are derived from one another, and from the results themselves, it may, with some reason, be presumed, that the coefficients of the terms and their signs are subject to the following laws:

(a) That each numerical expression of a simple radical exponent has unity for its coefficient;

(*b*) That each numerical expression of a compound radical exponent of m terms has the coefficient $1 . 2 . 3 \ldots\ldots m-1$;

(*c*) That the sign $-$ or $+$ may be given to every numerical expression whatever, according as the number of the terms of its radical exponent is odd or even.

Thus if these rules be correct, the term (α) $(\beta+\gamma)$ $(\delta+\varepsilon+\zeta)$ $(\eta+\vartheta+\iota+\kappa)$ has the coefficient $1 \times 1 \times 1 . 2 \times 1 . 2 . 3$, or merely $1 . 2 \times 1 . 2 . 3$, with the sign $+$, because it has two radical exponents of an even, and two of an odd, number of terms.

That these rules are correct for $(\alpha\beta)$, $(\alpha\beta\gamma)$, $(\alpha\beta\gamma\delta)$, one can be readily convinced of by inspection. It only remains now, by a very common method in mathematics, to prove the rule, that when they obtain for any solution, they likewise necessarily obtain for the following one.

3. For instance, let

$$(A) \ldots\ldots (\alpha+\beta+\gamma+\ldots+\kappa)\ (\lambda+\mu+\nu \ldots +\psi)$$

be any term in the solution of $(\alpha\beta\gamma \ldots\ldots \psi)$. The radical exponent of the first numerical expression in (A) contains m terms, that of the second n terms. Consequently the coefficient of the product, according to the hypothesis $= 1 . 2 . 3 \ldots\ldots m-1 \times 1 . 2 . 3 \ldots\ldots n-1$.

4. Now let $(\alpha\beta\gamma\ldots\psi\,\omega)$ be another numerical expression, which contains more radical exponents than the preceding by ω. For its solution, the term (A) by 6, § XXIII gives the three following terms:

$$(\omega)(\alpha+\beta+\gamma+\ldots+\kappa)(\lambda+\mu+\nu+\ldots+\psi)$$
$$-(\alpha+\beta+\gamma+\ldots+\kappa+\omega)(\lambda+\mu+\nu+\ldots+\psi)$$
$$-(\alpha+\beta+\gamma+\ldots+\kappa)(\lambda+\mu+\nu+\ldots+\psi+\omega)$$

5. The first of these terms is obtained from the term (A), by multiplying the latter by (ω), and consequently it has the same coefficient and the same sign; which agrees with the hypothesis.

6. The second term in 4 arises from the substitution of $\omega+\alpha$, $\omega+\beta$, $\omega+\gamma \ldots\ldots \omega+\kappa$ for α, β, γ, …… κ, (6. § XXIII) and occurs m times. In like manner the third term arises from the substitution of $\omega+\lambda$, $\omega+\mu$, $\omega+\nu$, … $\omega+\psi$ for λ, μ, ν, …… ψ, and $\therefore$ occurs n times. Consequently the second term must contain m, and the third n, more coefficients than the term (A).

7. Hence it follows, that the coefficient of the second term in 4. $= 1 . 2 . 3 \ldots . m \times 1 . 2 . 3 \ldots n-1$, and the coefficient of the third term $= 1 . 2 . 3 \ldots . m-1 \times 1 . 2 . 3 \ldots . n$. Likewise these terms have a different sign, from that of the term (A).

8. Since this agrees with the hypothesis, so it may be concluded, that, when the hypothesis is true for the term (A), it is necessarily true for the terms derived from it in the following solution.

9. Although here the term (A) has only been assumed as a product of two numerical expressions, it is sufficiently evident from the manner in which the proof has

been managed, that it may be extended to any other number of factors.

10. If ∴ the solution of $[\alpha\beta\gamma \ldots \psi]$ be subject to the assumed rules, so likewise is that of $[\alpha\beta\gamma \ldots \psi\omega]$ which follows it. But they obtain for the first solutions, they ∴ also obtain for all those which follow.

Corollary. In order ∴ to exhibit a numerical expression of the form $[\alpha\beta\gamma \ldots \lambda]$, in which all the radical exponents are different, completely solved, we only require to perform the involution given in § XXIV, and affix to each term the coefficient determined from 1 of this § with its sign. The following example will serve as an illustration.

Example. The complete solution of $[\alpha\beta\gamma\delta\varepsilon]$, when the brackets are omitted in the terms, is as follows:

$+1,\ \varepsilon \mid \delta \mid \gamma \mid \beta \mid \alpha$	$+2,\ \varepsilon+\gamma+\alpha,\ \delta,\ \beta$
$-1,\ \varepsilon \mid \delta \mid \gamma \mid \beta+\alpha$	$+1,\ \varepsilon+\beta,\ \delta,\ \gamma+\alpha$
$-1,\ \varepsilon \mid \delta \mid \gamma+\beta,\ \alpha$	$-2,\ \varepsilon+\delta,\ \gamma+\beta+\alpha$
$-1,\ \varepsilon \mid \delta \mid \gamma+\alpha,\ \beta$	$-6,\ \varepsilon+\gamma+\beta+\alpha,\ \delta$
$+2,\ \varepsilon \mid \delta \mid \gamma+\beta+\alpha$	$+2,\ \varepsilon+\delta+\gamma,\ \beta,\ \alpha$
$-1,\ \varepsilon \mid \delta+\gamma,\ \beta,\ \alpha$	$+1,\ \varepsilon+\beta,\ \delta+\gamma,\ \alpha$
$-1,\ \varepsilon \mid \delta+\beta,\ \gamma,\ \alpha$	$+1,\ \varepsilon+\alpha,\ \delta+\gamma,\ \beta$
$-1,\ \varepsilon \mid \delta+\alpha,\ \gamma,\ \beta$	$+2,\ \varepsilon+\delta+\beta,\ \gamma,\ \alpha$
$+1,\ \varepsilon \mid \delta+\gamma,\ \beta+\alpha$	$+1,\ \varepsilon+\gamma,\ \delta+\beta,\ \alpha$
$+2,\ \varepsilon \mid \delta+\beta+\alpha,\ \gamma$	$+1,\ \varepsilon+\alpha,\ \delta+\beta,\ \gamma$
$+2,\ \varepsilon \mid \delta+\gamma+\beta,\ \alpha$	$+2,\ \varepsilon+\delta+\alpha,\ \gamma,\ \beta$
$+1,\ \varepsilon \mid \delta+\alpha,\ \gamma+\beta$	$+1,\ \varepsilon+\gamma,\ \delta+\alpha,\ \beta$
$+2,\ \varepsilon \mid \delta+\gamma+\alpha,\ \beta$	$+1,\ \varepsilon+\beta,\ \delta+\alpha,\ \gamma$
$+1,\ \varepsilon \mid \delta+\beta,\ \gamma+\alpha$	$-2,\ \varepsilon+\delta+\gamma,\ \beta+\alpha$
$-6,\ \varepsilon \mid \delta+\gamma+\beta+\alpha$	$-2,\ \varepsilon+\beta+\alpha,\ \delta+\gamma$

$-1,\ \varepsilon+\delta,\ \gamma,\ \beta,\ \alpha$	$-6,\ \varepsilon+\delta+\beta+\alpha,\ \gamma$
$-1,\ \varepsilon+\gamma,\ \delta,\ \beta,\ \alpha$	$-2,\ \varepsilon+\gamma,\ \delta+\beta+\alpha$
$-1,\ \varepsilon+\beta,\ \delta,\ \gamma,\ \alpha$	$-6,\ \varepsilon+\delta+\gamma+\beta,\ \alpha$
$-1,\ \varepsilon+\alpha,\ \delta,\ \gamma,\ \beta$	$-2,\ \varepsilon+\alpha,\ \delta+\gamma+\beta$
$+1,\ \varepsilon+\delta,\ \gamma,\ \beta+\alpha$	$-2,\ \varepsilon+\delta+\alpha,\ \gamma+\beta$
$+1,\ \varepsilon+\gamma,\ \delta,\ \beta+\alpha$	$-2,\ \varepsilon+\gamma+\beta,\ \delta+\alpha$
$+2,\ \varepsilon+\beta+\alpha,\ \delta,\ \gamma$	$-6,\ \varepsilon+\delta+\gamma+\alpha,\ \beta$
$+1,\ \varepsilon+\delta,\ \gamma+\beta,\ \alpha$	$-2,\ \varepsilon+\beta,\ \delta+\gamma+\alpha$
$+2,\ \varepsilon+\gamma+\beta,\ \delta,\ \alpha$	$-2,\ \varepsilon+\delta+\beta,\ \gamma+\alpha$
$+1,\ \varepsilon+\alpha,\ \delta,\ \gamma+\beta$	$-2,\ \varepsilon+\gamma+\alpha,\ \delta+\beta$
$+1,\ \varepsilon+\delta,\ \gamma+\alpha,\ \beta$	$+24,\ \varepsilon+\delta+\gamma+\beta+\alpha$

We have $\therefore$ $[\alpha\beta\gamma\delta\varepsilon] = [\varepsilon]\,[\delta]\,[\gamma]\,[\beta]\,[\alpha] -$
$[\varepsilon]\,[\delta]\,[\gamma]\,[\beta+\alpha]-[\varepsilon]\,[\delta]\,[\gamma+\beta]\,\alpha-$&c.

SECTION XXVI.

If there are more radical exponents equal to one another in the numerical expression $[\alpha\beta\gamma \ldots\ldots \lambda]$, or if it takes the form $[\alpha^a\beta^b\gamma^c \ldots\ldots \kappa^k]$, the solution admits of reduction, because in this case more terms than one are equal to one another. By the application of the rules (a) and (b) in 2, § XXIV, we have only to take care that no term occurs more than once, and with this view it is only necessary in performing the involution, always to revert to the combinations already found, and to omit all those which occurred once before. Thus we find the involution for the terms of $[\alpha^3\beta^3]$ to be as follows:

$\beta \mid \beta \mid \beta \mid \alpha \mid \alpha \mid \alpha$	$2\beta,\ \beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta \mid \alpha \mid 2\alpha$	$2\beta,\ \beta+\alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta \mid 3\alpha$	$2\beta,\ \beta+2\alpha,\ \alpha$
$\beta \mid \beta \mid \beta+\alpha,\ \alpha,\ \alpha$	$2\beta,\ \beta+3\alpha$
$\beta \mid \beta \mid \beta+\alpha,\ 2\alpha$	$3\beta,\ \alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta+2\alpha,\ \alpha$	$3\beta,\ \alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta+3\alpha$	$3\beta,\ 3\alpha$
$\beta \mid 2\beta,\ \alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid 2\beta,\ \alpha,\ 2\alpha$	$\beta+\alpha,\ 2\beta+\alpha,\ \alpha$
$\beta \mid 2\beta,\ 3\alpha$	$\beta+\alpha,\ \beta+\alpha,\ \beta+\alpha$
$\beta \mid 2\beta+\alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ 2\alpha$
$\beta \mid \beta+\alpha,\ \beta+\alpha,\ \alpha$	$\beta+2\alpha,\ 2\beta+\alpha$
$\beta \mid 2\beta+\alpha,\ 2\alpha$	$2\beta+2\alpha,\ \beta+\alpha$
$\beta \mid \beta+2\alpha,\ \beta+\alpha$	$3\beta+2\alpha,\ \alpha$
$\beta \mid 2\beta+2\alpha,\ \alpha$	$3\beta+3\alpha$
$\beta \mid 2\beta+3\alpha$	

This involution contains every possible analysis of $3^\alpha+3^\beta$. In the same manner we obtain generally by the involution for $[\alpha^a\beta^b\gamma^c......\kappa^k]$ every possible analysis of $a\alpha+b\beta+c\gamma+...+k\kappa$. It may ∴ be used with advantage in all those cases where it is required to represent analysis of this kind, with facility and without the danger of omission. In the numerical expression $[a^a]$, the operation is merely reduced to a numerical analysis, respecting which information is given in the rule of combinations.

But as to the coefficients of the terms, it is easily conceived, that the rules (*a*) and (*b*), in 1, § XXV, in the case where the numerical expression $[\alpha\beta\gamma...\lambda]$ is changed to $[\alpha^a\beta^b\gamma^c...\kappa^k]$, must undergo many modifications, and first, for this reason, because in this case

more terms than one of the solution vanish, secondly, for the reason given in § XVI. But in order that we may not be obliged in the sequel, to break the thread of the inquiry by extraneous matter, we shall premise with the following auxiliary rules.

SECTION XXVII.

I.—*AUXILIARY RULE.*

PROB. Find in how many ways a given number of things may be placed in a determinate number of divisions, in such a way, that in each of these divisions there is a given number of these things.

1. *Solution for two divisions.*

Let A be the number of all the things, a the number of these things which are to enter into the first of these divisions; $\therefore$ $A-a$ the number in the second.

It is evident, that this is merely to find the number of combinations of the ath class in A things. Now this is

$$=\frac{A\;.\;A-1\;.\;A-2\;\ldots\ldots\ldots\ldots\;A-a+1}{1\;.\;2\;.\;3\;\ldots\ldots\ldots\ldots\ldots\ldots\;a};$$

or if we multiply numerator and denominator by $1\;.\;2\;.\;3\ldots A-a$, and then in the denominator substitute a' for $A-a$:

$$\frac{1\;.\;2\;.\;3\;.\;4\ldots\ldots\ldots\ldots\ldots A-1\;.\;A}{1\;.\;2\;.\;3\ldots\ldots\ldots\ldots a\times 1\;.\;2\;.\;3\ldots a'}$$

in which a and a' denote the numbers contained in the two divisions.

2. *Solution for three divisions.*

Again, let A be the number of things; further, a, a',

a'', the number in the first, second and third division, $\therefore a+a'+a''=A$.

But in the first division the number a may enter in as many different ways as there are combinations of the ath class in A things; the number of these ways is

$$=\frac{A\,.\,A-1\,\ldots\ldots\ldots\ldots\ldots A-a+1}{1\,.\,2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}$$

So in like manner the number of ways, in which the remaining $A-a$ things may be arranged in the second division in the number a'

$$=\frac{A-a\,.\,A-a-1\ldots\ldots\ldots A-a-a'+1}{1\,.\quad 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a'}.$$

Each of these last combinations may be associated with each of the first, and the number of combinations which is thus obtained is $\therefore$ the product of those two numbers. The number of ways, in which A things may be arranged in the first, second and third divisions, in the numbers a, a,' a,'' is consequently

$$=\frac{A\,.\,A-1\,.\,A-2\ldots\ldots\ldots\ldots A-a-a'+1}{1\,.\,2\,.\,3\ldots\ldots a\times 1\,.\,2\,.\,3\ldots\ldots\ldots\ldots a'},$$

or, when the numerator and denominator are multiplied by $1, 2\ldots a''=1\,.\,2\ldots\ldots A-a-a'$,

$$=\frac{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A-1\,.\,A}{1\,.\,2\ldots\ldots a\times 1\,.\,2\ldots\ldots a'\times 1\,.\,2\ldots\ldots a''}.$$

3. *Solution for four divisions.*

Again, let A be the number of all the things, and a, a,' a,'' a,''' the numbers which are separately contained in the first, second, third and fourth division, $\therefore a+a'+a''+a'''=A$. The number of cases, in which A things may be arranged in the number a,

$$=\frac{A\,.\,A-1\ldots\ldots\ldots\ldots A-a+1}{1\,.\quad 2\,\ldots\ldots\ldots\ldots\ldots\ldots a}$$

The number of cases, in which the remaining $A-a$ things may be arranged in the number a'

$$=\frac{A-a\,.\,A-a-1\ \ldots\ldots\ A-a-a'+1}{1\quad .\quad 2\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a'};$$

consequently the number of cases in which A things may be arranged, first in the number a, and then a'

$$=\frac{A\,.\,A-1\ \ldots\ldots\ldots\ldots\ldots\ A-a-a'+1}{1\,.\,2\ldots\ldots\ldots\ldots a\times 1\,.\,2\ldots\ldots\ldots\ldots a'},$$

The number of cases in which the number a'' may, in the third division, be arranged from the remaining $A-a-a'$ things

$$=\frac{A-a-a'\,.\,A-a-a'\,1\ldots\ldots A-a-a'-a''+1}{1\,.\quad 2\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a''}.$$

Each of these cases may be combined with each of the preceding, $\therefore$ the number of cases, in which the A things are arranged in the four divisions

$$=\frac{A\,.\,A-1\ \ldots\ldots\ldots\ldots\ldots\ A-a-a'-a''+1}{1\,.\,2\ldots a\times 1\,.\,2\ldots\ldots a'\times 1\,.\,2\ldots\ldots a''},$$

or, when we multiply numerator and denominator by $1\,.\,2\ldots\ldots a'''=1\,.\,2\ldots\ldots A-a-a'-a''$

$$=\frac{1\,.\,2\,.\,3\,.\,4\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A}{1\,.\,2\ldots a\times 1\,.\,2\ldots a'\times 1\,.\,2\ldots . a''\times 1\,.\,2\ldots . a'''}.$$

4. *General Solution.*

The conclusions drawn in 1, 2, 3, may easily be extended to any number of divisions. The number of ways in which A things may be arranged in n divisions, so that the first may contain the number a, the second

the number a', the third the number a'', &c.; lastly, the nth division the number $a^{(n-1)}$ of these things, is $\therefore$

$$=\frac{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ldots\ldots\ A-1\ .\ A}{1.2\ldots a\times1.2\ldots a'\times1.2\ldots a''\times\ldots\times1.2\ldots a^{(n-1)}},$$

EXAMPLE. Place 16 balls in four divisions of 6, 5, 3, and 2: in how many ways can this be done?—Here $A=16$, $a=6$, $a'=5$, $a''=3$, $a'''=2$; consequently the required number

$$=\frac{1.2.3.4.5.6\ \ 7.8.9.10\ \ 11.12.13.14.15.16}{1.2.3.4.5.6\times1.2.3.4.5\times1.2.3\times1.2}$$
$$=20180160.$$

SECTION XXVIII.

II.—Auxiliary Rule.

PROB. The numbers A, B, C, &c. of different kinds of things are given: find in how many different ways these numbers may be arranged in a given number, when in each determinate division there shall be a given number of things of each kind.

Solution 1. For the sake of perspicuity, I shall assume the particular case, that there are only three numbers of a different kind given, which are to be arranged in four divisions, so that in the first division there are a things of the first, b things of the second, and c things of the third kind; and that a', b', c'; a'', b'', c''; a''', b''', c''', denote the same for the second, third, and fourth divisions, as a, b, c, do for the first, $\therefore a+a'+a''+a'''=A$, $b+b'+b''+b'''=B$, $c+c'+c''+c'''=C$. Further, let

$$\lambda = \frac{1 \, . \, 2 \, . \, 3 \, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, A - 1 \, . \, A}{1 \, . \, 2 \ldots a \times 1 \, . \, 2 \ldots a' \times 1 \, . \, 2 \ldots a'' \times 1 \, . \, 2 \ldots a'''}$$

$$\lambda' = \frac{1 \, . \, 2 \, . \, 3 \, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, B - 1 \, . \, B}{1 \, . \, 2 \ldots b \times 1 \, . \, 2 \ldots b' \times 1 \, . \, 2 \ldots b'' \times 1 \, . \, 2 \ldots b'''}$$

$$\lambda'' = \frac{1 \, . \, 2 \, . \, 3 \, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \, C - 1 \, . \, C}{1 \, . \, 2 \ldots c \times 1 \, . \, 2 \ldots c' \times 1 \, . \, 2 \ldots c'' \times 1 \, . \, 2 \ldots c'''}$$

2. Then, from the preceding §, λ is the number of ways in which A can be arranged in four divisions of a, a', a'', a''' things, λ' the number of ways in which B things can be arranged in four divisions of b, b', b'', b''' things, and λ'' the number of ways in which C things can be arranged in four divisions of c, c', c'', c'''.

3. But it is evident, that each of these divisions may be combined with each of the other two divisions in all possible ways. Now, since the number of these combinations is equal to the product $\lambda\,\lambda'\,\lambda''$, in like manner the number which arises as often as the numbers A, B, C, agreeably to the proposed conditions, are combined together $=\lambda\,\lambda'\,\lambda''$.

4. What has been here proved for three numbers and four divisions, can, in a similar way, be proved for every other number. If ∴ λ', λ'', λ''', λ^{iv}, λ^{v}, &c. are similar expressions for the numbers B, C, D, E, F, &c. as λ is for the number A, then the required number is always $=\lambda\,\lambda'\,\lambda''\,\lambda'''\,\lambda^{iv}.\lambda^{v}$, &c.

Example. There are 40 balls of four colours, viz. 10 red, 14 blue, 9 green, and 7 white: show

in how many different ways these 40 balls may be arranged in three divisions, so that in the first division there may be 7 red, 5 blue, 3 green, and 2 white; in the second division 2 red, 6 blue, 4 green, and 1 white; and in the third 1 red, 3 blue, 2 green, and 4 white.

Here $A=10$, $B=14$, $C=9$, $D=7$; $a=7$, $b=5$, $c=3$, $d=2$; $a'=2$, $b'=6$, $c'=4$, $d'=1$; $a''=1$, $b''=3$, $c''=2$, $d''=4$; $\therefore$

$$\lambda = \frac{1.2.3.4.5.6.7.8.9.10}{1.2.3.4.5.6.7 \times 1.2 \times 1} = 360$$

$$\lambda' = \frac{1.2.3.4.5.6.7.8.9.10.11.12.13.14}{1.2.3.4.5.6 \times 1.2.3.4.5 \times 1.2.3} = 168168$$

$$\lambda'' = \frac{1.2.3.4.5.6.7.8.9}{1.2.3.4 \times 1.2.3 \times 1.2} = 1260$$

$$\lambda''' = \frac{1.2.3.4.5.6.7}{1.2.3.4 \times 1.2 \times 1} = 105$$

The required number of possible divisions is $\therefore$

$$= \lambda\lambda'\lambda''\lambda''' = 8009505504000.$$

SECTION XXIX.

III.—Auxiliary Rule.

PROB. Let there be more numbers A, B, C, &c. of things of different kinds. It is required to arrange these things in $\mu+\mu'+\mu''+\mu'''+$ &c. rows, so that in each of the μ rows there may be a things of the number A, b things of the number B, c things of the number C, and so on; in each of the μ' rows, a' things of the number A, b' things of the number B, c' things of the number C, and so on; in each of the μ'' rows, a'' things of the number A, b'' things of the number B, c'' things of the

number C, and so on. Find the number of all the possible divisions.

Solution 1. If all the rows were different from one another, as, for instance, when they are represented by different numbers, then we could have applied the formulæ of the foregoing § to this case. Then

$$\lambda = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A - 1 \,.\, A}{(1 \,.\, 2 \ldots a)^{\mu} \times (1 \,.\, 2 \ldots a')^{\mu'} \times (1 \,.\, 2 \ldots a'')^{\mu''} \times \&c.}$$

$$\lambda' = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots B - 1 \,.\, B}{(1 \,.\, 2 \ldots b)^{\mu} \times (1 \,.\, 2 \ldots b')^{\mu'} \times (1 \,.\, 2 \ldots b'')^{\mu''} \times \&c.}$$

$$\lambda'' = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots C - 1 \,.\, C}{(1 \,.\, 2 \ldots c)^{\mu} \times (1 \,.\, 2 \ldots c')^{\mu'} \times (1 \,.\, 2 \ldots c'')^{\mu''} \times \&c.}$$

&c.

and the number of all the possible divisions, as in the preceding §, $= \lambda\, \lambda'\, \lambda''\, \lambda'''$ &c.

2. But since the problem only requires in general, that the μ rows should be arranged with the combinations of a, b, c, &c. things, the μ' others with the combinations of a', b', c', &c. we must $\therefore$, as we know from the rule of combinations, multiply the number just found by $1 \,.\, 2 \ldots \mu \times 1 \,.\, 2 \ldots \mu' \times 1 \,.\, 2 \ldots \mu'' \times 1 \,.\, 2 \ldots \mu'''$ $\times$ &c.

3. The required number of all the possible divisions, according to the conditions of the problem, is $\therefore$

$$= \frac{\lambda\, \lambda'\, \lambda''\, \lambda''' \;\&c.}{1 \,.\, 2 \ldots \mu \times 1 \,.\, 2 \ldots \mu' \times 1 \,.\, 2 \ldots \mu'' \times \&c.}$$

Remark. Any one of the numbers a, b, c, &c. a', b', c', &c., &c. may be $= 0$. This happens, for instance, when

in a certain row, or in more rows at the same time, one or other of the different kinds of things is altogether wanting. In this case, it is only necessary, for self-evident reasons, to omit from the denominators of λ, λ', λ'', &c. those of the products $1 \cdot 2 \ldots a$, $1 \cdot 2 \ldots b$, &c. which refer to the deficient numbers.

SECTION XXX.

PROB. Find the coefficients and the signs of the terms in the solution of $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$.

Solution 1. If in the numerical expression $[\alpha\beta\gamma\delta\ldots\omega]$ more radical exponents than one are equal to one another, consequently, when this expression assumes the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$, then the terms of the solution have the following general form:

$$(\psi)\ldots\ldots\left\{\begin{array}{l} (a\alpha + b\beta + c\gamma + \ldots + f\zeta) \\ \times\ (a'\alpha + b'\beta + c'\gamma + \ldots + l'\lambda) \\ \times\ (a''\alpha + b''\beta + c''\gamma + \ldots + r''\rho) \\ \times\ \&c.. \end{array}\right\}.$$

2. Since in each term of the solution of $[\alpha\beta\gamma\delta\ldots\omega]$, all the letters α, β, γ, ... ω are divided into simple and compound radical exponents (Remark, § 24); then, likewise, in the case when this numerical expression assumes the form $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}} \ldots \kappa^{\mathfrak{k}}]$, in each term (ψ)

$a + a' + a'' + \&c. = \mathfrak{a}$, $b + b' + b'' + \&c. = \mathfrak{b}$,

$c + c' + c'' + \&c. = \mathfrak{c}$, &c.

or, in other words, the radical exponents of the numerical expressions, which occur in each term as factors, are no other than the divisions of $\mathfrak{a}\alpha + \mathfrak{b}\beta + \mathfrak{c}\gamma + \ldots + \mathfrak{k}\kappa$,

$-1,\ \varepsilon+\delta,\ \gamma,\ \beta,\ \alpha$	$-6,\ \varepsilon+\delta+\beta+\alpha,\ \gamma$
$-1,\ \varepsilon+\gamma,\ \delta,\ \beta,\ \alpha$	$-2,\ \varepsilon+\gamma,\ \delta+\beta+\alpha$
$-1,\ \varepsilon+\beta,\ \delta,\ \gamma,\ \alpha$	$-6,\ \varepsilon+\delta+\gamma+\beta,\ \alpha$
$-1,\ \varepsilon+\alpha,\ \delta,\ \gamma,\ \beta$	$-2,\ \varepsilon+\alpha,\ \delta+\gamma+\beta$
$+1,\ \varepsilon+\delta,\ \gamma,\ \beta+\alpha$	$-2,\ \varepsilon+\delta+\alpha,\ \gamma+\beta$
$+1,\ \varepsilon+\gamma,\ \delta,\ \beta+\alpha$	$-2,\ \varepsilon+\gamma+\beta,\ \delta+\alpha$
$+2,\ \varepsilon+\beta+\alpha,\ \delta,\ \gamma$	$-6,\ \varepsilon+\delta+\gamma+\alpha,\ \beta$
$+1,\ \varepsilon+\delta,\ \gamma+\beta,\ \alpha$	$-2,\ \varepsilon+\beta,\ \delta+\gamma+\alpha$
$+2,\ \varepsilon+\gamma+\beta,\ \delta,\ \alpha$	$-2,\ \varepsilon+\delta+\beta,\ \gamma+\alpha$
$+1,\ \varepsilon+\alpha,\ \delta,\ \gamma+\beta$	$-2,\ \varepsilon+\gamma+\alpha,\ \delta+\beta$
$+1,\ \varepsilon+\delta,\ \gamma+\alpha,\ \beta$	$+24,\ \varepsilon+\delta+\gamma+\beta+\alpha$

We have $\therefore\ [\alpha\beta\gamma\delta\varepsilon] = [\varepsilon]\,[\delta]\,[\gamma]\,[\beta]\,[\alpha] -$
$[\varepsilon]\,[\delta]\,[\gamma]\,[\beta+\alpha]-[\varepsilon]\,[\delta]\,[\gamma+\beta]\,\alpha -$&c.

SECTION XXVI.

If there are more radical exponents equal to one another in the numerical expression $[\alpha\beta\gamma\ \ldots\ldots\ \lambda]$, or if it takes the form $[\alpha^a\beta^b\gamma^c\ \ldots\ldots\ \kappa^k]$, the solution admits of reduction, because in this case more terms than one are equal to one another. By the application of the rules (a) and (b) in 2, § XXIV, we have only to take care that no term occurs more than once, and with this view it is only necessary in performing the involution, always to revert to the combinations already found, and to omit all those which occurred once before. Thus we find the involution for the terms of $[\alpha^3\beta^3]$ to be as follows:

$\beta \mid \beta \mid \beta \mid \alpha \mid \alpha \mid \alpha$	$2\beta,\ \beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta \mid \alpha \mid 2\alpha$	$2\beta,\ \beta+\alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta \mid 3\alpha$	$2\beta,\ \beta+2\alpha,\ \alpha$
$\beta \mid \beta \mid \beta+\alpha,\ \alpha,\ \alpha$	$2\beta,\ \beta+3\alpha$
$\beta \mid \beta \mid \beta+\alpha,\ 2\alpha$	$3\beta,\ \alpha,\ \alpha,\ \alpha$
$\beta \mid \beta \mid \beta+2\alpha,\ \alpha$	$3\beta,\ \alpha,\ 2\alpha$
$\beta \mid \beta \mid \beta+3\alpha$	$3\beta,\ 3\alpha$
$\beta \mid 2\beta,\ \alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ \alpha,\ \alpha$
$\beta \mid 2\beta,\ \alpha,\ 2\alpha$	$\beta+\alpha,\ 2\beta+\alpha,\ \alpha$
$\beta \mid 2\beta,\ 3\alpha$	$\beta+\alpha,\ \beta+\alpha,\ \beta+\alpha$
$\beta \mid 2\beta+\alpha,\ \alpha,\ \alpha$	$3\beta+\alpha,\ 2\alpha$
$\beta \mid \beta+\alpha,\ \beta+\alpha,\ \alpha$	$\beta+2\alpha,\ 2\beta+\alpha$
$\beta \mid 2\beta+\alpha,\ 2\alpha$	$2\beta+2\alpha,\ \beta+\alpha$
$\beta \mid \beta+2\alpha,\ \beta+\alpha$	$3\beta+2\alpha,\ \alpha$
$\beta \mid 2\beta+2\alpha,\ \alpha$	$3\beta+3\alpha$
$\beta \mid 2\beta+3\alpha$	

This involution contains every possible analysis of $3^{\alpha}+3^{\beta}$. In the same manner we obtain generally by the involution for $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$ every possible analysis of $\mathfrak{a}\alpha+\mathfrak{b}\beta+\mathfrak{c}\gamma+\ldots+\mathfrak{k}\kappa$. It may $\therefore$ be used with advantage in all those cases where it is required to represent analysis of this kind, with facility and without the danger of omission. In the numerical expression $[\alpha^{\mathfrak{a}}]$, the operation is merely reduced to a numerical analysis, respecting which information is given in the rule of combinations.

But as to the coefficients of the terms, it is easily conceived, that the rules (*a*) and (*b*), in 1, § XXV, in the case where the numerical expression $[\alpha\beta\gamma\ldots\lambda]$ is changed to $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\kappa^{\mathfrak{k}}]$, must undergo many modifications, and first, for this reason, because in this case

then the number of divisions (§ XXIX.) which the numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, &c. admit of, is

$$N = \frac{1}{1 . 2 \ldots \mu \times 1 . 2 \ldots \mu' \times 1 . 2 \ldots \mu'' \times \&c.} \times$$
$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots \mathfrak{a} - 1 . \mathfrak{a}}{(1 . 2 \ldots a)^{\mu} \times (1 . 2 \ldots a')^{\mu'} \times (1 . 2 \ldots a'')^{\mu''} \times \&c.} \times$$
$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots \mathfrak{b} - 1 . \mathfrak{b}}{(1 . 2 \ldots b)^{\mu} \times (1 . 2 \ldots b')^{\mu'} \times (1 . 2 \ldots b'')^{\mu''} \times \&c.} \times$$
$$\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots \mathfrak{b} - 1 . \mathfrak{c}}{(1 . 2 \ldots c)^{\mu} \times (1 . 2 \ldots c')^{\mu'} \times (1 . 2 \ldots c'')^{\mu''} + \&c.} \times$$
$$\&c.$$

9. But from § XXV. 1 . (b), the coefficient of every term which is combined with the term (ψ)

$$= (1 . 2 \ldots m - 1)^{\mu} \times (1 . 2 \ldots m' - 1)^{\mu} \times (1 . 2 \ldots m'' - 1)^{\mu''} \times \ldots$$

Further, as in § VI.

$$\kappa = 1 . 2 \ldots \mathfrak{a} \times 1 . 2 \ldots \mathfrak{b} \times 1 . 2 \ldots \mathfrak{c} \times \ldots$$

10. We ∴ have for this case

$$K = \frac{\left[\begin{array}{c}(1 . 2 \ldots m - 1)^{\mu} \times (1 . 2 \ldots m' - 1)^{\mu'} \times \\ (1 . 2 \ldots m'' - 1)^{\mu''} \times \&c.\end{array}\right]}{1 . 2 \ldots \mathfrak{a} \times 1 . 2 \ldots \mathfrak{b} \times 1 . 2 \ldots \mathfrak{c} \times \&c.} . N$$

or, when for N its value in 8 is substituted,

$$K = \frac{\left[\begin{array}{c}(1 . 2 \ldots m - 1)^{\mu} + (1 . 2 \ldots m' - 1)^{\mu'} \times \\ (1 . 2 \ldots m'' - 1)^{\mu''} \times \&c.\end{array}\right]}{\left[\begin{array}{l}\quad 1 . 2 \ldots \mu \times 1 . 2 \ldots \mu' \times 1 . 2 \ldots \mu'' \times \&c. \\ \times (1 . 2 \ldots a)^{\mu} \times (1 . 2 \ldots a')^{\mu'} \times (1 . 2 \ldots a'')^{\mu''} \times \&c. \\ \times (1 . 2 \ldots b)^{\mu} \times (1 . 2 \ldots b')^{\mu'} \times (1 . 2 \ldots b'')^{\mu''} \times \&c. \\ \times (1 . 2 \ldots c)^{\mu} \times (1 . 2 \ldots c')^{\mu'} \times (1 . 2 \ldots c'')^{\mu''} \times \&c. \\ \times \ldots\ldots\ldots\ldots \&c.\end{array}\right]}$$

or

11.

$$K=\frac{\left[\begin{array}{c}(1\,.\,2\ldots m-1)^{\mu}\times(1\,.\,2\ldots m'-1)^{\mu'}\times\\ (1\,.\,2\ldots m''-1)^{\mu''}\times \&c.\end{array}\right]}{\left[\begin{array}{l}\quad 1\,.\,2\ldots{}^{\mu}\times 1\,.\,2\ldots{}^{\mu'}\times 1\,.\,2\ldots{}^{\mu''}\times\&c.\\ \times\ (1\,.\,2\ldots a\times 1\,.\,2\ldots b\times 1\,.\,2\ldots c\times\&c.)^{\mu}\\ \times\ (1\,.\,2\ldots a'\times 1\,.\,2\ldots b'\times 1\,.\,2\ldots c'\times\&c.)^{\mu'}\\ \times\ (1\,.\,2\ldots a''\times 1\,.\,2\ldots b''\times 1\,.\,2\ldots c''\times\&c.)^{\mu''}\\ \times\ \ldots\ldots\ldots\ \&c.\end{array}\right]}$$

12. This expression for K includes that in 7, because we obtain the former when we substitute $\mu=\mu'=\mu''=$&c. $=1$ in the latter; it is consequently quite general, and obtains for all imaginable cases.

In the case where one of the numbers m, m', m'', &c. say m, $=1$, we must, instead of the product $1\,.\,2\ldots m-1$, merely put 1.

13. The sign of the term (ψ) is always the same as that of the expression $(\overline{+m})^{\mu'}\times(\overline{+m'})^{\mu'}\times(\overline{m''})^{\mu''}$, when every time we merely give the numbers m, m', m'', &c., the sign $-$ when even, and the sign $+$ when odd. The reason of this follows from § XXV., 1. (c.)

Example. Determine the coefficient and the sign of the term

$$(3\alpha+7\beta\times 2\gamma+4\delta)\,(5\alpha+4\gamma+\delta)^{3}\,(5\alpha)^{4}$$

of the solution of $(\alpha^{38}\beta^{7}\gamma^{14}\delta^{7})$.

Here $a=3$, $b=7$, $c=2$, $d=4$; $a'=5$, $b'=0$, $c'=4$, $d'=1$; $a''=5$; further $\mu=1$, $\mu'=3$, $\mu''=4$; $\therefore m=a+b+c+d=16$, $m'=a'+b'+c'+d'=10$, $m''=a''=5$. We have consequently from the formula in 11.

$$K=\frac{(1\,.\,2\,.\,3\ldots15)^1\times(1\,.\,2\,.\,3\ldots9)^3\times(1\,.\,2\,.\,3\,.\,4)^4}{\left[\begin{array}{l}1\times1\,.\,2\,.\,3\times1\,.\,2\,.\,3\,.\,4\\ \times\;(1\,.\,2\,.\,3\times1\,.\,2\ldots7\times1.2\times1\,.\,2\,.\,3.\,4)^1\\ \times\;(1\,.\,2\,.\,3\,.\,4\,.\,5\times1\,.\,2\,.\,3\,.\,4\times1)^3\\ \times\;(1\,.\,2\,.\,3\,.\,4\,.\,5)^4\end{array}\right]}$$

$$=\frac{500594094}{25}.$$

The sign of the term is the same as that of the expression $(-16)^1 \times (-10)^3 \times (+5)^4$, consequently $+$.

SECTION XXXI.

In order to render what has been already advanced more intelligible, I shall here give the complete solution of the numerical expression $(a^3\beta^3)$, which has already been made use of in § XXVI. as an example for the representation of involution. The terms are arranged as they are there found.

$$\begin{aligned}
(a^3\beta^3) = {} & \\
& \tfrac{1}{36}(a)^3(\beta)^3-\tfrac{1}{12}(a)(2a)(\beta)^3+\tfrac{1}{18}(3a)(\beta)^3\\
& -\tfrac{1}{4}(a)^2(\beta)^2(a+\beta)+\tfrac{1}{4}(2a)(\beta)^2(a+\beta)+\tfrac{1}{1}(a)(\beta)^2(2a+\beta)\\
& -\tfrac{1}{2}(\beta)^2(3^a+\beta)-\tfrac{1}{12}(a)^3(\beta)(2\beta)+\tfrac{1}{4}(a)(2a)(\beta)(2\beta)\\
& -\tfrac{1}{6}(3a)(\beta)(2\beta)+\tfrac{1}{2}(a)^2(\beta)(a+2\beta)+\tfrac{1}{2}(a)(\beta)(a+\beta)^2\\
& -\tfrac{1}{2}(2a)(\beta)(a+2\beta)-(\beta)(a+\beta)(2a+\beta)-\tfrac{3}{2}(a)(\beta)(2a+2\beta)\\
& +2(\beta)(3a+2\beta)+\tfrac{1}{4}(a)^2(2\beta)(a+\beta)-\tfrac{1}{4}(2a)(2\beta)(a+\beta)\\
& -\tfrac{1}{2}(a)(2\beta)(2a+\beta)+\tfrac{1}{2}(2\beta)(3a+\beta)+\tfrac{1}{18}(a)^3(3\beta)\\
& -\tfrac{1}{6}(a)(2a)(3\beta)+\tfrac{1}{9}(3a)(3\beta)-\tfrac{1}{2}(a)^2(a+3\beta)\\
& -(a)(a+\beta)(a+2\beta)-\tfrac{1}{6}(a+\beta)^3+\tfrac{1}{2}(2a)(a+3\beta)\\
& +(a+2\beta)(2a+\beta)+\tfrac{1}{2}(a+\beta)(2a+2\beta)+2(a)(2a+3\beta)\\
& -\tfrac{10}{3}(3a+3\beta).
\end{aligned}$$

From this example we shall clearly perceive, how we are to proceed in every other case, and it seems to me unnecessary to add any thing more.

III.—On the values of the functions of the roots of an equation which are not symmetrical, and on the method by which to make these values dependant on equations.

SECTION XXXII.

SYMMETRICAL functions differ from the others in this, that in the first place they contain all the roots of the given equation; and secondly, these roots are so combined with one another, that the functions in each transposition are changed. For this kind of functions, it is sufficient only to mention in general the form of the combination, without referring to the roots themselves, because we are always sure before-hand to obtain the same results in the composition. Thus, for instance, the expression (12) is fully determined, although by the notation nothing more is indicated, than that we are to take the sum of all the products which arise from the combination of every root with the square of another. A function of this kind ∴ can only have a single value, and this value is no other than that, the way to find which, was shown in the foregoing §. It is, as we have already seen, in reference to the coefficients of the given equation, always rational, and it must necessarily be so, because otherwise the function would have more values.

But this is not the case with the other functions. If

we merely wished to give the form of their combination, we should not by this means alone be able to determine the function. If, for instance, a, b, c, be the three roots of an equation of the third degree, then indeed the sum of all the three roots $a+b+c$, can only be expressed in one way; on the other hand, the sum of two roots, in three different ways, viz. by $a+b$, $a+c$, $b+c$; and the difference of two roots may be expressed in as many as six different ways, viz. by $a-b$, $b-a$, $a-c$, $c-a$, $b-c$, $c-b$. A function of this kind consequently has more values, which arise partly from the substitution of the roots it contains for the remaining ones, and partly from the transposition of these roots. None of these, so long as the roots a, b, c, &c. are undetermined, can be found by themselves without the others, because otherwise there would be no reason why we should exactly find this or that value, and not the others likewise. Hence it follows, that the value of every function which is not symmetrical, can only be expressed by an equation which, at the same time, includes all the values which the function can contain by substitution and transposition of the roots.

Whatever may seem obscure in these general observations, the following problems will render clear. I here remind my readers, once for all, that I shall not in future denote the roots of the given equation, as heretofore, by a, b, c, d, &c., but by x', x'', x''', x^{IV}, &c. This I do, partly, because this notation has been adopted by nearly the whole of the modern analysts, and because it is always desirable to retain the mode of notation already adopted, if it can be done without disadvantage; partly

also, because we are accustomed to associate with the first letters of the alphabet, the idea of determinate numerical values; while, on the contrary, here for instance, x' denotes neither this nor that determinate root, but generally any root whatever, and the dashes over the x are put for the sake of distinction.

SECTION XXXIII.

PROB. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

determine the value of the function $x'x''$, without knowing the roots of the equation.

Solution 1. Since x', x'', x''', are the roots of the given equation, we have

$$x' + x'' + x''' = A$$
$$x'x'' + x'x''' + x''\,x''' = B$$
$$x'x''x''' = C$$

and from these three equations we must endeavour to determine the value of $x'x''$.

2. With this view, put $x'\,x'' = t$, substitute this value in the second and third equations, and multiply the first equation by x'''; this gives

$$x'x''' + x''x''' + x'''^2 = Ax'''$$
$$t + x'x''' + x''x''' = B$$
$$x'''t \qquad = C$$

If we subtract the first of these equations from the second, and then substitute for x''' its value $\frac{C}{t}$ from the third equation, we obtain the following equations for t:

$$t^3 - Bt^2 + ACt - C^2 = 0$$

The value of $x'x''$ can only be found by the solution of an equation of the third degree.

3. If we had put $x'x'''$, or $x''x''' = t$, we should have found the same equation. Hence we may safely conclude, that the values $x'x''$, $x'x'''$, $x''x'''$, must be the three roots of the equation found.

4. This consequence might very easily have been foreseen. Since, then, there is no reason why the equation for t should give exactly the product $x'x''$, and not also the product $x'x'''$, or $x''x'''$, as the three roots x', x'', x''', are in some way contained in the given equation, consequently it must necessarily be of the third degree.

5. We could ∴ also have found this equation in a direct way. Since, for instance, $x'x''$, $x'x'''$, $x''x'''$, must be the three roots of the required equation, consequently it can be no other than the following equation:

$$(t - x'x'')\,(t - x'x''')\,(t - x''x''') = 0.$$

If we actually multiply the three factors in the first part, we obtain

$$\begin{aligned} &t^3 - (x'x'' + x'x''' + x''x''')\,t^2 \\ &\quad + (x'^2x''x''' + x'x''^2x''' + x'x''x'''^2)t \\ &\quad - x'^2x''^2x'''^2 = 0 \end{aligned}$$

or since $x'x'' + x'x''' + x''x'''=B$, $x'^2x''x''' + x'x''^2x''' + x'x''x'''^2 = (x' + x'' + x''')\,x'x''x''' = AC$, $x'^2x''^2x'''^2 = (x'x''x''')^2 = C^2$, the same equation as we have above.

Example. In the equation $x^3 + x^2 - 3^2\,x - 60 = 0$, $A = -1$, $B = -32$; $C = 60$; ∴ we have

$$t^3 + 32t^2 - 60t - 3600 = o,$$

an equation, whose roots are the product of every two roots of the former equation. Thus the roots of this equation are $+ 10, - 12, - 30$, and the roots of the former are $- 2, - 5, + 6$.

PROB. From the given equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = o$$

determine the value of the function x'^2.

Solution 1. Since all the roots are contained in the given equation in the same way, and for the root whose square is required, nothing nearer can be determined; consequently the square of one of the roots cannot be found, without at the same time finding the squares of all the remaining ones. Therefore the equation by which x'^2 is expressed, must necessarily be of the fourth degree.

2. If ∴ we put $x'^2 = t$, then the equation which gives the value of t must be of the fourth degree, and its roots must be x'^2, x''^2, x'''^2 $x^{\text{IV}2}$. This equation is ∴ composed of the four particular equations

$$t - x'^2 = o, \quad t - x''^2 = o$$
$$t - x'''^2 = o, \quad t - x^{\text{IV}2} = o$$

and consequently is no other than the product of these last. By actual multiplication we have

$$t^4 - (x'^2 + x''^2 + x'''^2 + x^{\text{IV}2})\, t^3 +$$
$$(x'^2x''^2 + x'^2x'''^2 + x'^2x^{\text{IV}2} + x''^2x'''^2 + x''^2x^{\text{IV}2} + x'''^2x^{\text{IV}2})t^2$$
$$- (x'^2x''^2x'''^2 + x'^2x''^2x^{\text{IV}2} + x'^2x'''^2x^{\text{IV}2} + x''^2x'''^2x^{\text{IV}2})\, t$$
$$+ x'^2x''^2x'''^2x^{\text{IV}2} = o$$

It only remains now to express the coefficients of this equation by the coefficients of the given one.

3. But this equation, when we use brackets, may be thus represented:

$$t^4 - (2)\,t^3 + (2^2)\,t^2 - (2^3)\,t + (2^4) = 0$$

and the values of the numerical expressions are obtained directly from the annexed Tables. Thus, since the coefficients E, F, &c. $= 0$, we find

$$(2) = A^2 - 2B, \quad (2^2) = B^2 - 2AC + 2D,$$
$$(2^3) = C^2 - 2BD, (2^4) = D^2,$$

By substituting these values, we have

$$t^4 - (A^2 - 2B)\,t^3 + (B^2 - 2AC + 2D)\,t^2 -$$
$$(C^2 - 2BD)\,t + D^2 = 0$$

which is the equation sought.

4. We could also have found this equation in the following way. Put $x^2 = t$, or $x = \sqrt{t}$, substitute this value of x in the given equation, and place all the terms in which $\sqrt{t}$ is involved on one side of the equation. Hence we obtain

$$t^2 + Bt + D = (At \mp C)\,\sqrt{t}.$$

If this equation be squared, and properly arranged, we obtain the same equation as in 3.

Example. In the equation $x^4 + 10x^3 + 25x^2 - 2x - 12 = 0$, we have $A = -10$, $B = 25$, $C = 2$, $D = -12$. The equation, whose roots are the squares of the roots of this equation, is $\therefore$

$$t^4 - 50\,t^3 + 641\,t^2 - 604\,t + 144 = 0.$$

The roots of the former equation are $-3+\sqrt{5}$, $-3-\sqrt{5}$, $-2+\sqrt{7}$, $-2-\sqrt{7}$; the roots of the latter are $14-6\sqrt{5}$, $14+6\sqrt{5}$, $11-4\sqrt{7}$, $11+4\sqrt{7}$; these last are the squares of the former.

SECTION XXXV.

PROB. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

find the value of the function $ax'x''+bx'''$.

Solution 1. The equation, by which the function $ax'x''+bx'''$ is expressed, or on which it depends, must contain all its possible values. Since in this function all the three roots occur, it is only necessary, in order to find its different values, to transpose these roots in all possible ways, and retain those of the results which differ from one another. But the results of these transpositions are

$$ax'x''+bx''',\ ax'x'''+bx'',\ ax''x'''+bx'$$
$$ax''x'+bx''',\ ax'''x'+bx'',\ ax'''x''+bx'$$

amongst which there are only three which are different from one another, viz.

$$ax'x''+bx''',\ ax'x'''+bx'',\ ax''x'''+bx'.$$

The required equation must consequently have these three functions for roots.

2. If ∴ we denote each of these undetermined functions by t, we then obtain the three particular equations.

$$t-(ax'x''+bx''')=0$$
$$t-(ax'x'''+bx'')=0$$
$$t-(ax''x'''+bx')=0$$

and the product of these gives the equation, from which the value of each of the three functions must be determined. It may be represented by

$$t^3 - A't^2 + B't - C' = 0.$$

3. Then

$$A' = (ax'x'' + bx''') + (ax'x''' + bx'') + (ax''x''' + bx')$$
$$= a(1^2) + b(1)$$
$$B' = (ax'x'' + bx''')(ax'x''' + bx'')$$
$$+ (ax'x'' + bx''')(ax''x''' + bx')$$
$$+ (ax'x''' + bx'')(ax''x''' + bx')$$
$$= a^2(1^2 2) + ab(1\ 2) + b^2(1^2)$$
$$C' = (ax'x'' + bx''')(ax'x''' + bx'')(ax''x''' + bx')$$
$$= a^3(2^3) + a^2b(1^2 3) + ab^2(2^2) + b^3(1^3)$$

If for the numerical expressions we substitute their values from the annexed Tables, we then obtain, since D, E, &c. $= 0$,

$$A' = aB + bA$$
$$B' = a^2AC + ab(AB - 3C) + b^2B$$
$$C' = a^3C^2 + a^2b(A^2C - 2BC) + ab^2(B^2 - 2AC) + b^3C,$$

and consequently the equation on which the required function depends, is

$$t^3 - (aB + bA)t^2 + [a^2AC + ab(AB - 3C) + b^2B]t$$
$$- [a^3C^2 + a^2b(A^2C - 2BC) + ab^2(B^2 - 2AC) + b^3C] = 0$$

SECTION XXXVI.

PROB. From the given equation of the third degree

$$x^3 - Ax^2 + Bx - C = 0$$

find the equation on which the function $x' - x''$ depends.

Solution. The function $x' - x''$ contains the six follow-

ing values, which arise partly from the substitution, and partly from the transposition of the roots:

$$x'-x'',\ x'-x''',\ x''-x'''$$
$$x''-x',\ x'''-x',\ x'''-x''.$$

The required equation is $\therefore$ the product of the following six particular equations:

$$t-(x'-x'')=0,\quad t+(x'-x'')=0$$
$$t-(x'-x''')=0,\quad t+(x'-x''')=0$$
$$t-(x''-x''')=0,\quad t+(x''-x''')=0$$

or, when every two of the opposite equations are multiplied together, the product of the three following equations:

$$t^2-(x'-x'')^2=0$$
$$t^2-(x'-x''')^2=0$$
$$t^2-(x''-x''')^2=0$$

from which it follows, that it only contains even powers of t. Let $\therefore$

$$t^6-A't^4+B't^2-C'=0$$

be this equation; then

$$A'=(x'-x'')^2+(x'-x''')^2+(x''-x''')^2$$
$$=2(2)-2(1^2)$$
$$B'=(x'-x'')^2(x'-x''')^2+(x'-x'')^2(x''-x''')^2$$
$$+(x'-x''')^2(x''-x''')^2$$
$$=(4)+3(2^2)-2(13)$$
$$C'=(x'-x'')^2(x'-x''')^2(x''-x''')^2$$
$$=(24)-2(3^2)-6(2^3)+2(123)-2(1^24)$$

Now, if we take the values of the numerical expressions from the Tables, after the proper reduction, we find

$$A'=2A^2-6B$$
$$B'=A^4-6A^2B+9B^2$$
$$C'=A^2B^2-4B^3-4A^3C+18ABC-27C^2$$

and $\therefore$ the required equation is

$$t^6 - (2A^2 - 6B)\,t^4 + (A^4 - 6A^2B + 9B^2)\,t^2$$
$$- (A^2B^2 - 4B^3 - 4A^3C + 18ABC - 27C^2) = 0,$$

whose roots are the differences of the roots of the given equation.

EXAMPLE. In the equation $x^3 - 8x^2 + 19x - 12 = 0$ $A = 18$, $B = 19$, $C = 12$; $\therefore A' = 14$, $B' = 49$, $C' = 36$, and consequently the equation for the differences is

$$t^6 - 14\,t^4 + 49\,t^2 - 36 = 0.$$

The roots of the first equation are $+1$, $+3$, $+4$; the roots of the last are $+1$, -1, $+2$, -2, $+3$, -3, as required.

SECTION XXXVII.

PROB. From the given equation

$$t^3 - Ax^2 + Bx - C = 0,$$

find the equation for the fraction $\dfrac{x'}{x''}$.

Solution 1. The function $\dfrac{x'}{x''}$, by the substitution and transposition of the roots, contains the six following values:

$$\frac{x'}{x''},\ \frac{x''}{x'},\ \frac{x'}{x'''},\ \frac{x'''}{x'},\ \frac{x''}{x'''},\ \frac{x'''}{x''}.$$

The equation by which these functions are expressed, is $\therefore$ of the sixth degree.

2. This equation is represented by

$$t^6 - A't^5 + B't^4 - C't^3 + D't^2 - E't + F' = 0,$$

consequently A' is the sum of the functions given in 1, B' the sum of their products, two and two, C' the sum of

their products three and three, and so on. Hence, after the usual reduction, we get the following values:

$$A' = \frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$= \frac{(12)}{(1^3)}$$

$$B' = 3 + \frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$+ \frac{x'^3x''^3 + x'^3x'''^3 + x''^3x'''^3 + x'x''x'''^4 + x'x''^4x''' + x'^4x''x'''}{x'^2x''^2x'''^2}$$

$$= 3 + \frac{(12)}{(1^3)} + \frac{(3^2) + (1^24)}{(1^3)^2}$$

$$C' = 2 + 2.\frac{x'x''^2 + x'^2x'' + x'x'''^2 + x'^2x''' + x''x'''^2 + x''^2x'''}{x'x''x'''}$$

$$+ \frac{x'^2x''^4 + x'^4x''^2 + x'^2x'''^4 + x'^4x'''^2 + x''^2x'''^4 + x''^4x'''^2}{x'^2x''^2x'''^2}$$

$$= 2 + 2.\frac{(12)}{(1^3)} + \frac{(24)}{(1^3)^2}$$

If we proceed further with the calculation, we find the same expression for D' as for B', and for E' the same expression as for A'; further, F', as the product of all the above six functions, $= 1$.

3. Now, if we substitute for the numerical expressions their values taken from the Tables, we obtain

$$A' = E' = \frac{AB - 3C}{C}$$

$$B' = D' = \frac{B^3 - 5ABC + A^3C + 6C^2}{C^2}$$

$$C' = \frac{6ABC - 7C^2 + A^2B^2 - 2B^3 - 2A^3C}{C^2}$$

$$F' = 1$$

and these are the coefficients of the assumed equation for t.

Example. In the equation $x^3 + 2x - x - 2 = 0$, we have $A = -2$, $B = -1$, $C = 2$. Hence we find $A' = E' = -2$, $B' = D' - \frac{13}{4}$, $C' = \frac{17}{2}$. Consequently the required equation is

$$t^6 + 2t^5 - \frac{13}{4}t^4 - \frac{17}{2}t^3 - \frac{13}{4}t^2 + 2t + 1 = 0$$

The roots of this equation are -1, -1, -2, $-\frac{1}{2}$, $+2$, $+\frac{1}{2}$, which must be the case, since $+1$, -1, -2 are the roots of the given equation.

SECTION XXXVIII.

The equations for t, which we found in the foregoing §, and which may be found in a similar way for all other functions, may, in reference to those which are given, be called transformed equations. The degree and form of these last depend upon the functions which we assume for t. In the functions of which we have hitherto treated, the transformed equation has always been either of a higher or of the same degree as the given one. But there are functions for which the given equation is of a lower degree; and in this case it can sometimes serve to solve the given equation, as will be shown by the two following examples for equations of the fourth degree.

SECTION XXXIX.

Prob. From the given equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

find the value of the function $x'x'' + x'''x^{iv}$.

Solution 1. Since in the function $x'x'' + x'''x^{iv}$ all the roots occur at once, they can only alter their value by transposition. But the four roots x', x'', x''', x^{iv}, may be transposed in $1 \,.\, 2 \,.\, 3 \,.\, 4 = 24$ ways, and the results thus obtained are

$$\begin{array}{lll}
x'x'' + x'''x^{iv}, & x'x''' + x''x^{iv}, & x'x^{iv} + x''x''' \\
x'x'' + x^{iv}x''', & x'x''' + x^{iv}x'', & x'x^{iv} + x'''x'' \\
x''x' + x'''x^{iv}, & x'''x' + x''x^{iv}, & x^{iv}x' + x''x''' \\
x''x' + x^{iv}x''', & x'''x' + x^{iv}x'', & x^{iv}x' + x'''x'' \\
x'''x^{iv} + x'x'', & x''x^{iv} + x'x''', & x''x''' + x'x^{iv} \\
x^{iv}x''' + x'x'', & x^{iv}x'' + x'x''', & x'''x'' + x'x^{iv} \\
x'''x^{iv} + x''x', & x''x^{iv} + x'''x', & x''x''' + x^{iv}x' \\
x^{iv}x''' + x''x', & x^{iv}x'' + x'''x', & x'''x'' + x^{iv}x'
\end{array}$$

It is immediately seen, that eight of these results, which are in the same vertical column, are always equal to one another, and that also the function can have no more than the three following different values:

$$x'x'' + x'''x^{iv},\; x'x''' + x''x^{iv},\; x'x^{iv} + x''x'''$$

The transformed equation is consequently of the third degree, and its roots are the values just mentioned.

2. Let this equation be expressed by

$$t^3 - A't^2 + B't - C' = 0;$$

then, from the nature of equations,

$$A' = (x'x'' + x'''x^{\text{iv}}) + (x'x''' + x''x^{\text{iv}}) + (x'x^{\text{iv}} + x''x''')$$
$$= (1^2)$$
$$B' = (x'x'' + x'''x^{\text{iv}})\ (x'x''' + x''x^{\text{iv}})$$
$$+\ (x'x'' + x'''x^{\text{iv}})\ (x'x^{\text{iv}} + x''x''')$$
$$+\ (x'x''' + x''x^{\text{iv}})\ (x'x^{\text{iv}} + x''x''')$$
$$= (1^2 2)$$
$$C' = (x'x'' + x'''x^{\text{iv}})\ (x'x''' + x''x^{\text{iv}})\ (x'x^{\text{iv}} + x''x''')$$
$$= (2^3) + (1^3 3)$$

3. Now if we take the values of the numerical expressions from the annexed Tables, and then substitute for A', B', C', their values found in the assumed equation, we obtain the required transformed equation

$$t^3 - Bt^2 + (AC - 4D)\ t - (C^2 - 4BD + A^2D) = 0$$

I shall now proceed to show what use can be made of this equation in the general solution of equations of the fourth degree.

SECTION XL.

Let it be assumed, that we can find a root of the transformed equation; let t' be this root, $\therefore\ x'x'' + x'''x^{\text{iv}} = t'$. It only remains now, from this equation, together with the others, which express the known relations between the roots and the coefficients, to determine the values of x', x'', x''', x^{iv}.

For this purpose, combine, first, the two equations

$$x'x'' + x'''x^{\text{iv}} = t',\ x'x''x'''x^{\text{iv}} = D.$$

These give

$$(x'x'' - x'''x^{\text{iv}})^2 = (x'x'' + x'''x^{\text{iv}})^2 - 4x'x''x'''x^{\text{iv}}$$
$$= t'^2 - 4D$$
$$x'x'' - x'''x^{\text{iv}} = \surd\,(t'^2 - 4D)$$

$\therefore$

$$x'x'' = \frac{t' + \sqrt{(t'^2 - 4D)}}{2}, \quad x'''x^{IV} = \frac{t' - \sqrt{(t'^2 - 4D)}}{2}$$

Combine the two equations

$$x'''x^{IV}(x' + x'') + x'x''(x''' + x^{IV}) = B$$
$$(x' + x'') + \quad (x''' + x^{IV}) = A$$

These give

$$x' + x'' = \frac{Ax'x'' - B}{x'x'' - x'''x^{IV}} = \frac{At' - 2B}{2\sqrt{(t'^2 - 4D)}} + \frac{A}{2}$$

$$x''' + x^{IV} = \frac{Ax'''x^{IV} - B}{x'''x^{IV} - x'x''} = \frac{At' - 2B}{2\sqrt{(t'^2 - 4D)}} + \frac{A}{2}$$

Now we know the values of $x'x''$, $x' + x''$, $x'''x^{IV}$, $x''' + x^{IV}$. From the two first of these values we can determine the roots x', x'', and from the two last the roots x''', x^{IV}, merely by the solution of quadratic equations.

In this solution it is only sufficient to know one root of the transformed equation.

SECTION XLI.

Prob. From the given equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0,$$

determine the value of the function $(x' + x'' - x''' - x^{IV})^2$.

Solution 1. If we proceed with this function, as we have already done with the function $x'x'' + x'''x^{IV}$, we shall then find no more than the three following different values:

$$(x' + x'' - x''' - x^{IV})^2, \qquad (x' + x''' - x'' - x^{IV})^2$$
$$(x' + x^{IV} - x'' - x''')^2$$

which consequently likewise depend on an equation of the third degree. This equation we could find in the usual

way: the following method, however, in this case, leads much more readily to the desired object.

2. Thus if we put $(x' + x'' - x''' - x^{IV})^2 = t$;
then

$$t = x'^2 + x''^2 + x'''^2 + x^{IV2} + 2x'x'' - 2x'x''' - 2x'x^{IV} - 2x''x''' - 2x''x^{IV} + 2x'''x^{IV}$$
$$= (x' + x'' + x''' + x^{IV})^2 - 4(x'x'' + x'x''' + x'x^{IV} + x''x''' + x''x^{IV} + x'''x^{IV}) + 4(x'x'' + x'''x^{IV})$$
$$= [1]^2 - 4[1^2] + (4\, x'x'' + x'''x^{IV})$$
$$= A^2 - 4B + 4(x'x'' + x'''x^{IV})$$

and ∴,

$$x'x'' + x'''x^{IV} = \frac{t - A^2 + 4B}{4}$$

3. Now since $x'x'' + x'''x^{IV}$ is exactly the function for which in § XXXIX the equation

$$t^3 - Bt^2 + (AC - 4D)\,t - (C^2 - 4BD + A^2D) = 0$$

was found, consequently in this equation it is only necessary to put $\frac{t - A^2 + 4B}{4}$ for t. Hence we find the equation

$$t^3 - (3A^2 - 8B)\,t^2 + (3A^4 - 16A^2B + 16B^2 + 16AC - 64D)\,t - (A^3 - 4AB + 8C)^2 = 0,$$

whose roots are the functions given in 1.

Hence also we may obtain, as in the preceding §, a solution of equations of the fourth degree, which may be seen immediately; only it is here assumed, that all the three roots of this equation are already found.

SECTION XLII.

Let t', t'', t''', be the three roots of the transformed equation in the foregoing §, then we have

$$(x' + x'' - x''' - x^{iv})^2 = t'$$
$$(x' + x''' - x'' - x^{iv})^2 = t''$$
$$(x' + x^{iv} - x'' - x''')^2 = t'''$$

and consequently

$$x' + x'' - x''' - x^{iv} = \sqrt{t'}$$
$$x' + x''' - x'' - x^{iv} = \sqrt{t''}$$
$$x' + x^{iv} - x' - x''' = \sqrt{t'''}$$

If we combine these three equations with

$$x' + x'' + x''' + x^{iv} = A$$

we obtain, merely by addition and subtraction, the following expressions for the roots:

$$x' = \frac{A + \sqrt{t'} + \sqrt{t''} + \sqrt{t'''}}{4}$$

$$x'' = \frac{A + \sqrt{t'} - \sqrt{t''} - \sqrt{t'''}}{4}$$

$$x''' = \frac{A - \sqrt{t'} + \sqrt{t''} - \sqrt{t'''}}{4}$$

$$x^{iv} = \frac{A - \sqrt{t'} - \sqrt{t''} + \sqrt{t'''}}{4}$$

Instead of which .·. we have, so soon as we have solved the expression for t, the four roots of the given equation.

But here there appears an evident difficulty. Thus, since in the values of x', x'', x''', x^{iv}, there are three roots $\sqrt{t'}$, $\sqrt{t''}$, $\sqrt{t'''}$, and these roots may be assumed to be either positive or negative,—the question is, how we are to proceed in order to determine the signs. For this purpose consider the last term of the transformed equation. Since this term must be the product of its three roots, we have

$$(A^3 - 4AB + 8C)^2 = t't''t''';$$

$$\therefore \quad A^3 - 4AB + 8C = \sqrt{t't''t'''} = \sqrt{t'}\,.\,\sqrt{t''}\,.\,\sqrt{t'''}.$$

Now if $A^3 - 4AB + 8C$ be positive, then also the product $\sqrt{t'}\,.\,\sqrt{t''}\,.\,\sqrt{t'''}$ must be positive, and consequently the signs can only be combined together in the four following ways:

$$\begin{array}{l}
+\sqrt{t'},\ +\sqrt{t''},\ +\sqrt{t'''} \\
+\sqrt{t'},\ -\sqrt{t''},\ -\sqrt{t'''} \\
-\sqrt{t'},\ +\sqrt{t''},\ -\sqrt{t'''} \\
-\sqrt{t'},\ -\sqrt{t''},\ +\sqrt{t'''}
\end{array}$$

and these combinations give the above values for x', x'', x''', x^{iv}.

If, on the other hand, $A^3 - 4AB + 8C$ be negative, then the product $\sqrt{t'}\,.\,\sqrt{t''}\,.\,+t'''$ is also negative, and consequently in this case the signs can only be combined in the four following ways:

$$\begin{array}{l}
+\sqrt{t'},\ +\sqrt{t''},\ -\sqrt{t'''} \\
+\sqrt{t'},\ -\sqrt{t''},\ +\sqrt{t'''} \\
-\sqrt{t'},\ +\sqrt{t''},\ +\sqrt{t'''} \\
-\sqrt{t'},\ -\sqrt{t''},\ -\sqrt{t'''}
\end{array}$$

and these combinations give the following values for x', x'', x''', x^{iv}:

$$x' = \frac{A + \sqrt{t'} + \sqrt{t''} - \sqrt{t'''}}{4}$$

$$x'' = \frac{A + \sqrt{t'} - \sqrt{t''} + \sqrt{t'''}}{4}$$

$$x''' = \frac{A - \sqrt{t'} + \sqrt{t''} + \sqrt{t'''}}{4}$$

$$x^{iv} = \frac{A - \sqrt{t'} - \sqrt{t''} - \sqrt{t'''}}{4}$$

Example. In the equation $x^4 - 3x^3 - 15x^2 + 19x + 30 = 0$, $A = 3, B = -15, C = -19$; $D = 30$. These values, when substituted in the transformed equation, give

$$t^3 - 147t^2 + 3171t - (+55)^2$$

The roots of this equation are 1, 25, 121. Now since here $A^3 - 4AB + 8C = +55$, positive consequently, $\therefore$ the four first values for x', x'', x''', x^{IV}, must be taken, and these give $x' = 5$, $x'' = -3$, $x''' = -1$, $x^{IV} = +2$.

SECTION XLIII.

Prob. From the given equation of the indeterminate nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0,$$

find another equation for the squares of its roots.

Solution 1. Let the required equation have the roots $x'^2, x''^2, x'''^2, x^{IV2}$, &c. it must consequently be composed of the n simple equations $t - x'^2 = 0$, $t - x''^2 = 0$, $t - x'''^2 = 0$, &c. Hence we have, as in § XXXIV, the transformed equation

$$t^n - (2)t^{n-1} + (2^2)t^{n-2} - (2^3)t^{n-3} + (2^4)t^{n-4}) + \ldots\ldots$$
$$\ldots\ldots \pm (2^{n-1})t \mp (2^n) = 0$$

The numerical expressions may either be taken immediately from the Tables, or may be found by the method given in the two preceding chapters.

2. But this equation may also be found by the second method in § XXXIV. Put for instance, $x^2 = 1$, $\therefore$ $x = \sqrt{t}$, and substitute this value of x in the given

equation. Here there are two distinct cases; viz. one when n is even, the other when n is odd.

3. First, let $n=2m$. Arrange the given equation thus:

$$x^{2m}+Bx^{2m-2}+Dx^{2m-4}+Fx^{2m-6}+\&c.$$
$$=Ax^{2m-1}+Cx^{2m-3}+Ex^{2m-5}+Gx^{2m-7}+\&c.$$

Substitute $\sqrt{t}$ for x; this gives

$$t^{m}+Bt^{m-1}+Dt^{m-2}+Ft^{m-3}+\&c.$$
$$=(At^{m-1}+Ct^{m-2}+Et^{m-3}+Gt^{m-4}+\&c.)\ \sqrt{t};$$

or, when both sides of this equation are squared, and the terms properly arranged,

$$t^{2m}+(2B-A^{2})\,t^{2m-1}+(2D-2AC+B^{2})\,t^{2m-2}$$
$$+(2F-2AE+2BD-C^{2})\,t^{2m-3}+\&c.=0.$$

4. Let $n=2m+1$. In this case, when $\sqrt{t}$ is put for x, we have

$$(t^{m}+Bt^{m-1}+Dt^{m-2}+Ft^{m-3}+\&c.)\ \sqrt{t}$$
$$=At^{m}+Ct^{m-1}+Et^{m-2}+\&c.$$

and when both sides of the equation are squared, and the terms properly arranged, we obtain the same equation as in 3, except that we get $2m+1$, instead of $2m$.

5. For both cases of the equation we consequently have

$$t^{n}+(2B-A^{2})\,t^{n-1}+(2D-2AC+B^{2})\,t^{n-1}$$
$$+(2F-2AE+2BD-C^{2})\,t^{n-2}+\&c.=0.$$

Remark. If we had not already known how to find the expressions (2), (2^2), (2^3), &c. by another method, we could have found them immediately by these means, by placing the equations in 1 and 5, opposite one another, and

putting the coefficients of the same powers of t equal to one another. Thus for instance we obtain

$$-[2] = 2B - A^2$$
$$[2^2] = 2D - 2AC + B^2$$
$$-[2^3] = 2F - 2AE + 2BD - C^2$$
$$[2^4] = 2H - 2AG + 2BF - 2CF + D^2$$
&c.

from which the law is easily seen.

If we denote the coefficients of the given equation, in order to denote the places which they occupy in it, by $\overset{1}{A}, \overset{2}{A}, \overset{3}{A}, \overset{4}{A}$, &c. instead of A, B, C, D, &c. the law will be still more easily perceived. Then we have

$$-[2] = 2\overset{2}{A} - \overset{1}{A}\overset{1}{A}$$
$$[2^2] = 2\overset{4}{A} - 2\overset{3}{A}\overset{1}{A} + \overset{2}{A}\overset{2}{A}$$
$$-[2^3] = 2\overset{6}{A} - 2\overset{5}{A}\overset{1}{A} + 2\overset{4}{A}\overset{2}{A} - \overset{3}{A}\overset{3}{A}$$
$$[2^4] = 2\overset{8}{A} - 2\overset{7}{A}\overset{1}{A} + 2\overset{6}{A}\overset{2}{A} + \overset{5}{A}\overset{3}{A}\,\overset{4}{A}\overset{4}{A}$$

and in general

$$\pm [2^n] = 2\overset{2n}{A} - 2\overset{2n-1}{A}\overset{1}{A} + 2\overset{2n-2}{A}\overset{2}{A} - 2\overset{2n-3}{A}\overset{3}{A} + \ldots$$
$$\ldots\ldots \mp \overset{n+1}{A}\overset{n-1}{A} \pm \overset{n}{A}\overset{n}{A}$$

the upper sign obtains when n is even, and the lower when n is odd.

Euler uses these formulæ for finding the impossible roots of an equation [Complete Introduction to the Differential Calculus, translated by Michelsen, Part III. p. 135], but he gives no proof of them, but merely says, that they may be found by the theory of combinations. A proof of these formulæ different from the above, may be seen in Klügel's Mathematical Dictionary, Art. Combination, p. 469, &c.

SECTION XLIV.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0,$$

find the equation for the *m*th powers of its roots.

Solution. The roots of the required equation are x'^m, x''^m, x'''^m, &c. Hence we obtain in the same way as in the foregoing §, the equation

$$t^n - [m]\,t^{n-1} + [m^2]\,t^{n-2} - [m^3]\,t^{n-3} + [m^4]\,t^{n-4} - \ldots\ldots$$
$$\ldots\ldots \pm [m^{n-1}]\,t \mp [m^n] = 0$$

The numerical expressions here are all of the form $[a^\alpha]$, and ∴ may easily be found.

SECTION XLV

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0,$$

find the equation for the differences of its roots.

Solution 1. The number of the different values which the function $x' - x''$ contains by the substitution and transposition of the roots, is equal to the number of variations (in the sense in which Hindenburg uses this word)* of n different things of the second class, ∴ $= n\,.\,(n-1)$. The required equation is consequently of the $n\,.\,(n-1)$th degree. Further it is evident from § XXXVI, that this equation contains only even powers of t, and that ∴, when for the sake of brevity we put $n\,.\,(n-1) = 2m$, it has the following form:

$$t^{2m} - A't^{2m-2} + B't^{2m-4} - C't^{2m-6} + \&c. = 0;$$

* See vol. 1. p. 83, *note*.

and the roots of these equations are $(x'-x'')^2$, $(x'-x''')^2$, $(x'-x^{v})^2$, &c, $(x''-x''')^2$, $(x''-x^{v})^2$, &c. &c.

The coefficients A', B', C', &c. may be determined in the same way as in § XXXVI. yet the calculation by this method is attended with many difficulties, and besides, the law of the terms cannot be easily discovered. The following method, which will be frequently used in the sequel, is more simple and general.

2. For this purpose, put

$$S1=(x'-x'')^2+(x'-x''')^2+(x'-x^{v})^2+\ldots+(x''-x''')^2+\ldots$$
$$S2=(x'-x'')^4+(x'-x''')^4+(x'-x^{v})^4+\ldots+(x''-x''')^4+\ldots$$
$$S3=(x'-x'')^6+(x'-x''')^6+(x'-x^{v})^6+\ldots+(x''-x''')^6+\ldots$$

&c.

then the expressions $S1$, $S2$, $S3$, &c. are no other than the sum, the sum of the squares, the sum of the cubes &c., of the roots of the equation for t.

3. Since the expressions $S1$, $S2$, $S3$, &c. are the same for the transformed equation, as the expressions (1), (2), (3), &c. are for the given equation, consequently the formulæ found in §. IX. are equally applicable to the coefficients A', B', C', &c., when throughout $S1$, $S2$, $S3$, &c. are put for (1), (2), (3), &c. and also $-A'$, $+B'$, $-C'$, for A, B, C, &c. Thus we have

$$A'=S1$$

$$B'=\frac{A'S1-S2}{2}$$

$$C'=\frac{B'S1-A'S2+S3}{3}$$

$$D'=\frac{C'S1-B'S2+A'S3-S4}{4}$$

&c.

If we had calculated the expressions $S1$, $S2$, $S3$, &c., we should have found, by means of these equations, the coefficients A', B', C', D', &c.

4. If the expressions $S1$, $S2$, $S3$, &c. are solved, we obtain

$$S1 = (n-1)(x'^2 + x''^2 + x'''^2 + \&c.) - 2(x'x'' + x'x''' + x''x''' + \&c.)$$

$$= (n-1)(2) - 2(1^2)$$

$$S2 = (n-1)(x'^4 + x''^4 + x'''^4 + \&c.) - 4(x'x''^3 + {}'^3x'' + x'x'''^3 + x'^3x''' + x''x'''^3 + x''^3x''' + \&c.) + 6(x'^2x''^2 + x'^2x'''^2 + x''^2x'''^2)$$

$$= (n-1)(4) - 4(13) + 6(2^2)$$

$$S3 = (n-1)(x'^6 + x''^6 + x'''^6 + \&c.) - 6(x'x''^5 + x'^5x'' + x'x'''^5 + x'^5x''' + x''x'''^5 + x''^5x''' + \&c.) + 15(x'^2x''^4 + {}'^4x''^2 + x'^2x'''^4 + x'^4x'''^2 + x''^2x'''^4 + x''^4x'''^2 + \&c.) - 20(x'^3x''^3 + x'^3x''^3 + x''^3x''^3 + \&c.)$$

$$= (n-1)(6) - 6(15) + 15(24) - 20(3^2)$$

&c.

These values of $S1$, $S2$, $S3$, &c. need only be substituted in the equations in 3, in order to find the coefficients A', B', C', &c.

5. But these values may at any time be reduced, by means of the two equations

$$(\alpha\beta) = (\alpha)(\beta) - (\alpha + \beta)$$

$$2(\alpha^2) = (\alpha)^2 - (\overline{2\alpha}),$$

to sums of powers only, and then we have

$$S1 = (n-1)(2) - 2\left[\frac{(1)^2 - (2)}{2}\right]$$

$$S2 = (n-1)(4) - 4[(1)(3) - (4)] + 6\left[\frac{(2)^2 - (4)}{2}\right]$$

$$S3 = (n-1)(6) - 6[(1)(5) - (6)] + 15[(2)(4) - (6)]$$
$$- 20\left[\frac{(3)^2 - (6)}{2}\right]$$

&c.

or after the proper reduction,

6.

$$S1 = n(2) - 2\frac{(1)^2}{2}$$

$$S2 = n(4) - 4(3)(1) + 6\frac{(2)^2}{2}$$

$$S3 = n(6) - 6(5)(1) + 15(4)(2) - 20\frac{(3)^2}{2}$$

and in general

$$S\mu = n(2\mu) - 2\mu(2\mu-1(1) + \frac{2\mu.2\mu-1}{1\,.\,2}(2\mu-2)(2) - \ldots.$$
$$\ldots\ldots\ldots\ldots + \frac{2\mu\,.\,2\mu-1\,.\,2\mu-2\ldots\ldots\mu+1}{1\,.\,2\,.\,3\ldots\ldots\mu}\frac{(\mu)^2}{2}$$

These formulæ will be of great use in the sequel in the theory of imaginary roots of equations.

REMARK. From Elementary Books on Algebra it is known, that when in the equation $x^n - Ax^{n-1} + \&c.$, $x + \frac{A}{n}$ is substituted for x, its second term vanishes. But since by this substitution, all the roots of this equation are diminished in an algebraical sense by the magnitude $\frac{A}{n}$, $\therefore$ their differences remain the same, consequently also the equation for their differences undergoes no change.

We obtain, however, this advantage by losing the second term, that the values of the expressions (1), (2), (3), &c. $S1$, $S2$, $S3$, &c. are much more simple. Thus we have

$$(1) = 0$$
$$(2) = -2B$$
$$(3) = 3C$$
$$(4) = 2B^2 - 4D$$
$$(5) = -5BC + 5E$$
$$(6) = -2B^3 + 3C^2 + 6BD$$
&c.

Further, from 6, because $(1) = 0$, we get

$$S1 = n(2)$$
$$S2 = n(4) + 3(2)^2$$
$$S3 = n(6) + 15(4)(2) - 10(3)^2$$
&c.

These formulæ may be used with advantage in the case, when in the values of $S1$, $S2$, $S3$, &c., which have been found in 4, there are such numerical expressions as exceed the limits of the annexed tables. If this be not the case, it will be better to retain the former formulæ.

SECTION XLVI.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation for the sums of every two of its roots.

Solution 1. The required equation must have the following roots:

$x' + x''$, $x' + x'''$, $x' + x^{IV}$, ... $x'' + x'''$, $x'' + x^{IV}$......

and the number of these roots is equal to the number of

combinations of n things taken two and two, consequently $= \frac{n \, . \, n-1}{1 \, . \, 2}$, for which, for the sake of brevity, I shall put m. The required equation will consequently be of the mth degree; it may be represented by

$$t^m - A' t^{m-1} + B' t^{m-2} - C' t^{m-3} + \&c. = o.$$

The coefficients A', B', C', &c. may be most easily determined from the method used in the foregoing §.

2. For this purpose, then, put

$S1 =$
$(x' + x'') + (x' + x''') + (x' + x^{v}) + .. + (x'' + x''') + ...$

$S2 =$
$(x' + x'')^2 + (x' + x''')^2 + (x' + x^{v})^2 + .. + (x'' + x''')^2 + ...$

$S3 =$
$(x' + x'')^3 + (x' + x''')^3 + (x' + x^{v})^3 + .. + (x'' + x''')^3 + ...$

&c.

so that the expressions $S1$, $S2$, $S3$, &c. denote the sums of the first, second, third, &c. powers of the roots. If we have determined the values of these equations in any way, then the coefficients in 3 of the foregoing § give the coefficients A', B', C', &c.

3. If we solve these expressions, we get

$$S1 = (n-1)\,(1)$$
$$S2 = (n-1)\,(2) + 2(1^2)$$
$$S3 = (n-1)\,(3) + 3(12)$$
$$S4 = (n-1)\,(4) + 4(13) + 6(2^2)$$
$$S5 = (n-1)\,(5) + 5(14) + 10(23)$$

&c.

whence the law may very easily be discovered.

4. If we wish to represent the values of the expressions $S1$, $S2$, $S3$, &c. immediately in sums of powers, we only need proceed as in 5 of the foregoing §. We then obtain

$$S1 = (n-1)\,(1)$$
$$S2 = (n-1)\,(2)+2\left[\frac{(1)^2-(2)}{2}\right]$$
$$S3 = (n-1)\,(3)+3\,[(1)\,(2)-(3)]$$
$$S4 = (n-1)\,(4)+4\,[(1)\,(3)-(4)]+6\left[\frac{(2)^2-(4)}{2}\right]$$
$$S5 = (n-1)\,(5)+5[(1)(4)-(5)]+10[(2)(3)-(5)]$$
&c.

or after the proper reduction

$$S1 = (n-1)\;(1)$$
$$S2 = (n-2)\;(2) + 2\frac{(1)^2}{2}$$
$$S3 = (n-2^2)\,(3) + 3\,(2)\,(1)$$
$$S4 = (n-2^3)\,(4) + 4\,(3)\,(1) + 6\frac{(2^2)}{2}$$
$$S5 = (n-2^4)\,(5) + 5\,(4)\,(1) + 10\,(3)\,(2)$$
&c.

and in general

$$S\mu=(n-2^{\mu-1})\,(\mu)+\mu\,(\mu-1)\,(1)+\frac{\mu\,.\,\mu-1}{1\,.\,2}(\mu-2)\,(2)$$
$$+\frac{\mu\,.\,\mu-1\,.\,\mu-2}{1\;.\;2\;.\;3}(\mu-3)\,(3)+\ldots\ldots$$

and the last term of this series is either

$$\frac{\mu\,.\,\mu-1\,.\,\mu-2\,\ldots\ldots\ldots\,\frac{\mu}{2}+1}{1\;.\;2\;.\;3\,\ldots\ldots\ldots\ldots\ldots\,\frac{\mu}{2}}\;\frac{(\frac{\mu}{2})}{2}$$

or

$$\frac{\mu \cdot \mu-1 \cdot \mu-2 \ldots\ldots \frac{\mu+3}{2}}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots \frac{\mu-1}{2}} \left(\frac{\mu+1}{2}\right)\left(\frac{\mu-1}{2}\right)$$

according as m is an even or an odd number.

The abbreviations in the note to the preceding §, may, moreover, be also applied here.

SECTION XLVII.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation for the function $(ax' + bx'')^p$; when p is a whole positive number.

Solution 1. Since in the function $(ax' + bx'')^p$, we can put every other root of the given equation for the roots x', x'', consequently the number of the values, which the function can contain, $= n \cdot n - 1$, for which I shall substitute m. The required equation is consequently only of the mth degree, and may ∴ be represented by the equation

$$t^m - A't^{m-1} + B't^{m-2} - C't^{m-3} + \&c. = 0.$$

2. The method which has been made use of in the two preceding sections, for determining the coefficients A', B', C', &c. may likewise be applied here. Thus, if we denote by $S1$, $S2$, $S3$, &c. the sums of the first, second, third, &c. powers of the roots of the transformed equation, we have

$$S1 = (ax' + bx'')^p + (ax'' + bx')^p + (ax' + bx''')^p + \ldots$$
$$S2 = (ax' + bx'')^{2p} + (ax'' + bx')^{2p} + (ax' + bx''')^{2p} + \ldots$$
$$S3 = (ax' + bx'')^{3p} + (ax'' + bx')^{3p} + (ax' + bx''')^{3p} + \ldots$$
&c.

Having determined these equations, the equations in 3, § 45, give the values of the coefficients A', B', C'. It is now only necessary to determine the expression $S1$; having found this, we then obtain the remaining ones, $S2$, $S3$, &c. when we successively substitute $2p$, $3p$, &c. for p.

To find $S1$, we shall arrive most readily at the object in the following way.

3. Make the new expression ...... (ϕ) ......
$$\Sigma = (ax' + bz)^p + (ax'' + bz)^p + (ax''' + bz)^p + (ax^{lv} + bz)^p + \text{\&c.}$$
in which z denotes any unknown magnitude hitherto undetermined. If we solve this expression by means of the binomial theorem according to the powers of z, we obtain
$$\Sigma = a^p\,(x'^p + x''^p + x'''^p + x^{lv\,p} + \text{\&c.}) + pa^{p-1}b\,(x'^{p-1} + x''^{p-1} + x'''^{p-1} + x^{lv\,p-1}\ \text{\&c.})\,z + \text{\&c.}$$
or .... (ψ) ....
$$\Sigma = a^p(p) + pa^{p-1}b(p-1)\,.\,z + \frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2(p-2)\,.\,z^2 + \text{\&c.}$$
This equation must always be true, whatever we substitute for z.

4. Now, if we successively put x', x'', x''', x^{lv}, &c. for

z, and denote that which Σ becomes by these means, by Σ', Σ'', Σ''', Σ^{iv}, &c., we have in the first place from the equation (ϕ)

$$\Sigma' = (a+b)^p x'^p + (ax''+bx')^p + (ax'''+bx')^p \quad + \text{\&c.}$$
$$\Sigma'' = (ax'+bx'')^p + (a+b)^p x''^p + (ax'''+bx'')^p + \text{\&c.}$$
$$\Sigma''' = (ax'+bx''')^p + (ax''+bx''')^p + (a+b)^p x'''^p + \text{\&c.}$$
&c.

Hence it follows that

$$\Sigma' + \Sigma'' + \Sigma''' + \text{\&c.} = (a+b)^p [p] + S1$$

Further, from the equation (ψ) we obtain

$$\Sigma' = a^p[p] + pa^{p-1}b[p-1] \, . \, x' \quad + \frac{p \, . \, p-1}{1 \, . \, 2} a^{p-2}b^2[p-2] \, . \, x'^2 \quad + \text{\&c.}$$

$$\Sigma'' = a^p[p] + pa^{p-1}b[p-1] \, . \, x'' \quad + \frac{p \, . \, p-1}{1 \, . \, 2} a^{p-2}b^2[p-2] \, . \, x''^2 \quad + \text{\&c.}$$

$$\Sigma''' = a^p[p] + pa^{p-1}b[p-1] \, . \, x''' \quad + \frac{p \, . \, p-1}{1 \, . \, 2} a^{p-2}b^2[p-2] \, . \, x'''^2 + \text{\&c.}$$
&c.

and hence

$$\Sigma' + \Sigma'' + \Sigma''' + \text{\&c.} = na^p[p] + pa^{p-1}b[p-1]\;[1] + \frac{p \, . \, p-1}{1 \, . \, 2} a^{p-2}b^2[p-2]\;[2] \quad + \text{\&c.}$$

5. If we put the two values found for $\Sigma' + \Sigma'' + \Sigma'''$&c. equal to one another, we obtain

$$[a+b]^p\,[p] + S1 = na^p[p] + pa^{p-1}b[p-1][1] + \frac{p \, . \, p-1}{1 \, . \, 2} a^{p-2}b^2\;[p-2]\;[2] + \text{\&c.}$$

and hence

$$S1 = [na^p - (a+b)^p]\,[p] + pa^{p-1}b\,[p-1]\,[1] +$$

$$\frac{p\,.\,p-1}{1\,.\,2}a^{p-2}b^2\,[p-2]\,[2] + \frac{p\,.\,p-1\,.\,p-2}{1\,.\,2\,.\,3}a^{p-3}b^3[p-3]\,[3]$$

$$+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{p\,.\,p-1\,.\,p-2\ldots1}{1\,.\,2\,.\,3\ldots\ldots p}b^p[0]\,[p]$$

6. If in this expression for $S1$, we combine the first and the last terms, the second and the last but one, and generally every two terms, of which one is as distant from the first, as the other is from the last, and at the same time keep in mind that $[0] = x'^0 + x''^0 + x'''^0 + \&c. = n$, we then obtain

$$S\,1 = (n\,(a^p + b^p) - (a+b)^p)\,[p] +$$
$$p\,(a^{p-1}b + ab^{p-1})\,[p-1]\,[1] +$$
$$\frac{p\,.\,p-1}{1\,.\,2}(a^{p-2}b^2 + a^2b^{p-2})\,[p-2]\,[2] +$$
$$\frac{p\,.\,p-1\,.\,p-2}{1\,.\,2\,.\,3}(a^{p-3}b^3 + a^3b^{p-3})\,[p-3]\,[3] +$$
&c.

The last term of the expression, when p is an even number, is

$$\frac{p\,.\,p-1\ldots\ldots\frac{p}{2}+1}{1\,.\,2\,\ldots\ldots\ldots\ldots\frac{p}{2}}\,.\,a^{\frac{p}{2}}\,.\,b^{\frac{p}{2}}\,.\,\left[\tfrac{p}{2}\right]$$

But if p be an odd number, then the last term is

$$\frac{p\,.\,p-1\ldots\frac{p+3}{2}}{1\,.\,2\ldots\frac{p-1}{2}}\left(a^{\frac{p+1}{2}}\,.\,b^{\frac{p-1}{2}} + a^{\frac{p-1}{2}}\,.\,b^{\frac{p+1}{2}}\right)\cdot\left[\tfrac{p+1}{2}\right]\left[\tfrac{p-1}{2}\right]$$

7. If in the expression for $S\,1$, we substitute for p,

$2p$, $3p$, $4p$, &c. successively, we obtain the values of the expressions $S2$, $S3$, $S4$, &c. and the substitution of these values in the formulæ in 3, § 45, gives the values of the assumed coefficients $A,'$ B', C', &c.

SECTION XLVIII.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation for the function $(ax' + bx'')^{-p}$, when p is a whole positive number.

Solution. From the equation found in the preceding § for t, which has the roots $(ax' + bx'')^p$, $(ax'' + bx')^p$, $(ax' + bx''')^p$, &c. another may be derived, which has (§ 10) the reciprocal roots $(ax' + bx'')^{-p}$, $(ax'' + bx')^{-p}$, $(ax' + bx''')^{-p}$, &c. and this will be the equation which is here sought.

SECTION XLIX.

From the foregoing problems it is sufficiently seen, what must be done, in order to find the equation on which a given function of the roots of an equation depends. By these, then, we arrive merely at the two following points:

1. To find all the possible values of which the given function is capable.

2. From these values to form the required equation. I shall begin with the first.

In order to find all the possible values of a function,

we must transpose its roots in as many different ways as possible with the other roots of the given equation, and with each other; and of all the results or values thus obtained, we only retain those, which are actually different from one another.

If all the roots of a given equation are in a function, it is only necessary to permute these roots in all possible ways. Consequently, if the given equation be of the nth degree, then a function of this kind generally contains $1 . 2 . 3 \ldots n$ values, because n things can be permuted this number of times. But if the form of the function is such, that more permutations than one generate equal results, then the number of the values is often less; and if all the values be equal, then the function is symmetrical.

If in the function there is only a number μ of the n roots of the given equation, then these n roots enter into the function in as many different ways as there are combinations in n things taken μ and μ; and the number of these combinations

$$= \frac{n . n-1 . n-2 \ldots\ldots n-\mu+1}{1 \ . \ 2 \ . \ 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mu}$$

Every such combination, however, allows of $1 . 2 . 3 \ldots \mu$ permutations of the roots it contains; consequently the number of the values which such a function generally contains is

$$= n . n-1 . n-2 \ldots\ldots n-\mu+1$$

$\therefore$ it is equal to the number of the variations of n things taken μ and μ. But if amongst these there are equal values, then this number will often be much less; although in

the assumed case they never can be less than n, because the number of variations never can be less than the number of elements. Consequently the transformed equation in this case can never be of a lower degree than the given one itself.

In the general inquiry respecting functions, it is always allowable to assume, that all the roots of the given equation are contained in them, because in the contrary case, only each of the roots which are wanting, considered with a coefficient $=0$, can be added to the function.

SECTION L.

EXPLANATION 1. Functions are said to be homogeneous, when they contain the same roots, and when in all the transpositions of these roots, they either at the same time change, or remain the same.

Let, for instance, the functions

$$x' + x'' - x''' - x^{\mathrm{iv}},\ x'x'' - x'''x^{\mathrm{iv}}$$

and likewise the functions

$$\frac{x''}{x'''} + \frac{x'''}{x'} + \frac{x'}{x''},\ x'^{p}x''^{q}x'''^{r} + x''^{p}x'''^{q}x'^{r} + x'''^{p}x'^{q}x''^{r}$$

be homogeneous. Then the first two have no more than the following 6 different and corresponding values:

$$\begin{array}{ll}
x' + x'' - x''' - x^{\mathrm{iv}}, & x'x'' - x'''x^{\mathrm{iv}} \\
x' + x''' - x'' - x^{\mathrm{iv}}, & x'x''' - x''x^{\mathrm{iv}} \\
x' + x^{\mathrm{iv}} - x'' - x''', & x'x^{\mathrm{iv}} - x''x''' \\
x''' + x^{\mathrm{iv}} - x' - x'', & x'''x^{\mathrm{iv}} - x'x''' \\
x'' + x^{\mathrm{iv}} - x' - x''', & x''x^{\mathrm{iv}} - x'x''' \\
x'' + x''' - x' - x^{\mathrm{iv}}, & x''x''' - x'x^{\mathrm{iv}}
\end{array}$$

and the two last no more than 2, viz.

$$\frac{x''}{x'''}+\frac{x'''}{x'}+\frac{x'}{x''},\ x'^{p}x''^{q}x'''^{r}+x''^{p}x'''^{q}x'^{r}+x'''^{p}x'^{q}x'''^{r}$$

$$\frac{x'''}{x''}+\frac{x''}{x'}+\frac{x'}{x'''},\ x'^{p}x'''^{q}x''^{r}+x'''^{p}x''^{q}x'^{r}+x''^{p}x'^{q}x'''^{r}$$

2. The letter f prefixed to the roots of an equation, or even to other magnitudes, in the sequel, always denotes a rational function of these roots or magnitudes. Thus f: (x') denotes a rational function of x', f: (x') (x'') a rational function of x' and x'', and in general f: (x') (x'') (x''') $(x^{\prime v})$ ...... $(x^{(\mu)})$ a rational function of $x', x'', x''', x^{\prime v}, \ldots\ldots x^{(\mu)}$, and so in like manner of other magnitudes. In order to distinguish the functions, sometimes also the letters F, ϕ, ψ are made use of instead of f.

In this notation of the functions, it is preferable with each set of brackets ( ), which follows the letter f, to attach a definite representation of the manner in which the magnitudes contained in it are combined with the others; so that when any permutation of these magnitudes under the symbol f is intended, one of the permutations corresponding to it in the expression represented by it, must be denoted.

Thus, if f: $(x')\,(x'')\,(x''') = (x'x''-x''')\,(x''-x')$, we then have

$$\begin{aligned}
f:\ (x')\,(x''')\,(x'') &= (x'x'''-x'')\,(x'''-x')\\
f:\ (x'')\,(x')\,(x''') &= (x''x'-x''')\,(x'-x'')\\
f:\ (x'')\,(x''')\,(x') &= (x''x'''-x')\,(x'''-x'')\\
f:\ (x''')\,(x')\,(x'') &= (x'''x'-x'')\,(x'-x''')\\
f:\ (x''')\,(x'')\,(x') &= (x'''x''-x')\,(x''-x''')
\end{aligned}$$

3. In order to distinguish the values which a given function contains by the permutation of its magnitudes, from the symbols by which this permutation is denoted, I shall call the latter *types*. Thus, the function $f:(x')(x'')(x''')$ has six types, viz.: $f:(x')(x'')(x''')$, $f:(x')(x''')(x'')$, $f:(x'')(x')(x''')$, $f:(x'')(x''')(x')$, $f:(x''')(x')(x'')$, $f:(x''')(x'')(x')$; and generally there are always as many types of a function as permutations of the magnitudes under the functional symbol.

The types are, as it were, the representatives of the values which a function contains, and in general investigations of functions may be used with very great advantage. If, for instance, we wished to show that any particular function had such a form, that the values arising from these or those transpositions were equal to one another, instead of actually expressing these permutations, which would often be attended with a great deal of trouble, it is only necessary to give the types, which correspond to the equal values.

SECTION LI.

When a function has such a form, that any two of its values are equal to one another, then the function must always necessarily have more than two equal values. Thus, if the function be such, that

$$f:(x')(x'')(x''')=f:(x')(x''')(x'')$$

then also must

$$f:(x'')(x')(x''')=f:(x'')(x''')(x')$$
$$f:(x''')(x')(x'')=f:(x''')(x'')(x')$$

For the first equation shows, that the value of the type $f:(x')\,(x'')\,(x''')$ remains unchanged, when the roots of the two last sets of brackets are transformed, ∴ the types $f:(x'')\,(x')\,(x''')$, $f:(x''')\,(x')\,(x'')$ in the same transformation of the brackets, must remain unchanged, because the equality, in the sense in which it is here taken, by which is meant no more than identity, is not derived from the numerical nature of the roots, but merely from the nature of their combination, consequently from the form of the functions.

SECTION LII.

Auxiliary Rule. If we combine a series of elements $a, b, c, d, e, \ldots\ldots p$, with as many numbers $1, 2, 3, 4, 5\ldots\ldots\pi$, arranged in any order according to a cipher, for instance, as follows:

$$\begin{matrix} 3 & 2 & 1 & 6 & 4 & 5 & 7 & 8 & 9\ldots\ldots\pi \\ a & b & c & d & e & f & g & h & i\ldots\ldots p \end{matrix}$$

after this, permute the elements in the manner denoted by the ciphers placed over them, and from the permutation thus obtained, derive another, from this again another, and generally from every permutation last found, derive a new one, always observing in the transposition the law denoted by the ciphers: now I affirm, that by continual permutation, we must necessarily return again to the first permutation.

Thus we obtain from the permutation $A_1 = \overset{583214796}{abcdefghi}$ for the figures placed over it no more than 9 permutations $A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}$

	5	8	3	2	1	4	7	9	6
$A_1 =$	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
$A_2 =$	*e*	*h*	*c*	*b*	*a*	*d*	*g*	*i*	*f*
$A_3 =$	*a*	*i*	*c*	*h*	*e*	*b*	*g*	*f*	*d*
$A_4 =$	*e*	*f*	*c*	*i*	*a*	*h*	*g*	*d*	*b*
$A_5 =$	*a*	*d*	*c*	*f*	*e*	*i*	*g*	*b*	*h*
$A_6 =$	*e*	*b*	*c*	*d*	*a*	*f*	*g*	*h*	*i*
$A_7 =$	*a*	*h*	*c*	*b*	*e*	*d*	*g*	*i*	*f*
$A_8 =$	*e*	*i*	*c*	*h*	*a*	*b*	*g*	*f*	*d*
$A_9 =$	*a*	*f*	*c*	*i*	*e*	*h*	*g*	*d*	*b*
$A_{10} =$	*e*	*d*	*c*	*f*	*a*	*i*	*g*	*b*	*h*

If we proceed with the last permutation in the same manner as we did with the preceding ones, we again obtain the first.

Proof. Let

$A_1, A_2, A_3, A_4, \ldots\ldots\ldots A_\mu, \ldots\ldots\ldots A_\nu, \ldots\ldots\ldots$

denote the permutations which may successively be derived from the expression $A_1, = a\,b\,c\,d\,e\,f \ldots\ldots p$ according to the law of any one cipher.

Since the number of transpositions, which an expression can generally contain is always limited, we must ∴ necessarily once come to a permutation A_ν, which is equal to one of the preceding A_μ. But if $A_\nu = A_\mu$, consequently also $A_{\nu-1} = A_{\mu-1}$; for if the permutations $A_{\nu-1}$, $A_{\mu-1}$, were not equal to one another, then also the permutations A_ν, A_μ, could not be equal, since A arises from $A_{\nu-1}$ by the same transformation of the elements as A_μ arises from $A_{\mu-1}$. In the same way we may further conclude from $A_{\nu-1} = A_{\mu-1}$, that also $A_{\nu-2} = A_{\mu-2}$, and hence again, that $A_{\nu-3} = A_{\mu-3}$, and so on. Consequently

$A_{\nu-(\mu-1)} = A_{\mu-(\mu-1)} = A_1$. We have $\therefore$ a permutation $A_{\nu-(\mu-1)}$ which is equal to the first. Q. E. D.

The contents of all the permutations obtained according to a given rule of transposition, are, for the sake of shortening the expression, called a period, because we always obtain the same permutations again, however long we continue the transposition.

SECTION LIII.

From the foregoing § we deduce the following propositions.

I.—That all the permutations of a period are different from one another.

For if in the period $A_1, A_2, A_3, A_4 \ldots\ldots A_\mu \ldots\ldots A_\nu \ldots\ldots A_n$ there are two equal permutations A_μ, A_ν, then also must $A_{\nu-1} = A_{\mu-1}$, $A_{\nu-2} = A_{\mu-2}$, and so on; consequently also $A_{\nu-(\mu-1)} = A_{\mu-(\mu-1)} = A_1$; but in this case A_n could not be the last permutation of the period.

II.—Let B denote any permutation different from A_1, now it may belong to the period $A_1, A_2, A_3, \ldots\ldots A_n$ or not; further, let B_2, B_3, B_4, &c. be the permutations derived from B_1, according to the same rule of transposition by which A_2, A_3, A_4, &c. was derived from A_1; I affirm, that in this case the two periods arising from A_1 and B_1 consist of the same number of permutations.

For since the rule of transposition, which is denoted by the figures, does not refer to the elements themselves, but only to their places, so it is quite the same, in reference to the number of the permutations of which a

period consists, which elements are in the different places of the first permutation.

III.—If B_1 is equal to any one of the permutations $A_1, A_2, A_3 \ldots\ldots\ldots A_\omega$ of the first period, then the two periods consist of the same permutations.

This proposition is an immediate consequence of the foregoing §.

IV.—If B_1 be not equal to any of the permutations $A_1, A_2, A_3 \ldots\ldots A_\omega$ of the first period, then the permutations of the two periods are all different from one another.

For if in the period $B_1, B_2, B_3 \ldots\ldots B_\kappa \ldots\ldots B_\omega$ there be any permutation B_κ, which is equal to a permutation A_μ of the period $A_1, A_2, A_3 \ldots\ldots A_\mu \ldots\ldots A_\omega$, then likewise must $B_{\kappa+1} = A_{\mu+1}$, because $B_{\kappa+1}$ is derived from B_κ by the same rule as $A_{\mu+1}$ is from A_μ; and when we further conclude in this way that $B_{\kappa+2} = A_{\mu+2}$, $B_{\kappa+3} = A_{\mu+3}$, and so on, lastly $B_{\omega+1} = A_{\mu+\omega-\kappa+1}$. But since $B_{\omega+1} = B_1$, $A_{\mu+\omega-\kappa+1} = A_{\mu-\kappa+1}$, then must $B_1 = A_{\mu-\kappa+1}$, which is contrary to the supposition that B_1 is different from all the permutations contained in the first period.

SECTION LIV.

Prob. Let a function be such, that any two given types are equal to one another: find all the equal values of the function which arise from this supposition (§ 51).

Solution. For the sake of perspicuity, I shall confine

myself to a single case, because it will be sufficiently clear from it how we are to proceed in every other one.

1. Let

$$f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$$

denote any function for which the two types

$$A_1 \ldots\ldots\ldots f:(x')\,(x'')\,(x''')\,(x^{\prime v}).(x^{v})$$
$$A_2 \ldots\ldots\ldots f:(x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'')$$

are equal to one another. Compare these types, and observe how the roots are transposed in the first, second, third, fourth, and fifth brackets, when A_2 is deduced from A_1. If we retain this transposition in our memory, and then derive from A_2 a new type by the same rule by which A_2 was generated from A_1, from this last derive another, and continue this proceeding till we return again to the first, we shall obtain the following period, consisting of five types:

$$A_1 \ldots\ldots\ldots f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$$
$$A_2 \ldots\ldots\ldots f:(x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'')$$
$$A_3 \ldots\ldots\ldots f:(x^{v})\,(x')\,(x'')\,(x''')\,(x^{\prime v})$$
$$A_4 \ldots\ldots\ldots f:(x'')\,(x''')\,(x^{\prime v})\,(x^{v})\,(x')$$
$$A_5 \ldots\ldots\ldots f:(x^{\prime v})\,(x^{v})\,(x')\,(x'')\,(x''')$$

2. These types must necessarily be all equal, because they have all been derived from one another by the same rule. Now, since the function $f:(x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v})$ has exactly as many types as there are transpositions in five magnitudes, consequently 120 types, it only remains that we take from the remaining 115 those which, under the hypothesis that $A_1 = A_2$, are equal to one another.

3. But this can be done very easily. For we only require to take away from the remaining types any one whatever, and from this derive a second period by the same rule, then again from the 110 types which still remain take away another, and by the same rule form a third period, and continue this till there are none remaining.

4. In this way we obtain 24 periods, each of which consists of five equal types. But if these be already found, then also the equal values, as soon as they are known, of the function itself corresponding to them, can be found.

Example. Suppose we have observed, that the function $x'x''^{2}x'''^{3} + x'''x^{\prime v 2}x^{v3} + x^{v}x'^{2}x''^{3} + x''x'''^{2}x^{\prime v 3} + x^{\prime v}x^{v2}x'^{2}$ remains the same, when in the terms in which the roots x', x'', x''', $x^{\prime v}$, x^{v} are found, we put the roots x''', $x^{\prime v}$, x^{v}, x', x'' respectively, or more briefly $f:\ (x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v}) = f:\ (x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'')$. Since this is exactly the equation which was used in the solution for the illustration of the operation, it is ∴ certain, that the function has 24 times five equal values. Thus the period in 1 contains the five following values:

$$f:\ (x')\,(x'')\,(x''')\,(x^{\prime v})\,(x^{v}) =$$
$$x'x''^{2}x'''^{3} + x'''x^{\prime v 2}x^{v3} + x^{v}x'^{2}x''^{3} + x''x'''^{2}x^{\prime v 3} + x^{\prime v}x^{v2}x'^{3}$$
$$f:\ (x''')\,(x^{\prime v})\,(x^{v})\,(x')\,(x'') =$$
$$x'''x^{\prime v 2}x^{v3} + x^{v}x'^{2}x''^{3} + x''x''^{2}x^{\prime v 3} + x^{\prime v}x^{v2}x'^{3} + x'x''^{2}x'''^{3}$$
$$f:\ (x^{v})\,(x')\,(x'')\,(x''')\,(x^{\prime v}) =$$
$$x^{v}x'^{2}x''^{3} + x''x''^{2}x^{\prime v 3} + x^{\prime v}x^{v2}x'^{3} + x'x''^{2}x'''^{3} + x'''x^{\prime v 2}x^{v3}$$
$$f:\ (x'')\,(x''')\,(x^{\prime v})\,(x^{v})\,(x') =$$

$$x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime v 3}+x^{\prime v}x^{v 2}x^{\prime 3}+x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime 3}+x^{\prime\prime\prime}x^{\prime v 2}x^{v 3}+x^{v}x^{\prime 2}x^{\prime\prime 3}$$

$$f:\ (x^{\prime v})\,(x^{v})\,(x^{\prime})\,(x^{\prime\prime})\,(x^{\prime\prime\prime}) =$$

$$x^{\prime v}x^{v 2}x^{\prime 3}+x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime 3}+x^{\prime\prime\prime}x^{\prime v 2}x^{v 3}+x^{v}x^{\prime 2}x^{\prime\prime 3}+x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime v 3}$$

which are evidently all equal.

SECTION LV.

Prob. Let there be a function such, that more than two of its types are equal: find the equal values of this function.

Solution. Let A, B, C, D, &c. denote the types, which, according to the hypothesis, are equal. In order from hence to find the equal values, proceed as follows.

1. First permute the type A, or even any other, according to the rule of transposition, that $A=B$, as was shown in the foregoing §. If the period, which we obtain from this, consist of μ types, then we have at once μ equal types.

2. Then permute each of these μ types in particular by the rule $A=C$. I shall assume that there are μ' types; then we have generally $\mu\mu'$ types, which are all equal.

3. Permute again each of the $\mu\mu'$ types obtained by the rule $A=D$, which may give μ'' types; then we have in all $\mu\mu'\mu''$ equal types.

4. In the same way we proceed, when we successively make use of the rules $A=E$, $A=F$, &c. $B=C$, $B=D$, &c. $C=D$, $C=E$, &c. or, in short, when we put the given

types A, B, C, &c. taken two and two in all possible ways, equal to one another.

5. Let the number of types, obtained according to the directions in 1, 2, 3, 4, $= \nu$. Take now from all the types, which arise from all the possible transpositions of x', x'', x''', &c. any other, which is not amongst those already found, and proceed with this according to the same directions, then we obtain ν equal types. If we continue this proceeding, till all the types are exhausted, we at length obtain a number of divisions, each consisting of ν equal types. But that the types in every such division are different from the types in all the other divisions, is an immediate consequence of § 53. IV.

Corollary. It follows from this solution, that the number of the different values which a function can contain, is always a sub-multiple of the number of all the values, which arise from all the transpositions of x', x'', x''', &c.

Example I. Let there be a function such, that

$$f: (x')\,(x'')\,(x''')\,(x^{\text{v}}) = f: (x'')\,(x''')\,(x')\,(x^{\text{v}})$$
$$= f: (x'')\,(x''')\,(x^{\text{v}})\,(x'):$$

required to find its equal values.

From the equation $A = B$, or $f: (x')\,(x'')\,(x''')\,(x^{\text{v}}) = f: (x'')\,(x''')\,(x')\,(x^{\text{v}})$, we obtain, in the first place, the period

$$f: (x')\,(x'')\,(x''')\,(x^{\text{v}}) = f: (x'')\,(x''')\,(x')\,(x^{\text{v}})$$
$$= f: (x''')\,(x')\,(x'')\,(x^{\text{v}})$$

From each of these three types we obtain, by the application of the equation $A=C$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x''')\ (x^{\prime\text{v}})\ (x')$, a period of four equal values, ∴ in all twelve equal types, viz.

$$f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x''')\ (x^{\prime\text{v}})\ (x') =$$
$$f:\ (x''')\ (x^{\prime\text{v}})\ (x')\ (x'') = f:\ (x^{\prime\text{v}})\ (x')\ (x'')\ (x''') =$$
$$f:\ (x'')\ (x')\ (x^{\prime\text{v}})\ (x'') = f:\ (x')\ (x^{\prime\text{v}})\ (x'')\ (x''') =$$
$$f:\ (x^{\prime\text{v}})\ (x'')\ (x''')\ (x') = f:\ (x'')\ (x''')\ (x')\ (x^{\prime\text{v}}) =$$
$$f:\ (x''')\ (x')\ (x'')\ (x^{\prime\text{v}}) = f:\ (x')\ (x'')\ (x^{\prime\text{v}})\ (x''') =$$
$$f:\ (x'')\ (x^{\prime\text{v}})\ (x''')\ (x') = f:\ (x^{\prime\text{v}})\ (x''')\ (x')\ (x'')$$

At length we obtain from these twelve types, by means of the equation $B=C$, or $f:\ (x'')\ (x''')\ (x')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x''')\ (x^{\prime\text{v}})\ (x')$, consequently by the permutation of the roots in the two last brackets, twelve other types, which, together with the former, give all the twenty-four types of the function $f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}})$. Hence it follows, that a function of the supposed nature must necessarily be symmetrical.

Example. II. Let a function be such, that

$$f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x')\ (x''')\ (x^{\prime\text{v}}) =$$
$$f:\ (x')\ (x'')\ (x^{\prime\text{v}})\ (x''') = f:\ (x''')\ (x^{\prime\text{v}})\ (x')\ (x''):$$

required to find its equal values.

The equation $A=B$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x')\ (x''')\ (x^{\prime\text{v}})$ gives no more than these two equal types. If to these we apply the equation $A=C$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x')\ (x'')\ (x^{\prime\text{v}})\ (x''')$, we obtain the four following equal types:

$$f:\ (x')\ (x'')\ (x''')\ (^{\prime\text{v}}) = f:\ (x')\ (x'')\ (x^{\prime\text{v}})\ (x''') =$$
$$f:\ (x'')\ (x')\ (x''')\ (x^{\prime\text{v}}) = f:\ (x'')\ (x')\ (x^{\prime\text{v}})\ (x''') =$$

From these we again obtain, by means of the equation $A=D$, or $f:\ (x')\ (x'')\ (x''')\ (x^{\mathrm{v}}) = f:\ (x''')\ (x^{\mathrm{v}})\ (x')\ (x'')$ the following eight equal types:

$$f:\ (x')\ (x'')\ (x''')\ (x^{\mathrm{v}}) = f:\ (x''')\ (x^{\mathrm{v}})\ (x')\ (x'') =$$
$$f:\ (x')\ (x'')\ (x^{\mathrm{v}})\ (x''') = f:\ (x^{\mathrm{v}})\ (x''')\ (x')\ (x'') =$$
$$f:\ (x'')\ (x')\ (x''')\ (x^{\mathrm{v}}) = f:\ (x''')\ (x^{\mathrm{v}})\ (x'')\ (x') =$$
$$f:\ (x'')\ (x')\ (x^{\mathrm{v}})\ (x''') = f:\ (x^{\mathrm{v}})\ (x''')\ (x'')\ (x') =$$

The equations $B=C$, $B=D$, $C=D$ always again give the same types. The type $f:\ (x')\ (x'')\ (x''')\ (x^{\mathrm{v}})$ consequently contains no more than seven equal types.

If now we take any other type of the remaining sixteen, viz. $f:\ (x''')\ (x')\ (x^{\mathrm{v}})\ (x'')$ and proceed with it as we previously did with $f:\ (x')\ (x'')\ (x''')\ (x^{\mathrm{v}})$, we again obtain eight equal types, viz.

$$f:\ (x''')\ (x')\ (x^{\mathrm{v}})\ (x'') = f:\ (x^{\mathrm{v}})\ (x'')\ (x''')\ (x') =$$
$$f:\ (x''')\ (x')\ (x'')\ (x^{\mathrm{v}}) = f:\ (x'')\ (x^{\mathrm{v}})\ (x''')\ (x') =$$
$$f:\ (x')\ (x''')\ (x^{\mathrm{v}})\ (x'') = f:\ (x^{\mathrm{v}})\ (x'')\ (x')\ (x''') =$$
$$f:\ (x')\ (x''')\ (x'')\ (x^{\mathrm{v}}) = f:\ (x'')\ (x^{\mathrm{v}})\ (x')\ (x''') =$$

If from among the eight remaining types, we select any one, for instance, $f:\ (x'')\ (x''')\ (x^{\mathrm{v}})\ (x')$, and proceed with it in the same way, we obtain them all.

Hence ∴ it follows, that the function of the supposed kind can have no more than three different values, viz.: $(x')\ (x'')\ (x''')\ (x^{\mathrm{v}})$, $f:\ (x''')\ (x')\ (x^{\mathrm{v}})\ (x'')$, $f:\ (x'')\ (x''')\ (x^{\mathrm{v}})\ (x')$, and that consequently such a function will not lead to any equation higher than the third degree. Of this nature are the functions $(x'+x''-x'''-x^{\mathrm{v}})^2$, $x'x''+x'''x^{\mathrm{v}}$, and innumerable others; the method to find which will be given hereafter. But if we wish to use a function of this kind to solve equations of the fourth

degree, it is not sufficient that the transformed equation should be of a lower degree than the given one; but we must likewise be enabled, from the known values of such a function, to determine the roots x', x'', x''', $x^{\prime v}$ by equations of a lower degree than the fourth, because otherwise the transformation of the equation will be of no use. In the functions $x'x''+x'''x^{\prime v}$, $(x'+x''-x'''-x^{\prime v})^2$, this is actually the case, as we have already seen in § 41 and 42. In the sequel the conditions will be given, under which it is generally possible, from the known value of a function $f: (x')\,(x'')\,(x''')\ldots\ldots(x^{(\mu)})$ to find the values of the roots x', x'', x''', ... $x^{(\mu)}$ by equations of a lower degree than the mth.

SECTION LVI.

Prob. Determine the degree of the equation on which a given function depends.

Solution 1. If in a function $f: (x')\,(x'')\,(x''')\ldots\ldots(x^{(\mu)})$ there are all the roots of the given equation, and if this function be such, that in each transposition of its roots it changes its value, then the transformed equation is necessarily of the degree $1 . 2 . 3 \ldots\ldots \mu$.

2. If the function be such that a number of types A, B, C, D, &c. are equal, and if the number of the equal types which can in general be derived from them, by the directions given in 1, 2, 3, 4, of the foregoing §, $=\nu$, then the number of the different values, of which such a function is capable, or the degree of the transformed equation,

$$=\frac{1 . 2 . 3 \ldots\ldots \ldots \mu}{\nu}$$

3. If the function remain the same, when m roots change their places in all possible ways, then the degree of the transformed equation

$$=\frac{1 . 2 . 3 \ldots\ldots \mu}{1 . 2 \ldots\ldots m}=\mu . \mu-1 \ldots\ldots m+1$$

4. If the function still remain the same, when m' other roots, and again m'' other roots, and so on, change their places, then the degree of the transformed equation

$$=\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mu}{1 . 2 \ldots m \times 1 . 2 \ldots m' \times 1 . 2 \ldots m'' \times \&c.}$$

5. If the function be such, that each time its value remains unchanged, when m roots, and again m' other roots, &c. change their places in all possible ways, and if, besides, a number of types A, B, C, D, &c. are equal, then the degree of the transformed equation

$$=\frac{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mu}{1 . 2 \ldots m \times 1 . 2 \ldots m' \times 1 . 2 \ldots m'' \times \&c. \times \nu}$$

when ν retains the signification it does in 2.

6. If all the roots of the given equation be not in the function $f: (x') (x'') (x''') \ldots (x'^{\mu})$, and if the equation be of the nth degree, then all the formulæ given in 1, 2, 3, 4, 5, must be multiplied by the factor.

$$\frac{n . n-1 . n-2 \ldots\ldots\ldots n-\mu+1}{1 . \ 2 . \ \ 3 \ldots\ldots\ldots\ldots\ldots\ldots \mu}$$

The reason of all this is sufficiently evident from what goes before.

SECTION LVII.

PROB. From the given equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

find the equation on which the function $f:\ (x')\ (x'')\ (x''')\ldots(x^{(\mu)})$ depends.

Solution 1. Seek first all the different values which this function contains, both by substituting for its roots the other roots of the given equation, and by their permutation. Let these different values be denoted by y', y'', y''', $y^{\text{IV}} \ldots\ldots y^{(\pi)}$.

2. Then form the equation

$$(t - y')\,(t - y'')\,(t - y''') \ldots (t - y^{(\pi)}) = 0$$

and actually multiply the factors in the first part. Then

$$t^\pi - A't^{\pi-1} + B't^{\pi-2} - C't^{\pi-3} + \&c. = 0$$

is the equation obtained from this operation, $\therefore$

$$A' = y' + y'' + y''' + y^{\text{IV}} + \&c.$$
$$B' = y'y'' + y'y''' + y'y^{\text{IV}} + y''y''' + \&c.$$
$$C' = y'y''y''' + y'y''y^{\text{IV}} + y'y'''y^{\text{IV}} + \&c.$$

&c.

3. The functions A', B', C', &c. are symmetrical in relation to y', y'', y''', ... $y^{(\pi)}$, and consequently no transposition of these magnitudes can effect any change in the values of the former functions. But the magnitudes y', y'', y''', ... $y^{(\pi)}$ are themselves again functions of the roots x', x'', x''', &c. and such too as merely transform

into one another, when the roots are transformed and permuted in every possible way. Consequently no further change takes place by transforming and permuting the roots in the above expressions for A', B', C', &c. than that y', y'', y''', &c. change their places. Now, since this effects no change in the values of A', B', C', &c. consequently these values remain unchanged by the transformation and permutation of x', x'', x''', &c. Therefore the coefficients A', B', C', &c. are necessarily symmetrical functions of the roots x', x'', x''', &c.

4. In the two first sections, however, it was shown, that every symmetrical function of the roots of an equation, may always be expressed rationally by the coefficients of this equation. Therefore also the coefficients A', B', C', &c. may always be expressed rationally by A, B, C, &c.

5. Consequently an equation may always be found, on which depends a given function of the roots of another equation, and the coefficients of the former will always be rational functions of the coefficients of the latter.

IV.—On Elimination, together with its Application to the Reduction of Equations.

Section LVIII.

Prob. Let there be n equations of the first degree given, which contain as many unknown magnitudes: required to reduce their solution to the solution of $n-1$ equations of the first degree only, which contain but $n-1$ of these unknown magnitudes.

Solution 1. Let

$$\begin{array}{l} ax + by + cz + \ldots\ldots + kv + lw = A \\ a_1x + b_1y + c_1z + \ldots\ldots + k_1v + l_1w = A_1 \\ a_2x + b_2y + c_2z + \ldots\ldots + k_2v + l_2w = A_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n-1}x + b_{n-1}y + c_{n-1}z + \ldots\ldots + k_{n-1}v + l_{n-1}w = A_{n-1} \end{array}$$

be the n given equations; $x, y, z \ldots\ldots v, w$, the n unknown magnitudes; $a, b, c, \ldots\ldots k, l$; $a_1, b_1, c_1, \ldots k_1, l_1$; $a_2, b_2, c_2, \ldots k_2, l_2$, &c.; likewise A_1, A_2, A_3, &c. the given magnitudes.

2. Assume $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots \Pi_{n-1}$ as $n-1$ magnitudes hitherto unknown, and multiply the second equation by Π_1, the third by Π_2, the fourth by Π_3, and so on; lastly, the last by Π_{n-1}; then add all these products to the first equation; hence arises the equation

$$
\begin{aligned}
A + A_1\Pi_1 + A_2\Pi_2 + \ldots\ldots + A_{n-1}\Pi_{n-1} = {} & \\
(a + a_1\Pi_1 + a_2\Pi_2 + \ldots\ldots + a_{n-1}\Pi_{n-1})\, x & \\
+ (b + b_1\Pi_1 + b_2\Pi_2 + \ldots\ldots + b_{n-1}\Pi_{n-1})\, y & \\
+ (c + c_1\Pi_1 + c_2\Pi_2 + \ldots\ldots + c_{n-1}\Pi_{n-1})\, z & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
+ (l + l_1\Pi_1 + l_2\Pi_2 + \ldots\ldots + l_{n-1}\Pi_{n-1})\, w &
\end{aligned}
$$

3. Now we try to determine the factors $\Pi_1, \Pi_2, \Pi_3, \ldots \Pi_{n-1}$, in such a way, that the coefficients of all the unknown magnitudes $x, y, z, \ldots\ldots v, w$, those excepted which we wish to find vanish. If, for instance, we wished to find x, we put

$$
\begin{aligned}
b + b_1\Pi_1 + b_2\Pi_2 + \ldots\ldots + b_{n-1}\Pi_{n-1} &= 0 \\
c + c_1\Pi_1 + c_2\Pi_2 + \ldots\ldots + c_{n-1}\Pi_{n-1} &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
l + l_1\Pi_1 + l_2\Pi_2 + \ldots\ldots + l_{n-1}\Pi_{n-1} &= 0
\end{aligned}
$$

4. Hence the equation in 2 is reduced to the following one:

$$
\begin{aligned}
A + A_1\Pi_1 + A_2\Pi_2 + \ldots\ldots + A_{n-1}\Pi_{n-1} = {} & \\
(a + a_1\Pi_1 + a_2\Pi_2 + \ldots\ldots + a_{n-1}\Pi_{n-1})\, x &
\end{aligned}
$$

and this gives

$$
x = \frac{A + A_1\Pi_1 + A_2\Pi_2 + \ldots\ldots + A_{n-1}\Pi_{n-1}}{a + a_1\Pi_1 + a_2\Pi_2 + \ldots\ldots + a_{n-1}\Pi_{n-1}}
$$

and the determination of the assumed magnitudes $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots \Pi_{n-1}$, depends upon the solution of the equations in 3.

5. In order ∴ to find the value of x, we must solve the $n-1$ equations in 3, from it determine the $n-1$ mag-

nitudes $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots \Pi_{n-1}$, and then substitute their values in the expression found for x.

6. If both in 3 and 4 we substitute a for b, we shall find y, and in the same manner we find z, when in 3 and 4 we substitute a for c, and so on.

Remark. The reduction of equations here given, may sometimes be used with advantage, as will be seen by an example given further on in this work. But if we intend merely to solve the given equations, we shall by these means attain our object but very slowly; in this case the method in the following § is preferable.

SECTION LIX.

Prob. Let the following n equations of the first degree be given:

$$
\begin{array}{l}
a_1x + b_1y + c_1z + \ldots\ldots + k_1v + l_1w = m_1 \\
a_2x + b_2y + c_2z + \ldots\ldots + k_2v + l_2w = m_2 \\
a_3x + b_3y + c_3z + \ldots\ldots + k_3v + l_3w = m_3 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_nx + b_ny + c_nz + \ldots\ldots + k_nv + l_nw = m_n
\end{array}
$$

in which there are n unknown magnitudes $x, y, z, \ldots\ldots v, w$: find the values of these magnitudes directly, and without any substitution or any other calculation.

Solution 1. If we merely had the two equations with two unknown magnitudes

$$
\begin{array}{l}
a_1x + b_1y = m_1 \\
a_2'x + b_2y = m_2
\end{array}
$$

we should have found them in the usual way

$$x = \frac{m_1b_2 - m_2b_1}{a_1b_2 - a_2b_1}\ y = \frac{a_1m_2 - a_2m_1}{a_1b_2 - a_2b_1}$$

2. If we had the three equations with the three unknown magnitudes

$$a_1x + b_1y + c_1z = m_1$$
$$a_2x + b_2y + c_2z = m_2$$
$$a_3x + b_3y + c_3z = m_3$$

we then should find

$$x = \frac{m_1b_2c_3 - m_1b_3c_2 - m_2b_1c_3 - m_2b_3c_1 + m_3b_1c_2 - m_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}$$

$$y = \frac{a_1m_2c_3 - a_1m_3c_2 - a_2m_1c_3 + a_2m_3c_1 + a_3m_1c_2 - a_3m_2c_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}$$

$$z = \frac{a_1b_2m_3 - a_1b_3m_2 - a_2b_1m_3 + a_2b_3m_1 + a_3b_1m_2 - a_3b_2m_1}{a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1}$$

3. From the formulæ in 1 and 2, the rules for the solution of the above general equations follow by induction. In order to abbreviate them, I shall call the numbers which are affixed to the letters m, a, b, &c. symbolical numbers.

(*a*) Take the product $a_1 b_2 c_3 \ldots\ldots k_{n-1} l_n$; then permute the symbolical numbers in all possible ways, while the letters themselves are not changed; the aggregate of all these $1 \cdot 2 \cdot 3 \ldots\ldots n$ products, then gives the common denominator in the values of $x, y, z \ldots\ldots v, w$.

(*b*) In order to find the signs of every one of the terms,

of which the denominator consists, try how often in such a term a lower symbolical number follows a higher one, mediately or immediately. If the number of these successions be even or O, then the sign of the term is +; if it be odd, the sign is—.

(*c*) If the common denominator be found, we obtain from it the numerator in the value of x, merely by substituting m for a; the numerator in the value of y, by substituting m for b; the numerator in the value of z, by substituting m for c; and so in like manner with the other unknown magnitudes.

Thus the denominator in the values of x, y, z, is merely the product $a_1 b_2 c_3$, with the 1 . 2 . 3 permutations of the symbolical numbers; and with respect to the signs, if, for instance, the term $a_2 b_3 c_1$, has the sign +, because it contains two successions of a lower symbolical number to a higher, viz. 21, 31; but in the term $a_3 b_2 c_1$, there are three such successions, viz. 32, 31, 21, and this term consequently has the sign —. Likewise the numerators are formed in the manner given in (*c*).

Example. From the four equations

$$a_1x + b_1y + c_1z + d_1u = m_1$$
$$a_2x + b_2y + c_2z + d_2u = m_2$$
$$a_3x + b_3y + c_3z + d_3u = m_3$$
$$a_4x + b_4y + c_4z + d_4u = m_4$$

we obtain for the common denominator in the values of x, y, z, u, the following expression:

$$
\begin{aligned}
& a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_3c_4d_2 \\
+ & a_1b_4c_2d_3 - a_1b_4c_3d_2 - a_2b_1c_3d_4 + a_2b_1c_4d_3 \\
+ & a_2b_3c_1d_4 - a_2b_3c_4d_1 - a_2b_4c_1d_3 + a_2b_4c_3d_1 \\
+ & a_3b_1c_2d_4 - a_3b_1c_4d_2 - a_3b_2c_1d_4 + a_3b_2c_4d_1 \\
+ & a_3b_4c_1d_2 - a_3b_4c_2d_1 - a_4b_1c_2d_3 + a_4b_1c_3d_2 \\
+ & a_4b_2c_1d_3 - a_4b_2c_3d_1 - a_4b_3c_1d_2 + a_4b_3c_2d_1
\end{aligned}
$$

SECTION LX.

Since the values of the unknown magnitudes in the solution of the foregoing § always appear in the form of fractions, it may sometimes happen, that the common denominator $= 0$, as, for instance, in the two equations $a_1b_2 - a_2b_1 = 0$, and the three equations $a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 = 0$. In this case, if also the numerator $= 0$, then we arrive at expressions of the form $\frac{0}{0}$. Such a form as this merely indicates, that the conditions given in the equations are not such, that the values of the unknown magnitudes can be determined by their means alone. Thus, if we had the two equations $3x + 5y = 16$, $6x + 10y = 32$, then would $a_1 = 3, b_1 = 5, a_2 = 6, b_2 = 10, m_1 = 16, m_2 = 32$, and consequently from the formulæ in 1 of the foregoing §,

$$x = \frac{16 \,.\, 10 - 32 \,.\, 5}{3 \,.\, 10 - 6 \,.\, 5} = \frac{0}{0}, y = \frac{3 \,.\, 32 - 6 \,.\, 16}{3 \,.\, 10 - 6 \,.\, 5} = \frac{0}{0},$$

the values of x and y ∴ would remain undetermined. But we immediately see why they must continue to be undetermined. For if we divide the second equation by 2, we obtain the first; consequently the latter is contained in the former, and we have ∴ in fact no more than one equation, from which neither x nor y can be determined.

But if the given equations be such, that the denominator vanishes in the value of an unknown magnitude, but not the numerator, consequently that we arrive at an expression of the form $\frac{a}{0} = \infty$; then this result always indicates, that the relations expressed by the equation are contradictory, and cannot obtain at the same time, while the unknown magnitudes, as is always assumed here, have only finite values. Thus, suppose we have the two equations $3x + 5y = 16$, $6x + 10y = 20$, we obtain from 1 of the foregoing §, $x = \frac{60}{0}$, $y = \frac{-36}{0}$; consequently, as we are convinced that there can be no other values but these, there must be contradictory relations in the given equations. This, indeed; is really the case; for if we multiply the first equation by 2, we get $6x + 10y = 32$, whereas, from the second equation $6x + 10y = 20$.

SECTION LXI.

The problems, § LVIII. and LXIX. contain all that relates to the elimination of equations of the first degree. I shall now direct my attention to the elimination of equations of higher degrees; and I shall first assume, that there are no more than two equations given with two, or even more unknown magnitudes. Here two cases must be considered: first, when the first equation, in reference to the magnitudes to be eliminated, is of the first degree, and the second of a higher; secondly, when both equations are of higher degrees.

There is no difficulty in the first case; for we only

require to find from the first equation the value of the magnitude to be eliminated, and substitute this value in the second, when we obtain an equation, in which this magnitude does not occur.

In the second case, by multiplying by proper factors, and by the requisite combination of the results thus obtained, we always try to reduce the degree of the equations in reference to the magnitude to be eliminated, till we come to an equation, which contains only the first power of this magnitude. If from this equation we find the value of the magnitude to be eliminated, and substitute it in that equation in which it occurs in the lowest power, we shall obtain the required final equation.

The following problems will sufficiently elucidate the foregoing.

SECTION LXII.

PROB. Let p, q, r, p', q', r', be functions of y; further, let the two equations

$$\text{I. } p + qx + rx^2 = 0$$
$$\text{II. } p' + q'x + r'x^2 = 0$$

be given between x and y: find the equation, which arises from the elimination of x.

Solution 1. Multiply the first equation by p', the second by p, then subtract the results thus obtained from one another, and divide by x; hence arises the equation

$$pq' - p'q + (pr' - p'r)\,x = 0$$

and this gives

$$x = \frac{p'q - pq'}{pr' - p'r}.$$

2. Further, multiply the first equation by r', and the second by r, and subtract; then we have

$$pr' - p'r + (qr' - q'r)\,x = 0$$

If in this equation we substitute for x its value obtained from 1, we get the equation

$$(\psi) \ldots (pr'-p'r)^2 + (p'q-pq')\,(qr'-q'r) = 0$$

which only contains y, and which consequently is the final equation sought.

3. If we had immediately substituted the value of x from 1, in one of the given equations, for instance in I, we should have found

$$p + \frac{q\,(p'q-pq')}{pr' - p'r} + \frac{r\,(p'q-pq')^2}{(pr'-p'r)^2} = 0$$

and if we multiply by $(pr'-p'r)^2$, and then divide by p, we get the same equation as in 2.

SECTION LXIII.

PROB. From the two equations

$$\text{I. } p + qx + rx^2 = 0$$
$$\text{II. } p' + q'x + r'x^2 + s'x^3 = 0$$

eliminate x, supposing that p, q, r, p', q', r', s', are such expressions as do not contain x.

Solution 1. Multiply the first equation by p', the second by p, and subtract the results, and we get the equation

$$pq' - p'q + (pr' - p'r)\,x + ps'x^2 = 0$$

2. If we combine the equation I. with this one, the

case in the preceding § enters here; only that $pq'-p'q$, $pr'-p'r$, ps', are here what p', q', r', were in the former. We only require ∴ to substitute the former values for the latter in the equation (ψ) of the foregoing §. If this be actually done, we obtain the equation

$$(p^2s'+qrp'-prq')^2+(pqs'-prr'+r^2p')\times$$
$$(pqq'-q^2p'-p^2r'+prp') = 0$$

3. If we solve this equation, and then divide by p, we obtain

$$p^3s'^2 + p^2rr'^2 + pr^2q'^2 + r^3p'^2 - qr^2p'q' +$$
$$(q^2-2pr)\ (rp'r'+pq's')+(3pqr-q^3)p's'-$$
$$pqrq'r'-p^2qr's'=0$$

an equation which does not contain x.

SECTION LXIV.

PROB. Again, let p, q, r, s, p', q', r', s', be functions which do not contain x: find the result of the elimination of x from the two equations

$$\text{I. } p + qx + rx^2 + sx^3 = 0$$
$$\text{II. } p' + q'x + r'x^2 + s'x^3 = 0$$

Solution 1. Multiply the first equation by p', the second by p, and subtract the results; after dividing by x, this gives

$$pq' - qp' + (pr' - rp')\ x + (ps' - sp')\ x^2 = 0$$

2. Further, multiply the first equation by s', the second by s, and again subtract; this gives

$$sp' - ps' + (sq' - qs')\ x + (sr' - rs')\ x^2 = 0.$$

3. It is not necessary to continue the reduction further; for since the equations found in 1 and 2 are similar to the equations I and II, in § 62, for which the result of the elimination was there found, it is only necessary in the equation (ψ) of that section, to make the following substitutions:

$$pq'-qp' \text{ for } p,\quad sp'-ps' \text{ for } p'$$
$$pr'-rp' \text{ for } q,\quad sq'-qs' \text{ for } q'$$
$$ps'-sp' \text{ for } r,\quad sr'-rs' \text{ for } r'$$

4. By this substitution we obtain, after the proper solution,

$$(pq'-qp')^2\ (sr'-rs')^2 - 2(pq'-qp')\ (ps'-sp')\ (sp'-ps')\ (sr'-rs') + (ps'-sp')^2\ (sp'-ps')^2 + (pr'-rp')^2\ (sp'-ps')\ (sr'-rs') - (pq'-qp')\ (pr'-rp')\ (sq'-qs')\ (sr'-rs') - (pr'-rp')\ (ps'-sp')\ (sp'-ps')\ (sq'-qs') + (pq'-qp')\ (ps'-sp')\ (sq'-qs')^2 = 0$$

5. The first part of this equation consists of seven terms, of which five are divisible by $sp'-ps'$. The other two, viz. the first and fifth, taken together, give

$$(pq'-qp')\ (sr'-rs') \times$$
$$[(pq'-qp')\ (sr'-rs') - (pr'-rp')\ (sq'-qs')]$$
$$= (pq'-qp')\ (sr'-rs')\ (pqr's'+rsp'q'-prq's'-qsp'r')$$
$$= (pq'-qp')\ (sr'-rs')\ (sp'-ps')\ (rq'-qr')$$

and consequently the sum of these two terms is also divisible by $sp'-ps'$.

6. If ∴ the equation in 4 be divided by $sp'-ps'$, we at last obtain the equation

$$(pq' - qp')(sr' - rs')(rq' - qr') + 2(pq' - qp')$$
$$(sp' - ps')(sr' - rs') + (sp' - ps')^3 + (pr' - rp')^2$$
$$(sr' - rs') + (pr' - rp')(sp' - ps')(sq' - qs') -$$
$$(pq' - qp')(sq' - qs')^2 = 0$$

SECTION LXV.

PROB. From the two equations

$$\text{I.}\quad p + qx + rx^2 + sx^3 + tx^4 = 0$$
$$\text{II.}\quad p' + q'x + r'x^2 + s'x^3 + t'x^4 = 0$$

eliminate the magnitude x.

Solution 1. Multiply the first equation by p', the second by p, and subtract; after dividing by x, this gives

$$pq' - qp' + (pr' - rp')x + (ps' - sp')x^2$$
$$+ (pt' - tp')x^3 = 0$$

2. Further, multiply the first equation by t', the second by t, and again subtract; this gives

$$pt' - tp' + (qt' - tq')x + (rt' - tr')x^2$$
$$+ (st' - ts')x^3 = 0$$

3. Since the equations in 1 and 2 are both of the third degree, in order to save the trouble of carrying on the operation, we need only use immediately the equation found in 6 of the foregoing §, when we make the following substitutions in it:

$pq' - qp'$ for p, $pt' - tp'$ for p'
$pr' - rp'$ for q, $qt' - tq'$ for q'
$ps' - sp'$ for r, $rt' - tr'$ for r'
$pt' - tp'$ for s, $st' - ts'$ for s'

and the result of this substitution is the final equation sought.

SECTION LXVI.

Prob. From the two general equations

I. $p + qx + rx^2 + sx^3 + \ldots\ldots + vx^m = 0$

II. $p' + q'x + r'x^2 + s'x^3 + \ldots\ldots + v'x^n = 0$

eliminate x.

Solution 1. I shall assume, that $m < n$. Multiply then one equation by the first term of the second, *i. e.* by p', and the other by the first term of the first, *i. e.* by p; then subtract, and divide the remainder by x; we then obtain an equation of the form

III. $A + A_1x + A_2x^2 + A_3x^3 + \ldots\ldots + A_{-1}x^{n-1} = 0$

2. If again $m < n-1$, we proceed with the equations I and III, exactly as we did before with the equations I and II.

3. In this way we continue to diminish the degree of the resulting equation, till we arrive at an equation of the mth degree. Let

IV. $B + B_1x + B_2x^2 + B_3x^3 + \ldots\ldots + B\ \ x^m = 0$

be this equation.

4. Combine the equation IV with the equation I in a two-fold way, viz. 1. when we multiply the first by p, the second by B; subtract the results, and divide the remainder by x; 2. when we multiply the first by v, the second by B_m, and again subtract the results from one

another. By this operation we obtain two equations of the $m-1$th degree.

V. $C + C_1 x + C_2 x^2 + C_3 x^3 + \ldots\ldots + C_{m-1} x^{m-1} = 0$
VI. $D + D_1 x + D_2 x^2 + D_3 x^3 + \ldots\ldots + D_{m-1} x^{m-1} = 0$

5. If we proceed with the two equations so obtained, in the same way as we did before with I and IV, we shall again obtain two equations of the $m-2$th degree. In this manner we reduce the degree of the equations less and less, till we arrive at two equations of the first degree. Let

$$K + K_1 x = 0\,,\ L + L_1 x = 0$$

be these two equations; then we have

$$x = -\frac{K}{K_1} = -\frac{L}{L_1}$$

and consequently

$$KL_1 - LK_1 = 0$$

and this, since it does not contain x, is the final equation sought.

6. But it is by no means necessary to continue throughout the elimination to equations of the first degree; thus, if we have already found the results of the elimination for equations of a certain degree, then it is sufficient, as we have shown in the foregoing §, to continue the reduction to this degree only.

SECTION LXVII.

The method of elimination which has been applied in the foregoing §§, which Euler makes use of in the 19th

chapter of the 2nd Book of his Introduction, is perfectly general, but is essentially deficient in this respect, that it does not always give the final equation in its simple form. Thus, for instance, the equation in 2, § LXIII, was divisible by p, and then gave the equation in 3; in the same way, after dividing the equation in 4, § LXIV, by $sp'-ps'$ we obtained the equation in 6. This also obtains in the higher equations, and the divisors in this case are difficult to find. We shall however see, in the sequel, that these divisors are actually superfluous, and by no means belong to the final equation. Should we not ∴ probably be able to find such a divisor, we should have to solve not only a higher equation for y, than was actually required, but there would be also amongst its roots such as do not belong to the equations I and II of the foregoing §, consequently which were not so constituted, that we were enabled to find the corresponding values of x, which at the same time verify the two equations just mentioned.

Since the elimination of x from two equations between x and y has only this aim, to give one or more such values for y, that it may be possible to find one or more corresponding values for x, which are common to both equations, consequently every method which serves to attain this object, may also be applied to the elimination.

If ∴ we denote one of the values of x by a, which is common to both equations, then $x-a$ must be their common divisor; consequently it is merely necessary to find the conditions on which the possibility of a divisor of this kind depends. To effect this, we only require to proceed with the given equations, exactly as though we

sought their common divisor; the last remainder, which we get by the successive divisions, and which does not contain x, if it contains such a divisor, must necessarily vanish. If we put this remainder $= 0$, we then obtain the required conditional or final equation. The following problem will be sufficient to elucidate what has been said. The method to find the common divisor, is assumed to be known.

SECTION LXVIII.

PROB. From the two equations

$$\text{I.}\quad x^3 + 3x^2y + 3xy^2 - 98 = 0$$
$$\text{II.}\quad x^2 + 4xy - 2y^2 - 10 = 0$$

eliminate x by the method of the common divisor.

Solution. The calculation is as follows.

1. Dividend $x^3 + 3x^2y + 3xy^2 - 98$
   Divisor $x^2 + 4xy - 2y^2 - 10$
   Quotient $x - y$
   First Remainder $(9y^2 + 10)\,x - 2y^3 - 10y - 98$

2. Dividend $x^2 + 4xy - 2y^2 - 10$
   or rather $(9y^2 + 10)\,x^2 + 36xy^3 + 40xy -$
   $18y^4 - 110y^2 - 100$
   Divisor $(9y^2 + 10)\,x - 2y^3 - 10y - 98$
   Quotient $x + \dfrac{38y^3 + 50y + 98}{9y^2 + 10}$
   Second and last Remainder

$$-18y^4 - 110y^2 - 100 + \frac{(2y^3 + 10y + 98)(38y^3 + 50y + 98)}{9y^2 + 10}$$

3. If this remainder be put $=0$, and then multiplied by $9y^2+10$, we obtain

$$-86y^6-690y^4+3920y^3-1500y^2+5880y+8604=0$$

or, when we divide by two, and change all the signs

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0$$

and this is the conditional or final equation sought.

Remark. Having found a value of y from the conditional equation, we can find the value of x, which corresponds to it, by substituting that value in the two equations I and II, and then seeking the common divisor. Thus we shall find, that $y=3$ verifies the final equation; if $\therefore$ we substitute this value in I and II, we obtain the two equations $x^3+9x^2+27x-98=0$, $x^2+12x-28=0$, whose common divisor is $x-2$. Accordingly $x=2$ is the value of x, which belongs to $y=3$.

But we could also have found this value of x directly; for we know already, that always when amongst the remainders obtained in the divisions for finding the common divisor, that remainder which $=0$, is considered as the last, the preceding remainder is the required divisor. If we apply this to the present case, then $(9y^2+10)x-2y^3-10y-98$ is this divisor; and if we put $y=3$, then this divisor is $91x-182$, or $x-2$, which agrees with the foregoing.

But hence it follows, that we must also have obtained the final equation, if we had substituted the value $x=\dfrac{2y^3+10y+98}{9y^2+10}$, which we obtained from the equation $(9y^2+10)x-2y^3-10y-98=0$, immediately in the

equation II, as the lowest of the two given ones. But if we make the same substitution in the equation I, we arrive at an equation of the ninth degree, which consequently contains one factor more of the third degree. But we shall see in the sequel, that this factor is actually superfluous, and that ∴ the equation in z is the complete final equation.

Since the final equation is of the sixth degree, there are ∴, besides the value $y=3$, five other values of y, for each of which there is a corresponding value of x. Consequently it verifies the equations I and II in six ways. The same equation of the sixth degree we could also have obtained from the equation in z, § LXIII, if, as the present case requires, we had put $p=-2y^2-10$, $q=4y$, $r=1$, $p'=-98$, $q'=3y^2$, $r'=3y$ and $s'=1$.

SECTION LXIX.

From the method of the common divisor, we may derive another, which Euler gives in the work above mentioned.

Let

$$\text{I.}\quad x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c. = 0$$
$$\text{II.}\quad x^n + p'x^{n-1} + q'x^{n-2} + r'x^{n-3} + \&c. = 0$$

be the two equations, from which x is to be eliminated. If it happen that these two equations have any common divisor $x-a$, we can then put

$$\text{III.}\quad x^m + px^{m-1} + qx^{m-2} + \&c. = (x-a)\,\Pi$$
$$\text{IV.}\quad x^n + p'x^{n-1} + q'x^{n-2} + \&c. = (x-a)\,\Pi'$$

and then Π, Π' are the quotients, which arise from the

division of the first parts of the equations I, II by $x-a$. It is not necessary here actually to know these quotients; it is merely sufficient to observe, that they must necessarily have the following form:

$$\Pi = x^{m-1} + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \&c.$$
$$\Pi' = x^{n-1} + A'x^{n-1} + B'x^{n-2} + C'x^{n-3} + \&c.$$

and that the first contains $m-1$ undetermined magnitudes A, B, C, &c., and the second $n-1$ undetermined magnitudes A', B', C', &c.

Now, if we eliminate $x-a$ from the two equations III, IV, we then obtain the identical equation

$$(\phi) \quad \begin{aligned} &(x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c.)\,\Pi' = \\ &(x^n + p'x^{n-1} + q'x^{n-2} + r'x^{n-3} + \&c.)\,\Pi \end{aligned}$$

If we actually perform this multiplication, after having substituted for Π, Π', their assumed values, and then put the coefficients of the same powers of x in the two parts of the resulting equation equal to one another, we then obtain $m+n-1$ equations between the magnitudes p, q, r, &c. p', q', r', &c. A, B, C, &c. A', B', C', &c. which, in reference to the unknown magnitudes A, B, C, &c. A', B', C', &c. are all of the first degree only. Since $\therefore$ we have $m+n-1$ equations, and only $m+n-2$ undetermined magnitudes A, B, C, &c. A', B', C', &c. these can always be eliminated, and by this elimination we shall obtain an equation which contains no other magnitudes but the known functions p, q, r, &c. p', q', r', &c., and which consequently will be the required conditional or final equation.

The following problems will elucidate what has been said.

SECTION LXX.

Prob. From the two given equations

$$\text{I. } x^2 + px + q = 0$$

$$\text{II. } x^3 + p'x^2 + q'x + r' = 0$$

eliminate x by the method in the foregoing §.

Solution. Since here $m=2$, and $n=3$, then

$$\Pi = x + A,\ \Pi' = x^2 + A'x + B'.$$

The equation (ϕ) of the foregoing § will ∴ be

$$(x^2 + px + q)(x^2 + A'x + B') =$$
$$(x^3 + p'x^2 + q'x + r')(x + A)$$

Hence, by actual multiplication and equating the coefficients, we obtain

$$A' + p = A + p'$$
$$B' + pA' + q = p'A + q'$$
$$pB' + qA' = q'A + r'$$
$$qB' = r'A$$

Since there are four equations here, and only three undetermined magnitudes A, A', B', we can ∴ eliminate these magnitudes, which only occur in the first power, and we shall then, after the proper reduction, obtain the same equation as in § LXIII, by substituting $1, p, q, 1, p', q', r'$, for r, q, p, s', r', q', p', respectively, as the form of the equations here given, compared with the others, requires.

SECTION LXXI.

Prob. From the given equations

$$\text{I. } x^3 + px^2 + qx + r = 0$$

$$\text{II. } x^4 + p'x^3 + q'x^2 + r'x + s' = 0$$

eliminate x by the method in § LXIX.

Solution. Here $m=3$, $n=4$; we have $\therefore$

$\Pi = x^2 + Ax + B$, $\Pi' = x^3 + A'x^2 + B'x + C'$

If these values be substituted in the equation (ϕ) in § LXIX, we obtain

$$(x^3 + px^2 + qx + r)(x^3 + A'x^2 + B'x + C') =$$
$$(x^4 + p'x^3 + q'x^2 + r'x + s')(x^2 + Ax + B)$$

If we solve this equation, and equate the coefficients of its two parts, we get

$$\begin{aligned}
&A' + p = A + p' \\
&B' + pA' + q = B + p'A + q' \\
&C' + pB' + qA' + r' = p'B + q'A + r' \\
&pC' + qB' + rA' = q'B + r'A + s' \\
&qC' + rB' = r'B + s'A \\
&rC' = s'B
\end{aligned}$$

Since in these six equations there are only five undetermined magnitudes A, B, A', B', C', we can eliminate these, and thus we shall obtain an equation, which contains none but the unknown magnitudes p, q, r, p', q', r', s'', which consequently is the final equation sought.

SECTION LXXII.

Euler's second method, which is here elucidated, is at least quite as diffuse, if not more so, than the first; it is also quite as free as the first, from the fault of having superfluous factors, which we can easily convince ourselves of by the actual calculation of a few easy cases. Bezout, in his "Théorie générale des Equations Algébriques, Paris, 1799," has made use of a similar method; he has applied it to more than two equations, and to the elimination of more unknown magnitudes; he has shown

also, how we are to begin, in many cases, in order to find the complete final equation, without its including any thing superfluous. The work is, indeed, rather prolix, yet contains a great deal of matter, and is composed with much care. A revision of this work, with the application of the combination-analysis, would be a most useful undertaking.

From the hypothesis of the common divisor, we may also derive another method, which is not only much more simple, but also more direct, and more suited to the nature of equations than the preceding, inasmuch as it is founded on the theory of symmetrical functions. It has likewise this important advantage over the other one, that it always gives, at least for two equations and two unknown magnitudes, the complete final equation, without any thing heterogeneous.

For this purpose, we assume again the equations I, II, § LXIX. We suppose the equation I, with respect to x, is already solved; then its roots x'; x'', x''', &c. are functions of its coefficients, and consequently also functions of y. In like manner we suppose the second solved; then its roots, which I shall denote by x', x'', x''', &c. are also functions of y. If the two equations have a common divisor, then must at least one of the roots of the first equation be equal to one of the roots of the second. Put $x' = \mathrm{x}'$, then $x' - \mathrm{x}' = 0$ is the equation for y, which must be the case when the two determined roots x', x', are equal to one another. But since, equally well, every other two roots of both equations may be put equal to one another, we then obtain as many distinct equations $x' - \mathrm{x}' = 0$, $x' - \mathrm{x}'' = 0$, $x' - \mathrm{x}''' = 0 \ldots x'' - \mathrm{x}' = 0$, $x'' - \mathrm{x}'' = 0$, $x'' - \mathrm{x}''' = 0 \ldots$

$x''' - \mathrm{x}' = o$, $x''' - \mathrm{x}'' = o$, $x''' - \mathrm{x}''' = o$, &c. as the roots x', x'', x''', &c. may be combined with the roots x', x'' x''', &c. All these distinct equations, however, must at the same time be contained in the conditional or final equation sought, because there is no reason why it should exactly contain the one and not likewise the other; it must consequently be a product of these. The final equation is ∴ no other than

$$(\psi)\ldots\ldots\left\{\begin{matrix}(x' - \mathrm{x}')\,(x' - \mathrm{x}'')\,(x' - \mathrm{x}''')\ \ldots\ \times \\ (x'' - \mathrm{x}')\,(x'' - \mathrm{x}'')\,(x'' - \mathrm{x}''')\ \ldots\ \times \\ (x''' - \mathrm{x}')\,(x''' - \mathrm{x}'')\,(x''' - \mathrm{x}''')\ \ldots\ \times \\ \&c.\end{matrix}\right\} = o$$

The first part of this equation undergoes no change, however we transform the roots x', x'', x''', &c.; it is ∴ symmetrical in reference to x', x'', x''', &c. But since it also suffers no change when we transform x', x'' x''', &c. it is also symmetrical with reference to x', x'', x''', &c. Consequently the first part of the equation (ψ) may always be expressed rationally by the coefficients of the given equations.

I shall hereafter show how we can give the equation (ψ) a more convenient form for calculation; but I shall first explain what has been stated, by a few problems.

SECTION LXXIII.

Prob. From the two equations

$$\text{I. } x^2 - Ax + B = o$$
$$\text{II. } x^2 - A'x + B = o$$

in which A, B, A', B', are given functions of y, eliminate x' by the method in the foregoing §.

Solution 1. The equation (ψ) of the foregoing §, since both equations are of the second degree, in this case is

$$(x'-\text{x}')(x'-\text{x}'')(x''-\text{x}')(x''-\text{x}'')=0$$

or

$$[x'^2-\text{x}'(\text{x}'+\text{x}'')+\text{x}'\text{x}''][x''^2-x''(\text{x}'+\text{x}'')+\text{x}'\text{x}'']=0$$

or, since $\text{x}'+\text{x}''=A'$, $x'x''=B'$

$$(x'^2-x'A'+B')(x''^2-x''A'+B')=0$$

2. If we actually perform the multiplication, we obtain

$$x'^2x''^2-(x'x''^2+x'^2x'')A'+(x'^2+x''^2)B'$$
$$+x'x''A'^2-(x'+x'')A'B'+B'^2=0$$

or since $x'+x''=(1)=A$, $x'x''=(1^2)=B$, $x'x''^2+x'^2x''=(12)=AB$, $x'^2+x''^2=(2)=A^2-2B$,

$$B^2-ABA'+(A^2-2B)B'+BA'^2-AA'B'+B'^2=0$$

and this equation is the final equation sought, which we should also have obtained, if we had eliminated in the usual way.

SECTION LXXIV.

Prob. From the given equations

$$\text{I. } x^3-Ax^2+Bx-C=0$$
$$\text{II. } x^2-A'x+B'=0$$

eliminate x by the method in § LXXII.

Solution 1. The equation (ψ) § LXXII in this case, is

$$(x'-\text{x}')(x'-\text{x}'')(x''-\text{x}')(x''-\text{x}'')$$
$$(x'''-\text{x}')(x'''-\text{x}'')=0$$

or

$$[x'^2-x'(\text{x}'+\text{x}'')+\text{x}'\text{x}''][x''^2-x''(\text{x}'+\text{x}'')+\text{x}'\text{x}'']$$
$$[x'''^2-x'''(\text{x}'+\text{x}'')+\text{x}'\text{x}'']=0$$

or, since $x'+x''=A'$, $x'x''=B'$

$$(x'^2-x'A'+B')\,(x''^2-x''A'-B')$$
$$(x'''^2-x'''A'+B')=0$$

2. The actual multiplication of the three factors in the first part of this equation, gives

$$(1^3)^2-(12^2)A'+(2^2)B'+(1^22)A'^2-(12)A'B'+$$
$$(2)B'^2-(1^3)A'^3+(1^2)A'^2B'-(1)A'B'^2+B'^3$$
$$=0$$

or, when for the numerical expressions their values are substituted from the annexed tables,

$$C^2-BCA'+(B^2-2AC)B'+ACA'^2-(AB-3C)A'B'$$
$$+(A^2-2B)B'^2-CA'^3+BA'^2B'-AA'B'^2+B'^3$$
$$=0$$

which is the final equation sought.

SECTION LXXV.

Prob. From the given equations

$$\text{I. } x^m-Ax^{m-1}+Bx^{m-2}-Cx^{m-3}+\&c.=0$$
$$\text{II. } x^n-A'x^{n-1}+B'x^{n-2}-C'x^{n-3}+\&c.=0$$

eliminate x, by the method in § LXXII.

Solution 1. Since x', x'', x''', &c. are the roots of the equation II, then

$$(x-x')\,(x-x'')\,(x-x''')\ldots\ldots\ldots=$$
$$x^n-A'x^{n-1}+B'x^{n-2}-C'x^{n-3}+\&c.$$

If in this for x we substitute x', x'', x''', &c. successively, then

$$(x'-x')\,(x'-x'')\,(x'-x''')\,\&c.=x'^n-A'x'^{n-1}+\&c.$$
$$(x''-x')(x''-x'')(x''-x''')\&c.=x''^n-A'x''^{n-1}+\&c.$$
$$(x'''-x')(x'''-x'')(x'''-x''')\&c.=x'''^n-A'x'''^{n-1}+\&c.$$

2. If these values be substituted in the equation (ψ), § 72, the latter is transformed into

$$\left.\begin{cases} (x'^{n} - A'x'^{n-1} + B'x'^{n-2} - C'x'^{n-3} + \&c.) \\ \times\ (x''^{n} - A'x''^{n-1} + B'x''^{n-2} - C'x''^{n-3} + \&c.) \\ \times\ (x'''^{n} - A'x'''^{n-1} + B'x'''^{n-2} - C'x'''^{n-3} + \&c.) \\ \times\ \ \&c. \end{cases}\right\} =$$

The first part of this equation is no other than the product of all the expressions, which arise from substituting in the equation II for x all the roots x', x'', x''', &c. successively of the equation I.

3. But we immediately see, that the above product undergoes no change by any transformation of the roots x', x'', x''', &c. as in a transformation of this kind one factor is merely changed into another. The first part of the equation is ∴ necessarily a symmetrical function of the above roots, which consequently may always be omitted according to known rules. In this way we obtain an equation, which no longer contains x, and which ∴ is the final equation sought. From the operation itself, it follows besides, that it is complete, and contains nothing extraneous.

Remark. The problem from two given equations with two unknown magnitudes to eliminate one of these magnitudes, is consequently now solved in its most general form. The actual calculation involves many difficulties, and amongst these chiefly are the solution of the product in the first part of the equation in 2, and its reduction to numerical expressions. How these difficulties may be

removed by means of combinations, will be shown in the following §.

SECTION LXXVI.

Prob. Represent directly the result of the elimination of x from the two equations I and II of the foregoing §, fully solved.

Solution 1. To the equation II of the foregoing §, we can always, by dividing by its last term, give the following form:

$$1+(1)x+(2)x^2+(3)x^3+\ldots\ldots+(n)x^n=0$$

in which the coefficients (1), (2), (3), ... (n) denote given functions of y. This notation was chosen merely in order to facilitate the application of the combination-operations, and to make the law of the terms more evident. In order further to show, that two or more such coefficients are to be multiplied together, I shall merely put the numbers representing this operation in brackets near each other, and in these make use of repeating exponents. Thus ∴ (123), (2456), $(1^3 2^2)$, the first of these denotes the product of the coefficients (1), (2), (3); the second, the product of the coefficients (2), (4), (5), (6), and the third, the product of the third power of the coefficient (1) and the second power of the coefficient (2).

2. Put, as was done in the foregoing §, x', x'', x''', &c. successively for x, consequently the first part of the equation in 2 of that §, has the following form:

$$[1+(1)x'+(2)x'^{2}+(3)x'^{3}+(4)x'^{4}+(5)x'^{5}+\&c.]$$
$$\times[1+(1)x''+(2)x''^{2}+(3)x''^{3}+(4)x''^{4}+(5)x''^{5}+\&c.]$$
$$\times[1+(1)x'''+(2)x'''^{2}+(3)x'''^{3}+(4)x'''^{4}+(5)x'''^{5}+\&c.]$$
$$\&c.$$

The number of factors here is equal to the degree of the equation I, .·. $= m$.

3. First take the product of the two first factors, we then obtain

$$\begin{aligned}1 &+ (1)(x'+x'') + (2)+(x'^{2}+x''^{2}) + (3)(x'^{3}+x''^{3})\\ &\qquad\qquad + (1^{2})(x'x'') \qquad + (12)(x'x''^{2}+x'^{2}x'')\end{aligned}$$

$$\begin{aligned}&+ (4)(x'^{4}+x''^{4}) && + (5)(x'^{5}+x''^{5}) && + \&c.\\ &+ (13)(x'x''^{3}+x'^{3}x'') && + (14)(x'x''^{4}+x'^{4}x'') &&\\ &+ (2^{2})(x'^{2}x''^{2}) && + (23)(x'^{2}x''^{3}+x'^{3}x''^{2}) &&\end{aligned}$$

4. Hence, if the equation I were of the second degree only, then this product must be represented by numerical expressions, as follow:

$$\begin{aligned}1 &+ (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] + \&c.\\ &\qquad + (1^{2})[1^{2}] + (12)[12] + (13)[13](+(14)[14]\\ &\qquad\qquad\qquad + (2^{2})[2^{2}] + (23)[23]\end{aligned}$$

5. Now if we multiply the product in 3 by the third factor in 2, we then obtain, when the terms are properly arranged, the product

$$\begin{aligned}1 &+ (1)(x'+x''+x''') + (2)(x'^{2}+x''^{2}+x'''^{2})\\ &\qquad\qquad + (1^{2})(x'x''+x'x'''+x''x''')\\ &+ (3)(x'^{3}+x''^{3}+x'''^{3})\\ &+ (12)\begin{pmatrix}x'x''^{2}+x'^{2}x''+x'x'''^{2}+\\ x'^{2}x'''+x''x'''^{2}+x''^{2}x'''\end{pmatrix}\\ &+ (1^{3})(x'x''x''')\end{aligned}$$

$$+ (4)\ (x'^4 + x''^4 + x'''^4)$$

$$+ (13) \begin{pmatrix} x'x''^3 + x'^3x'' + x'x'''^3 + \\ x'^3x''' + x''x'''^3 + x''^3x''' \end{pmatrix}$$

$$+ (2^2)\ (x'^2x''^2 + x'^2x'''^2 + x''^2x'''^2)$$

$$+ (1^22)\ (x'x''x'''^2 + x'x''^2x''' + x'^2x''x''')$$

$$+ (5)\ (x'^5 + x''^5 + x'''^5)$$

$$+ (14) \begin{pmatrix} x'x''^4 + x'^4x'' + x'x'''^4 + \\ x'^4x''' + x''x'''^4 + x''^4x''' \end{pmatrix}$$

$$+ (23) \begin{pmatrix} x'^2x''^3 + x'^3x''^2 + x'^2x'''^3 + \\ x'^3x'''^2 + x''^2x'''^3 + x''^3x'''^2 \end{pmatrix}$$

$$+ (1^23)\ (x'x''x'''^3 + x'x''^3x''' + x'^3x''x''')$$

$$+ (12^2)\ (x'x''^2x'''^2 + x'^2x''x'''^2 + x'^2x''^2x''')$$

&c.

6. If ∴ the equation I be only of the third degree, then the product in 2 may be represented as follows:

$$\begin{aligned} 1 + (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] + \&c. \\ + (1^2)[1^2] + (12)[12] + (13)[13] + (14)[14] \\ + (1^3)[1^3] + (2^2)[2^2] + (23)[23] \\ + (1^22)[1^22] + (1^23)[1^23] \\ + (12^2)[12^2] \end{aligned}$$

7. It is not necessary to continue the multiplication further, as we may very easily perceive the law from the products already found. Thus we see, immediately, that the figures in the parentheses and brackets are always the same, and compounded in the same way. The included numerical expressions merely denote this, that they are all the possible numerical divisors for

the numbers 1, 2, 3, 4, ... mn. I say all the possible numerical divisions; because some vanish, as, for example, in the products in 4 and 6; this merely proceeds from this, that the numerical expressions for such divisions in the assumed degree of the equation I, are not possible, because more roots are required for their formation than this equation can possibly have. Generally, when we have to do with particular equations, all those divisions vanish, for which either the numerical expressions, or the products of the coefficients do not obtain.

8. How all the possible divisions of the numbers may be easily found, without the chance of omitting one, may be seen by the combination-analysis. In order $\therefore$ to find the elimination of x from the two equations,

I. $x^m - Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \&c. = 0$

II. $1 + (1)x + (2)x^2 + (3)x^3 + \ldots + (n)x^n = 0$

we must observe the following rules:

(a) Analyse the numbers 1, 2, 3, 4, ... mn in all possible ways.

(b) From every such analysis $\alpha\beta\gamma\delta\ldots$ make a term of the form $(\alpha\beta\gamma\delta\ldots)\ [\alpha\beta\gamma\delta\ldots]$.

(c) Then, if the sum of all these terms is represented by S, consequently

$$1 + S = 0$$

is the final equation sought.

9. All the numerical expressions relate to the equation I, and may partly be taken from the annexed tables, and partly may be calculated by the methods given in the two first sections. Nothing now remains but to elucidate this operation by a few examples.

SECTION LXXVII.

EXAMPLE 1. Find the result of the elimination of x from the two equations

I. $x^2 - Ax + B = 0$

II. $\mathfrak{A}x^5 + \mathfrak{B}x^4 + \mathfrak{C}x^3 + \mathfrak{D}x^2 + \mathfrak{E}x + \mathfrak{F} = 0.$

First give the equation II the form $1 + (1)x + (2)x^2 + (3)x^3 + (4)x^4 + (5)x^5$; consequently put $(1) = \frac{\mathfrak{E}}{\mathfrak{F}}$, $(2) = \frac{\mathfrak{D}}{\mathfrak{F}}$, $(3) = \frac{\mathfrak{C}}{\mathfrak{F}}$, $(4) = \frac{\mathfrak{B}}{\mathfrak{F}}$, $(5) = \frac{\mathfrak{A}}{\mathfrak{F}}$. Since the equation I is here only of the second degree, the divisions need not be continued further than the second class, because the numerical expressions for higher classes do not obtain (§ LXXVI, VII). The final equation ∴ has the following form:

$$\begin{aligned} 0 = 1 &+ (1)[1] + (2)[2] + (3)[3] + (4)[4] + (5)[5] \\ &+ (1^2)[1^2] + (12)[12] + (13)[13] + (14)[14] \\ &+ (2^2)[2^2] + (23)[23] \\ &+ (15)[15] + (25)[25] + (35)[35] + (45)[45] + (5^2)[5^2] \\ &+ (24)[24] + (34)[34] + (4^2)[4^2] \\ &+ (3^2)[3^2] \end{aligned}$$

Or, when we take the numerical expressions from the tables, for the numbers (1), (2), (3), (4), (5), substitute their values, and then multiply the whole equation by $\mathfrak{F}^2$,

$$
\begin{aligned}
0 = \mathfrak{F}^2 &+ \mathfrak{E}\mathfrak{F}A + \mathfrak{D}\mathfrak{F}(A^2-2B) + \mathfrak{C}\mathfrak{F}(A^3-3AB)\\
&\qquad\quad + \mathfrak{E}^2B \qquad\qquad + \mathfrak{D}\mathfrak{E}AB\\
&+\mathfrak{B}\mathfrak{F}(A^4-4A^2B+2B^2) + \mathfrak{A}\mathfrak{F}(A^5-5A^3B+5AB^2)\\
&+\mathfrak{C}\mathfrak{E}(A^2B-2B^2) \qquad + \mathfrak{B}\mathfrak{E}(A^3B-3AB^2)\\
&+\mathfrak{D}^2B^2 \qquad\qquad\qquad + \mathfrak{C}\mathfrak{D}AB^2\\
&+\mathfrak{A}\mathfrak{E}(A^4B-4A^2B^2+2B^3) + \mathfrak{A}\mathfrak{D}(A^3B^2-3AB^3)\\
&+\mathfrak{B}\mathfrak{D}(A^2B^2-2B^3) \qquad + \mathfrak{B}\mathfrak{C}AB^3\\
&+\mathfrak{C}^2B^3\\
&+\mathfrak{A}\mathfrak{C}(A^2B^2-2B^4)+\mathfrak{A}\mathfrak{B}AB^4+\mathfrak{A}^2B^5\\
&+\mathfrak{B}^2B^4
\end{aligned}
$$

EXAMPLE II. Let the two equations

$$
\begin{aligned}
&\text{I.}\quad x^3 - Ax^2 + Bx - C = 0\\
&\text{II.}\ \mathfrak{A}x^3 + \mathfrak{B}x^2 + \mathfrak{C}x + \mathfrak{D} = 0
\end{aligned}
$$

be given.

If we reduce the equation II to the form $1 + (1)x + (2)x^2 + (3)x^3$; then $(1) = \frac{\mathfrak{C}}{\mathfrak{D}}$, $(2) = \frac{\mathfrak{B}}{\mathfrak{D}}$, $(3) = \frac{\mathfrak{A}}{\mathfrak{D}}$

Since in this case the equation I is of the third degree, we do not require to continue the divisions beyond the third class. The final equation ∴ has the form

$$
\begin{aligned}
0 = 1 &+ (1)[1] + (2)[2] + (3)[3] + (13)[13] + (23)[23]\\
&+ (1^2)[1^2] + (12)[12] + (2^2)[2^2] + (1^23)[1^23]\\
&+ (1^3)[1^3] + (1^22)[1^22] + (12^2)[12^2]\\
&+ (3^2)\ [3^2] + (13^2)[13^2] + (23^2)[23^2] + (3^3)[3^3]\\
&+ (123)[123] + (2^23)[2^23]\\
&+ (2^3)\ [2^3]
\end{aligned}
$$

or, when we substitute for the numerical expressions and for the numbers (1), (2), (3), their values, and then multiply by $\mathfrak{D}^3$

$$\begin{array}{lll}
0=\mathfrak{D}^3+\mathfrak{C}\mathfrak{D}^2A & +\mathfrak{B}\mathfrak{D}^2(A^2-2B) & +\mathfrak{A}\mathfrak{D}^2(A^3-3AB+3C) \\
 & +\mathfrak{C}^2\mathfrak{D}B & +\mathfrak{B}\mathfrak{C}\mathfrak{D}\,(AB-3C) \\
 & & +\mathfrak{C}^3C
\end{array}$$

$$\begin{array}{ll}
+\mathfrak{A}\mathfrak{C}\mathfrak{D}(A^2B-2B^2-AC) & +\mathfrak{A}\mathfrak{B}\mathfrak{D}(AB^2-2A^2C-BC) \\
+\mathfrak{B}^2\mathfrak{D}\,(B^2-2AC) & +\mathfrak{A}\mathfrak{C}^2\,(A^2C-2BC) \\
+\mathfrak{B}\mathfrak{C}^2AC & +\mathfrak{B}^2\mathfrak{C}BC
\end{array}$$

$$\begin{array}{ll}
+\mathfrak{A}^2\mathfrak{D}(B^3-3ABC+3C^2) & +\mathfrak{A}^2\mathfrak{C}(B^2C-2AC^2) \\
+\mathfrak{A}\mathfrak{B}\mathfrak{C}(ABC-3C^2) & +\mathfrak{A}\mathfrak{B}^2AC^2 \\
+\mathfrak{B}^3C^2 &
\end{array}$$

$$+\mathfrak{A}^2\mathfrak{B}BC^2+\mathfrak{A}^3C^3$$

In order to show the use of this formula in a particular case, I shall assume that the two equations

$$x^3-2ax^2+4ayx-y^3=0$$
$$ax^2+y^2x-ay^2=0$$

are given. If we equate these with the equations I, II, we find $A=2a$, $B=4ay$, $C=y^3$, $\mathfrak{A}=0$, $\mathfrak{B}=a$, $\mathfrak{C}=y^2$, $\mathfrak{D}=-ay^2$. Since here $\mathfrak{A}=0$, the foregoing equation is reduced to

$$\begin{array}{lll}
0=\mathfrak{D}^3+\mathfrak{C}\mathfrak{D}^2A & +\mathfrak{B}\mathfrak{D}^2(A^2-2B) & +\mathfrak{B}\mathfrak{C}\mathfrak{D}(AB-3C) \\
 & +\mathfrak{C}^2\mathfrak{D}B & +\mathfrak{C}^3C \\
\multicolumn{3}{l}{\quad +\mathfrak{B}^2\mathfrak{D}\,(B^2-2AC)+\mathfrak{B}^2\mathfrak{C}BC+\mathfrak{B}^3C^2} \\
\multicolumn{3}{l}{\quad +\mathfrak{B}^2\mathfrak{C}AC}
\end{array}$$

and this equation obtains for every case, in which one of the given equations is of the third, and the other of the second degree. If in these we make the requisite substitutions, we obtain the required final equation

$$y^9+a^2y^7+6a^3y^6-12a^4y^5-12a^5y^4=0$$

or

$$y^5+a^2y^3+6a^3y^2-12a^4y-12a^5=0.$$

SECTION LXXVIII.

The method to find the final equation by means of the symmetrical functions, originated with Euler, who first made use of it in the Memoirs of the Berlin Academy for the year 1748, and applied it to a few easy examples. A short time after this, Cramer, in the second Appendix to his "Introduction à l'Analyse des Lignes Courbes," p. 660, &c. 1750, by a suitable method of notation (which I have partly adopted), made it more general, and the operation more easy. Both these great men chiefly endeavoured to prove by it, that two lines, one of which is of the mth, the other of the nth order, can be cut in no more than mn points. The proof of this does not belong to this work, but to the higher branches of geometry. It is quite enough to know, that this solely depends on the following rule, and is an easy and immediate consequence of it.

SECTION LXXIX.

Rule. When in the two equations

$$\text{I.}\quad x^m+\overset{1}{A}x^{m-1}+\overset{2}{A}x^{m-2}+\overset{3}{A}x^{m-3}+\ldots\ldots+\overset{m}{A}=0$$

$$\text{II.}\quad x^n+\overset{1}{A}{}'x^{n-1}+\overset{2}{A}{}'x^{n-2}+\overset{3}{A}{}'x^{n-3}+\ldots\ldots+\overset{n}{A}{}'=0$$

between two magnitudes x, y, the coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, ... $\overset{m}{A}$, $\overset{1}{A}{}'$, $\overset{2}{A}{}'$, $\overset{3}{A}{}'$ ... $\overset{n}{A}{}'$, are merely whole rational functions of y, and that $\overset{1}{A}$ and $\overset{1}{A}{}'$ are of the first degree, $\overset{2}{A}$ and $\overset{2}{A}{}'$ of the second, $\overset{3}{A}$ and $\overset{3}{A}{}'$ of the third, and so on; then the final equation in y, which arises from the elimination of x, can never exceed the degree mn.

PROOF 1. Let x', x'', x''', &c. be the roots of the equation I, then is $-[1]=\overset{1}{A}$, $[2]=\overset{1}{A}{}^2-2\overset{2}{A}$, $-[3]=\overset{1}{A}{}^3-3\overset{1}{A}\overset{2}{A}+3\overset{3}{A}$, and so on; from which we see, that [1] contains no other power of y than y^1, [2] no higher one than y^2, [3] no higher one than y^3; and it may also be satisfactorily proved with little trouble, from the nature of the formulæ, § VIII, that in the supposed nature of the coefficients $\overset{1}{A}$, $\overset{2}{A}$, &c. the numerical expression (μ) generally contains no higher power of y than y^μ.

2. Further, no expression of the form ($\alpha\beta\gamma\delta$ ...) can contain any higher power of y, than $y^{\alpha+\beta+\gamma+\delta}$.... The accuracy of this assertion appears from 1, together with the remark in § XXIV.

3. From § LXXV the first part of the final equation (the other=o) is the product of the m factors

$$x'^n+\overset{1}{A'}x'^{n-1}+\ldots\ldots+\overset{n-\mu}{A'}x'^\mu+\ldots\ldots+\overset{n}{A'}$$
$$x''^n+\overset{1}{A'}x''^{n-1}+\ldots\ldots+\overset{n-\nu}{A'}x''^\nu+\ldots\ldots+\overset{n}{A'}$$
$$x'''^n+\overset{1}{A'}x'''^{n-1}+\ldots\ldots+\overset{n-\pi}{A'}x'''^\pi+\ldots\ldots+\overset{n}{A'}$$

&c.

after we have eliminated in it the roots x', x'', x''', &c. by means of the coefficients of the equation I.

4. The general term of this product is

$$\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'},\ \text{\&c.}\ x'^\mu x''^\nu x'''^\pi\ldots\ldots$$

Now, since the product must be a symmetrical function of x', x'', x''', &c. this term necessarily belongs to

$$\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}\ \&c.\ [\mu\nu\pi\ldots\ldots]$$

5. But by 2, the highest exponent of y in $[\mu\nu\pi\ldots]$ is equal to the sum of the radical exponents $\mu+\nu+\pi+\ldots$; further, by the hypothesis, $n-\mu$ is the highest exponent of y in $\overset{n-\mu}{A'}$, $n-\nu$ the highest in $\overset{n-\nu}{A'}$, $n-\pi$ the highest in $\overset{n-\pi}{A'}$, &c. $\therefore$ $mn-(\mu+\nu+\pi+$ &c.) the highest in the product $\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}$ &c. But from both it follows, that mn is the highest exponent of y in the term $\overset{n-\mu}{A'}\ \overset{n-\nu}{A'}\ \overset{n-\pi}{A'}$ &c. $[\mu\nu\pi\ldots]$, and that consequently in this term there can be no higher power of y than y^{mn}.

6. Now, since what has been here proved of an indeterminate term, obtains for every term in particular, it follows $\therefore$ that in the final equation there can be no higher power of y than y^{mn}.

SECTION LXXX.

When more than two equations with more than two unknown magnitudes are given, then in general there is no other way but to combine these equations in the usual way, two and two, and thus get rid of one unknown magnitude after the other. But Bezout observes, with truth, in the above-mentioned work, that this method is very defective, because a number of useless factors enters into the successive eliminations, by which not only the operation is lengthened, but likewise the degree of the final

equation becomes much higher than it ought to be, and what is much more objectionable is, that these factors do not show themselves till the calculation is completed. Since, however, these difficulties can be got rid of even by Bezout's method in no other way than by considering a great number of single cases, (but neither the object nor the limits of this work allow of such a detail as this), I shall consequently not enter into these inquiries at present, but leave them for consideration till a future period.

Tschirnhausen, in the "Acta Eruditorum," for the year 1683, has given a method for solving those equations which are founded solely on eliminations. This method only requires to transform the given equation, by means of an assumed auxiliary one, into another, which contains any number of indeterminate magnitudes, by the proper determination of which it is possible to remove as many terms as we please, and by that means give it the form of an equation of two terms, of a quadratic, of a cubic, or of any other equation, whose solution is already known, or may be considered as known. Its inventor considered it as general, and so it is indeed; only its application often requires the solution of higher equations than the given one itself. The following problems will elucidate what has been just said.

SECTION LXXXI.

Prob. Let the two equations

$$\text{I.}\quad x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \&c. = 0$$

$$\text{II.}\quad y + A + Bx + Cx^2 + Dx^3 + \&c. = 0$$

be given, in which the coefficients $a, b, c,$ &c. $A, B, C,$ &c.

contain neither x nor y: determine the degree of the final equation, which is obtained by the elimination of x, in terms of y.

Solution. Let x', x'', x''', &c. be the roots of the equation I; then, according to § LXXV, the first part of the final equation (the other $= o$) is the product of the following m factors:

$$y + A + Bx' + Cx'^2 + Dx'^3 + \&c.$$
$$y + A + Bx'' + Cx''^2 + Dx''^3 + \&c.$$
$$y + A + Bx''' + Cx'''^2 + Dx'''^3 + \&c.$$
$$\&c.$$

Now, since in these factors y occurs in no other term but the first, consequently in the product there can be no higher power of y than y^m. The equation in terms of y is $\therefore$ necessarily of the mth degree, and consequently always of the same degree as the equation I, of which degree besides the equation II may also be.

Corollary. If $\therefore$ an equation

$$x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \&c. = o$$

be given, we can transform it into another of the same degree in numberless ways. For this purpose, we only need assume any auxiliary equation of the form

$$y + A + Bx + Cx^2 + Dx^3 + \&c.$$

and eliminate x from both equations. Now, since both the degree and the coefficients of the auxiliary equation are undetermined, we can always determine both, in the way required, by the form which we have determined on giving to the transformed equation. Thus, if we wish to

reduce the equation of the second degree $x^2 + ax + b = o$ to the form $y^2 + K = o$, we assume the auxiliary equation $y + A + x = o$. By eliminating x from this and the given equation, we get $y^2 + (2A - a)y + A^2 - aA + b = o$. Now, since the second term vanishes, we have $2A - a = o$, and $A = \frac{1}{2}a$; and by this value of A, the auxiliary equation is transformed into $y + \frac{1}{2}a + x = o$, and the transformed one into $y^2 - \frac{1}{4}a^2 + b = o$. The last gives $y = \pm \sqrt{(\frac{1}{4}a^2 - b)}$, and if this value be substituted in the first, we then obtain $x = -\frac{1}{2}a \mp \sqrt{(\frac{1}{4}a^2 - b)}$, as required.

SECTION LXXXII.

Prob. Reduce the general equation of the third degree

$$x^3 + ax^2 + bx + c = o$$

to an equation of the form $y^3 + K = o$.

Solution 1. Since the transformed equation is to have the form $y^3 + K = o$, in which the second and third term vanish, we must then assume an auxiliary equation with two indeterminate coefficients, in order, after having performed the elimination, to determine this one, in such a way as this condition requires. Let ∴

$$y + A + Bx + x^2 = o$$

be this auxiliary equation.

2. Now, in order to eliminate x from the two equations

$$\text{I.} \quad x^3 + ax^2 + bx + c = o$$

$$\text{II.} \quad y + A + Bx + x^2 = o$$

equate these with the equations I, II, in the second example, § LXXVII. This gives $A = -a$, $B = b$, $C = -c$, $A' = o$, $B' = 1$, $C' = B$, $D' = y + A$. Consequently the final equation in the above § is transformed into the following:

$$(y + A)^3 - (aB - a^2 + 2b)(y + A)^2 +$$
$$(bB^2 + (3c - ab)B + b^2 - 2ac)(y + A) - cB^3 + acB^2 - bcB$$
$$+ c^2 = o$$

3. If in this equation we solve the powers of $y + A$, we then obtain an equation of the form

$$y^3 + Hy^2 + Iy + K = o$$

and

$$H = 3A - aB + a^2 - 2b$$
$$I = 3A^2 - 2A(aB - a^2 + 2b) + bB^2$$
$$+ (3c - ab)B + b^2 - 2ac$$
$$K = A^3 - aA^2B + bAB^2 - cB^3 + (a^2 - 2b)A^2 +$$
$$(3c - ab)AB + acB^2 + (b^2 - 2ac)A - bcB + c^2$$

4. If we wish to reduce this equation to $y^3 + K = o$, we must put H and $I = o$; this gives the two equations

$$3A - aB + a^2 - 2b = o$$
$$3A^2 - 2A(aB - a^2 + 2b) + bB^2 + (3c - ab)B$$
$$+ b^2 - 2ac = o$$

Since the first of these equations is of the first, and the second of the second degree, in reference to A and B, we consequently can find these coefficients by the solution of an equation of the second degree, and the substitution of

these values in the expression for K gives K, and at the same time the reduced equation $y^3 + K = 0$.

Corollary. Having found the reduced equation, we are also enabled to find the roots of the given equation. Thus, from $y^3 + K = 0$, we obtain, when 1, α, β, denote the cube roots of unity, $y = -\sqrt[3]{K}$, $y = -\alpha\sqrt[3]{K}$, $y = -\beta\sqrt[3]{K}$. If these values of y, together with those of A and B, be substituted in the auxiliary equation, we then obtain the required value of x by the solution of equations of the second degree.

REMARK. Since the values of A and B depend on equations of the second degree, strictly speaking, we get 2×2 corresponding values of these magnitudes. Now, since every two corresponding values may be combined with each of the three values of y, these substitutions give six different equations of the second degree. Every one of these gives two values of x, and consequently we obtain generally twelve values of x, although the given equation can have no more than three roots.

We must, however, keep in mind, that only those values of x may be assumed, which verify at the same time the two equations I, II. In order to find these values, we need only $\therefore$ seek the common divisor of $x^3 + ax^2 + bx + c$ and $x^2 + Bx + A + y$ in the usual way. The division of the first expression by the second, gives the quotient $x + a - B$, and the remainder

$$(B^2 - aB + b - A - y)\,x + (B - a)\,(A + y) + c.$$

This remainder must vanish. We have consequently

$$(B^2 - aB + b - A - y)\,x + (B-a)\,(A+y) + c = o$$

and hence we obtain

$$x = -\frac{(B-a)\,(A+y)+c}{B^2 - aB + b - A - y}$$

If in this equation we substitute for A, B, their values from the equations in 4, also for y its three values successively, $-\sqrt[3]{K}, -\alpha\sqrt[3]{K}, -\beta\sqrt[3]{K}$, we obtain three values of x, which are the roots of the given equation. Moreover, it matters not, in this case, which we use of the corresponding values of A and B, because they always give the same values of x; of which we can easily persuade ourselves by actual calculation.

SECTION LXXXII.

PROB. Transform the general equation of the fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = o$$

into another of the form $y^4 + Hy^2 + K = o$.

Solution 1. Since the transformed equation is to have the form $y^4 + Hy^2 + K = o$, in which two terms vanish, viz. the second and fourth, we must consequently assume an auxiliary equation, with two arbitrary magnitudes. Let ∴

$$y + A + Bx + x^2 = o$$

be this auxiliary equation.

2. In order from

$$\text{I.}\quad y + A + Bx + x^2 = o$$

$$\text{II.}\quad y^4 + ax^3 + bx^2 + cx + d = o$$

to eliminate x, it is only necessary to compare these equations with those of the first example in § LXXVII; we then find $A'=0$, $B'=1$, $C'=a$, $D'=b$, $E'=c$, $F'=d$, $A=-B$, $B=y+A$. If we make these substitutions in the final equation in the above mentioned §, it is by these means transformed into an equation of the form

$$(y+A)^4+P(y+A)^3+Q(y+A)^2+R(y+A)+S=0$$

and then

$$\begin{aligned}
P &= -aB + a^2 - 2b \\
Q &= bB^2 + (3c-ab)\,B + b^2 - 2ac + 2d \\
R &= -cB^3 + (ac-4d)\,B^2 + (3ad-bc)\,B \\
&\qquad + c^2 - 2bd \\
S &= dB^4 - adB^3 + bdB^2 - cdB + d^2
\end{aligned}$$

3. If we arrange this equation according to y, we obtain

$$\begin{aligned}
&y^4 + (4A + P)y^3 + (6A^2 + 3PA + Q)y^2 \\
&\quad + (4A^3 + 3PA^2 + 2QA + R)y \\
&\quad + A^4 + PA^3 + QA^2 + RA + S = 0
\end{aligned}$$

in which any two terms may be eliminated at pleasure, by merely determining the letters A, B, conformably to it.

4. Now, in order, as the problem requires, to eliminate the second and fourth terms, we put

$$\begin{aligned}
&4A + P = 0 \\
&4A^3 + 3PA^2 + 2QA + R = 0
\end{aligned}$$

5. The first gives $A=-\frac{P}{4}$, and if we introduce this

value into the other equation, by omitting the fractions, we have

$$p^3 - 4PQ + 8R = 0$$

If in this equation we make use of the above values of P, Q, R, we then obtain for B an equation of the third degree, viz.

$$(4ab - a^3 - 8c)B^3 + (3a^4 - 14a^2b + 20ac + 8b^2 - 32d)\,B^2$$
$$+ (-3a^5 + 16a^3b - 16ab^2 - 20a^2c + 32ad + 16bc)\,B$$
$$+ a^6 - 6a^4b + 8a^3c - 8a^2d + 8a^2b^2 - 16abc + 8c^2$$
$$= 0$$

Having determined B from this equation, we need only substitute its value in the above expressions for P, Q, R, S, in order to find these coefficients also.

6. Further, the equation in 3, by putting $-\frac{P}{4}$ for A is transformed into

$$y^4 - \left(\frac{3P^2}{8} - Q\right)y^2 - \frac{3P^4}{256} + \frac{QP^2}{16} - \frac{RP}{4} + S = 0$$

or, when we substitute for R its values $\frac{PQ}{2} - \frac{P^3}{8}$ into

$$y^4 - \left(\frac{3P^2}{8} - Q\right)y^2 + \frac{5P^4}{256} - \frac{QP^2}{16} + S = 0;$$

and this equation has the form $y^4 + Hy^2 + K = 0$, as required.

Corollary. Now from this transformed equation we may find the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ in a similar way as in the foregoing § for the equation of the third degree. Thus, since the equation in 5 gives

three values for B, and the substitution of each of these values in the transformed equation gives four values of y, we $\therefore$ obtain generally twelve values for y. Each of these values of y, together with that of B when substituted in the auxiliary equation $x^2 + Bx + A + y = 0$, or $x^2 + Bx - \frac{P}{4} + y = 0$, gives two values for x, and we $\therefore$ obtain generally twenty-four values of x. Now, in order to learn which of these values are at the same time roots of the given equation, we must seek the common divisors of the two expressions $x^4 + ax^3 + bx^2 + cx + d$, $x^2 + Bx - \frac{P}{4} + y$. With this view, we divide the first expression by the last, until we come to a remainder, which contains x in the first power only; this remainder must $\therefore$ be $= 0$. In this way we obtain the equation

$$\left[B^3 - aB^2 + bB - c - (a - 2B)\left(y - \frac{P}{4}\right)\right]x$$

$$+ (B^2 - aB + b)\left(y - \frac{P}{4}\right) - \left(y - \frac{P}{4}\right)^2 - d = 0$$

and hence

$$x = \frac{d - (B^2 - aB + b)\left(y - \frac{P}{4}\right) + \left(y - \frac{P}{4}\right)^2}{B^3 - aB^2 + bB - c - (a - 2B)\left(y - \frac{P}{4}\right)}$$

Now, if we substitute in this expression of x for y its four values from the transformed equation, we thus obtain the roots of the given equation, and indeed we shall always find the same four values for x, whichever value of B we make use of.

REMARK. From this and the two foregoing sections we deduce at least this much, that Tschirnhausen's method leads to the actual solution of equations of the second, third, and fourth degree, although in a very laborious way. Whether, and how far this method is also applicable to higher degrees, will be the subject of inquiry hereafter.

V.—On the roots of the equation $x^n-1=0$, and its application to the elimination of surds from equations. A method, by which to find solvable equations, and some other subjects connected with it.

SECTION LXXXIV.

Prob. Find an equation, which merely contains the imaginary roots of the equation $x^n-1=0$.

Solution. Here two cases must be taken into consideration, viz. first, when n is an odd; secondly, when n is an even number.

1. Let n be an odd number. In this case there is no more than one real root, viz. $+1$, and consequently $x-1$ must be a divisor of x^n-1. If ∴ we divide the equation $x^n-1=0$ by $x-1$, and make the quotient $=0$, we obtain an equation which only contains the imaginary roots, and this is

$$x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots+x^2+x+1=0$$

2. Let n be an even number, ∴ the given equation is of the form $x^{2m}-1=0$. In this case it has necessarily two real roots, viz. $+1$ and -1, and no more. Consequently both $x-1$ and $x+1$ must be divisors of x^{n-1},

$\therefore$ also the product $(x-1)\ (x+1)=x^2-1$. If $\therefore$ we divide the equation $x^n-1=0$ by x^2-1, we thus obtain an equation, which only contains the imaginary roots, and this is

$$x^{n-2}+x^{n-4}+x^{n-6}+\ldots+x^4+x^2+1=0$$

Corollary. In order $\therefore$ to find all the roots of the equation $x^n-1=0$, we must, when n is an odd number, endeavour to solve the equation $x^{n-1}+x^{n-2}+\ldots+x+1=0$, and when n is an even number, the equation $x^{n-2}+x^{n-4}+\ldots+x^2+1=0$. The latter, because it only contains even powers of x, may always be reduced to an equation of the degree $\frac{n-2}{2}$ by substituting y for x^2.

Example I. The equation $x^3-1=0$ divided by $x-1$, gives

$$x^2+x+1=0$$

and when solved, $x=\frac{-1\pm\sqrt{-3}}{2}$. The three roots of this equation are consequently

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}$$

Example II. The equation $x^4-1=0$ divided by x^2-1, gives

$$x^2+1=0$$

whence we obtain $x=\pm\sqrt{-1}$. The four roots of the equation $x^4-1=0$ are consequently

$$+1,\ -1,\ +\sqrt{-1},\ -\sqrt{-1}$$

Example III. The equation $x^5-1=0$ divided by $x-1$, gives

$$x^4+x^3+x^2+x+1=0.$$

This equation may be analyzed into two quadratic equations

$$x^2+(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})\,x+1=0$$
$$x^2+(\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5})\,x+1=0$$

and the solution of these two equations gives the four following imaginary rooots:

$$\tfrac{1}{4}\left[-1-\sqrt{5}+\sqrt{(10-2\sqrt{5})}\,\sqrt{-1}\right]$$
$$\tfrac{1}{4}\left[-1-\sqrt{5}-\sqrt{(10-2\sqrt{5})}\,\sqrt{-1}\right]$$
$$\tfrac{1}{4}\left[-1+\sqrt{5}+\sqrt{(10+2\sqrt{5})}\,\sqrt{-1}\right]$$
$$\tfrac{1}{4}\left[-1+\sqrt{5}-\sqrt{(10+2\sqrt{5})}\,\sqrt{-1}\right]$$

Example IV. The equation $x^6-1=0$ divided by x^2-1, gives

$$x^4+x^2+1=0$$

and the solution of this equation gives

$$x=\pm\sqrt{\frac{-1\pm\sqrt{-3}}{2}}$$

The six roots of the equation $x^6-1=0$ are $\therefore$

$$+1 \qquad\qquad -1$$
$$+\sqrt{\frac{-1+\sqrt{-3}}{2}},\quad -\sqrt{\frac{-1+\sqrt{-3}}{2}}$$
$$+\sqrt{\frac{-1-\sqrt{-3}}{2}},\quad -\sqrt{\frac{-1-\sqrt{-3}}{2}}$$

SECTION LXXXV.

Prob. Reduce the equation $x^n-k=0$ to an equation of the form $y^n-1=0$.

Solution. Put $x=y\sqrt[n]{k}$, and substitute this value in the equation $x^n-k=0$, then this equation is transformed into $ky^n-k=0$, or when divided by k, into $y^n-1=0$.

Corollary. If ∴ we have in any way already solved the equation $y^n-1=0$, and denote by $1, \alpha, \beta, \gamma, \delta, \varepsilon$, &c. its n roots, or the value of y, we then obtain from $x=y\sqrt[n]{k}$ the n following roots of the equation $x^n-k=0$:

$$\sqrt[n]{k},\ \alpha\sqrt[n]{k},\ \beta\sqrt[n]{k},\ \gamma\sqrt[n]{k},\ \delta\sqrt[n]{k},\ \varepsilon\sqrt[n]{k},\ \&c.$$

SECTION LXXXVI.

PROB. Reduce the equation $x^{pq}-1=0$ to an equation of the form $y^q-1=0$.

Solution. Put $x^p=y$, then $x^{pq}=y^q$. If this value be substituted in the given equation, it is transformed into $y^q-1=0$.

Corollary. If we denote the roots of the equation $x^p-1=0$ by $1, \alpha, \beta, \gamma, \delta, \varepsilon$, &c., then the roots of the equation $x^p-y=0$ (foregoing §) are

$$\sqrt[p]{y},\ \alpha\sqrt[p]{y},\ \beta\sqrt[p]{y},\ \gamma\sqrt[p]{y},\ \delta\sqrt[p]{y},\ \varepsilon\sqrt[p]{y},\ \&c.$$

Now, since we can substitute for y each of the roots of the equation $y^q-1=0$, we obtain by these substitutions all the pq roots, of the equation $x^{pq}-1=0$.

EXAMPLE. To find the roots of the equation $x^{12}-1=0$, put $p=4$, $p=3$. We consequently have the two equations

$$x^4 - y = o, \; y^3 - 1 = o.$$

Now, the roots of the equation $x^4 - 1 = o$ (§ LXXXIV, example II) are $+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, $\therefore$ the roots of the equation $x^4 - y = o$ (foregoing §).

$$\sqrt[4]{y}, \; -\sqrt[4]{y}, \; +\sqrt{-1} \,.\, \sqrt[4]{y}, \; -\sqrt{-1} \,.\, \sqrt[4]{y}.$$

Further, the roots of the equation $y^3 - 1 = o$ (§ LXXXIV, example I).

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$$

If we successively substitute these values for y, we obtain the twelve following roots of the equation $x^{12} - 1 = o$:

$$1, \quad -1, \quad \sqrt{-1}, \quad -\sqrt{-1}$$

$$\sqrt[4]{\frac{-1+\sqrt{-3}}{2}}, -\sqrt[4]{\frac{-1+\sqrt{-3}}{2}}, \sqrt{-1} \,.\, \sqrt[4]{\frac{-1+\sqrt{-3}}{2}}, -\sqrt{-1} \,.\, \sqrt[4]{\frac{-1+\sqrt{-3}}{2}},$$

$$\sqrt[4]{\frac{-1-\sqrt{-3}}{2}}, -\sqrt[4]{\frac{-1-\sqrt{-3}}{2}}, \sqrt{-1} \,.\, \sqrt[4]{\frac{-1-\sqrt{-3}}{2}}, -\sqrt{-1} \,.\, \sqrt[4]{\frac{-1-\sqrt{-3}}{2}}.$$

SECTION LXXXVII.

PROB. Under the supposition that n is a prime number, from any one of the imaginary roots of the equation $x^n - 1 = o$, find all the remaining ones.

Solution 1. Let a denote one of the imaginary roots of the equation $x^n - 1 = o$, so that $a^n - 1 = o$, or $a^n = 1$.

2. Since $a^n = 1$, then also $(a^m)^n = (a^n)^m = 1$. If $\therefore$ a is a root of the equation $x^n - 1 = o$, then must also

a^m be one of its roots. Therefore the equation $x^n - 1 = 0$ has, besides a, the roots a^2, a^3, a^4, a^5, &c.

3. But since in this way we should find an infinite number of roots, and the equation $x^n - 1 = 0$ can only have n roots, we may safely presume, that in the series of powers a, a^2, a^3, a^4, a^5, &c. there must be an infinite number of equal roots. This likewise is really the case: for we find $a^{n+1} = a^n \cdot a = a$, $a^{n+2} = a^n \cdot a^2 = a^2$, $a^{n+3} = a^n \cdot a^3 = a^3$, &c.

4. Generally, when we exceed the nth power, we shall find only one of the n following roots

$$a, a^2, a^3, a^4, a^5, \ldots\ldots\ldots a^{n-1}, a^n$$

of which the last $= 1$. For let a^m be any power of a, and $m > n$. Further, let q denote the quotient, which we obtain after dividing m by n, and r the remainder, consequently $r < n$; then $m = nq + r$. We have $\therefore$ $a^m = a^{nq+r} = a^{nq} \cdot a^r = (a^n)^q \cdot a^r = 1^q \cdot a^r = a^r$. But a^r, since $r < n$, is necessarily one of the powers a, a^2, a^3, a^4, ... a^{n-1}.

5. The conclusions drawn hitherto obtain, whether n be a prime number or not. In the particular case, when n is a prime number, according to the supposition in the problem, it may be proved, that the roots $a, a^2, a^3, \ldots\ldots a^n$, are all different from one another. For we suppose two of these roots a^μ, a^ν, to be equal, and $\nu > \mu$. Then we divide the equation $a^\nu = a^\mu$ by a^μ, and obtain $a^{\nu-\mu} = 1$; but it may be shown, as follows, that this equation is impossible.

6. Thus, since n is a prime number, and $\nu - \mu < n$, the numbers $\nu - \mu$ and n are ∴ prime to one another. Consequently, as is already known from indeterminate analysis, two whole positive numbers t, u, may always be found, such that $(\nu - \mu)\, t = nu + 1$. If ∴ $a^{\nu-\mu} = 1$, then also must $a^{(\nu-\mu)t} = 1$, and consequently also $a^{nu+1} = 1$, or $a^{nu} \cdot a = 1$, or $a = 1$; which is impossible (1).

7. Since ∴ the roots $a, a^2, a^3, a^4, \ldots\ldots a^n$, as far as the number n, are all different from one another, then these are the n roots of the equation $x^n - 1 = 0$. If ∴ an imaginary root a be given, we then have likewise all the remaining ones.

Corollary. If ∴ we denote the imaginary roots of the equation $x^n - 1 = 0$ by a, β, γ, δ, &c., then, when n is a prime number, all the roots of this equation may be represented in one or other of the following ways:

either by $a, a^2, a^3 \ldots\ldots\ldots a^{n-1}, a^n$
or by $\beta, \beta^2, \beta^3 \ldots\ldots\ldots \beta^{n-1}, \beta^n$
or by $\gamma, \gamma^2, \gamma^3 \ldots\ldots\ldots \gamma^{n-1}, \gamma^n$
&c.

or, which is the same, we can substitute in the series of roots $a, a^2, a^3, \ldots\ldots a^{n-1}, a^n$, for a each imaginary root $a^2, a^3, a^4 \ldots\ldots a^{n-1}$, and then we shall always obtain the same n roots.

Example. When $n = 3$, the roots are a, a^2, a^3. If for the root a, we substitute the following one a^2, we

obtain a^2, a^4, a^6. But since $a^3 = 1$, then $a^4 = a$, and $a^6 = a^3$, and we have ∴ here a^2, a, a^3, as before. When $n = 5$, the roots are a, a^2, a^3, a^4, a^5. If we put a^2 for a, then, on the contrary, we have a^2, a^4, a^6, a^8, a^{10}, or, since $a^3 = 1$, a^2, a^4, a, a^3, a^5; consequently the same roots as before. In like manner, when a^3 is put for a, we find a^3, a^6, a^9, a^{12}, a^{15}, or a^3, a, a^4, a^2, a^5, and when a^4 is put for a, a^4, a^8, a^{12}, a^{16}, a^{20}, or a^4, a^3, a^2, a, a^5; consequently always the same roots, only in a different order.

SECTION LXXXVIII.

Rules.

I. When n is divisible by m, then all the roots of the equation $x^m - 1 = 0$, must also be roots of the equation $x^n - 1 = 0$.

Proof. Since n is divisible by m, then $\frac{n}{m} = q$ is a whole number, and $n = qm$. The equation $x^n - 1 = 0$ is consequently $x^{qm} - 1 = 0$, and if we put $x^m = y$, $y^q - 1 = 0$. Now, $y^q - 1 = 0$ is divisible by $y - 1$; consequently also, if again x^m is put for y, $x^{qm} - 1$ is divisible by $x^m - 1$; ∴ the roots of the equation $x^m - 1 = 0$ are also roots of the equation $x^{qm} - 1 = 0$, or $x^n - 1 = 0$. Q. E. D.

II. When a root (unity excepted) of the equation $x^n - 1 = 0$, which is also a root of the equation $x^m - 1 = 0$, is of a low degree, and that of the very lowest possible, then n must be divisible by m.

Proof. Let a be the common root, consequently $a^n - 1 = o$, and $a^m - 1 = o$. Now, if n be not divisible by m, then n divided by m gives the quotient q and the remainder r, so that $n = qm + r$, and $r < m$. Then $a^n = a^{qm+r} = a^{qm} \cdot a^r$. But $a^n = 1$, $a^{qm} = (a^m)^q = 1$; $\therefore 1 = a^r$; consequently a is also a root of the equation $x^r - 1 = o$, which is contrary to the hypothesis, that $x^m - 1 = o$ is the lowest equation, which contains the root a.

III. When m, n are two numbers, which have no common measure, then the equations $x^m - 1 = o$, $x^n - 1 = o$ have no common root, except unity.

Proof. If possible, let the two equations have a common root a, different from unity, then, at the same time, $a^m = 1$ and $a^n = 1$. Since m and n are prime to each other, we can always find two whole positive numbers, t, u, such, that $mt = nu + 1$. We have then the equation $a^{mt} = a^{nu+1} = a^{nu} \cdot a$. But according to the hypothesis $a^m = a^n = 1$, consequently $a^{mt} = a^{nu}$: $\therefore 1 = a$; which is contrary to the hypothesis.

IV. When the two equations $x^n - 1 = o$, $x^m - 1 = o$, have another root besides unity, common to both, then the exponents m, n, must have a common measure.

Proof. For if they have no common measure, then also they can have no common root except unity (III).

SECTION LXXXIX.

Prob. Let the equation $x^n-1=0$ be given, and n be a compound number: find all those roots of this equation, which do not belong to any equation of a lower degree of the same form.

Solution 1. Let p, q, r, &c. be the simple factors of the exponent n; further, let $\frac{n}{p}=\mu, \frac{n}{q}=\mu', \frac{n}{r}=\mu''$ &c. consequently n is divisible by μ, μ', μ'', &c.

2. Construct ∴ the equations

$$x^{\mu}-1=0,\ x^{\mu'}-1=0,\ x^{\mu''}-1=0,\ \&c.$$

then these equations must have all their roots in common with the equation $x^n-1=0$ (§ LXXXVIII, I.)

3. Now, I affirm, that each root of the equation $x^n-1=0$, which also belongs to a lower equation of this form, must necessarily be a root of one of the equations in 2. For, let a be a common root of the equations $x^n-1=0$, $x^k-1=0$, and the last the lowest of this form, to which this root can belong, then k must be a divisor of n (§ LXXXVIII, II.), consequently also assuredly a divisor of one of the numbers μ, μ', μ'', &c. Therefore, all the roots of $x^k-1=0$ must be contained in one of the equations in 2; and consequently also the root a.

4. If we divide the equation $x^n-1=0$ successively by $x^{\mu}-1$, $x^{\mu'}-1$, $x^{\mu''}-1$, &c., we obtain

$$x^{n-\mu} + x^{n-2\mu} + x^{n-3\mu} + \ldots + x^{2\mu} + x^{\mu} + 1 = 0$$
$$x^{n-\mu'} + x^{n-2\mu'} + x^{n-3\mu'} + \ldots + x^{2\mu'} + x^{\mu'} + 1 = 0$$
$$x^{n-\mu''} + x^{n-2\mu''} + x^{n-3\mu''} + \ldots + x^{2\mu''} + x^{\mu''} + 1 = 0$$
&c.

The first of these equations contains all those roots of $x^n - 1 = 0$, which are not contained in $x^{\mu} - 1 = 0$; the second all those roots of $x^n - 1 = 0$, which are not contained in $x^{\mu'} - 1 = 0$; &c.

5. A root β, which is common to all these equations, cannot be found in any one of the equations $x^{\mu} - 1 = 0$, $x^{\mu'} - 1 = 0$, $x^{\mu''} - 1 = 0$, &c. and $\therefore$ cannot be a root of an equation of two terms of a lower degree than $x^n - 1 = 0$ (3).

6. If $\therefore$ we seek the greatest common divisor of the equations in 4, then this must contain only such roots as are peculiar to the equation $x^{\mu} - 1 = 0$, and belong to none of a lower degree of this form. But it is also evident, that there can be no such root wanting in the greatest common divisor, because otherwise it could not be the greatest.

Example I. Let $x^4 - 1 = 0$ be the given equation, consequently $n = 4$. Since this number has only one simple factor, viz 2, $\therefore p = 2$; consequently $\mu = \dfrac{n}{p} = 2$. If $\therefore$ we divide the equation $x^4 - 1 = 0$ by $x^2 - 1$, we obtain the equation

$$x^2 + 1 = 0$$

whose roots $+\sqrt{-1}$ and $-\sqrt{-1}$ are such, that they do not become $+1$ till raised to the fourth power.

Example II. Let $x^{12}-1=0$ be the given equation, consequently $n=12$. This number has two simple factors, viz. 2 and 3. We have $\therefore p=2$, $q=3$, and consequently $\mu=\frac{n}{p}=6$, $\mu'=\frac{n}{q}=4$. Now, if we divide $x^{12}-1=0$ by $x^{6}-1$ and $x^{4}-1$, we obtain the two equations

$$x^{6}+1=0$$
$$x^{8}+x^{4}+1=0.$$

Their greatest common divisor is

$$x^{4}-x^{2}+1=0.$$

Hence we find

$$x=\pm\sqrt{\frac{1\pm\sqrt{-3}}{2}}=\pm\frac{\sqrt{3}\pm\sqrt{-1}}{2}$$

and these four roots are peculiar to the equation $x^{12}-1=0$, because they do not become $+1$ till raised to the twelfth power.

In order to find the above roots, it is only necessary to solve the equations $x^{6}-1=0$, $x^{4}-1=0$, and to take the common roots only once. The roots of the equations $x^{4}-1=0$, $x^{6}-1=0$ are in § LXXXIV. In this way we obtain the following eight roots.

$$\pm 1,\ \pm\sqrt{-1},\ \pm\sqrt{\frac{-1+\sqrt{-3}}{2}},\ \frac{\pm\sqrt{-1-\sqrt{-3}}}{2}$$

which together with the four preceding, give the twelve roots of $x^{12}-1=0$. This mode of expressing them, is, as we see, much more simple than that in § LXXXVI.

Remark. A root which is peculiar to the equation $x^{n}-1=0$, and which consequently belongs to no equation of

a lower degree of this form, is termed a primitive root of this equation.

SECTION XC.

PROB. Let n be a compound number, and a a given primitive root: find all the roots of this equation.

Solution 1. In § LXXXVII it has been proved, that for every n, though a may be any imaginary root, the powers a, a^2, $a^3 \ldots a^n$, are always roots of the equation $x^n - 1 = 0$.

2. I affirm, then, that when, as has been here supposed, a is a primitive root, in the series of magnitudes a, a^2, $a^3, \ldots\ldots a^n$, there are no two, which are equal to one another. For if $a^\mu = a^\nu$, then $a^{\mu-\nu} = 1$, consequently a is a root of the equation $x^{\mu-\nu} - 1 = 0$; therefore the root of an equation of the form $x^n - 1 = 0$, of a lower degree than n, and consequently no primitive root, which is contrary to the hypothesis.

3. Since ∴ the magnitudes a, a^2, a^3, $a^4, \ldots\ldots a^n$ are all roots of the equation $x^n - 1 = 0$, and all different from one another, they are the n roots of this equation, which were sought.

SECTION XCI.

PROB. Let n be a compound number, and a a primitive root of the equation $x^n - 1 = 0$, ∴ a, a^2, a^3, $a^4, \ldots . a^n$

all the roots of this equation (§ XC): find a criterion by which to distinguish the primitive roots of this equation from the others.

Solution 1. If two whole numbers m, n, have a common measure, there may always be found a whole number t, which is less than n, and such, that mt is divisible by n; on the other hand, if the numbers m, n, are prime to each other, then t cannot be less than n, if mt is divisible by n.

2. Now let a^m be any one of the magnitudes a, a^2, a^3, ... a^n. If this be a root of an equation $x^t - 1 = 0$, we then must have $a^{mt} - 1 = 0$, or $a^{mt} = 1$; $\therefore$ mt must be divisible by n.

3. From this condition and from 1 it follows, that when the numbers m, n, have a common measure, there can always be found an equation $x^t - 1 = 0$ of a lower degree than the nth, of which a^m is a root; but that no such equation can be found, when m, n, are prime to each other.

4. But hence it follows, that of the powers a, a^2, a^3, a^4, ... a^n, all those, without exception, are primitive roots of the equation $x^n - 1 = 0$, whose exponents have no common measure with n; and this $\therefore$ is the criterion by which the primitive numbers may be distinguished from the others.

EXAMPLE. Amongst all the roots a, a^2, a^3, a^4, a^5, a^6

a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, of the equation $x^{12} - 1 = 0$, there are no more than the four a, a^5, a^7, a^{11}, whose exponents have no common measure with 12, and consequently which are primitive roots of this equation; and these roots can be no other than the four which were found in the second example, § LXXXIX; viz.

$$\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1}, \quad \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1}$$
$$-\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1}, -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1}$$

In order to be convinced of this, assume one of them, viz. $\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1}$, for a; by actually raising this root to the fifth, seventh, and eleventh powers, we find:

$$\begin{aligned} a &= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1} \\ a^5 &= -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{-1} \\ a^7 &= -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1} \\ a^{11} &= \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{-1} \end{aligned}$$

and these are the same as the foregoing. We should have obtained the same result, if we had put every other of the four above-mentioned roots for a. That this must be the case, may, besides, be seen without actually completing the calculation; for if in a, a^5, a^7, a^{11}, we substitute a^5, a^7, a^{11}, successively for a, and omit in the exponents the multiples of 12, we then obtain

$$\begin{array}{llll} a^5,\ a^{25},\ a^{35},\ a^{55}, & \text{or } a^5,\ a,\ a^{11},\ a^7 \\ a^7,\ a^{35},\ a^{49},\ a^{77}, & \text{or } a^7,\ a^{11},\ a,\ a^5 \\ a^{11},\ a^{55},\ a^{77},\ a^{121}, & \text{or } a^{11},\ a^7,\ a^5,\ a, \end{array}$$

∴ always the same roots, only in a different order.

SECTION XCII.

If we compare the equation $x^n - 1 = 0$ with the general equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \ldots\ldots$

$\mp Sx \pm T = 0$, $A = 0$ $B = 0$, $C = 0, \ldots\ldots\ldots S = 0$, $T = \mp 1$. If $\therefore$ we denote the roots of this equation by $\alpha, \beta, \gamma, \delta$, &c. we have

$$\alpha + \beta + \gamma + \delta + \&c. = 0$$
$$\alpha\beta + \alpha\gamma + \&c. + \beta\gamma + \beta\delta + \&c. \&c. = 0$$
$$\alpha\beta\gamma + \alpha\beta\delta + \&c. + \beta\gamma\delta + \&c. \&c. = 0$$

and so on to the product of all the roots, which is $= -1$, or $= +1$, according as n is even or odd.

Since in the two first chapters, the letters, $\alpha, \beta, \gamma, \delta$, &c. are used to denote the radical exponents, in order to prevent mistakes, I shall once for all remind my readers, that these letters, when they are in the brackets [ ], always denote, as heretofore, radical exponents, but in every other case the roots themselves. Further, in order to indicate, that a numerical expression relates exclusively to the roots of an equation of the form $x^n - 1 = 0$, I shall place a dash over the left side of the bracket thus $'[\alpha\beta\gamma\delta \ldots\ldots \kappa]$ in reference to the equation $x^n - 1 = 0$, denotes a numerical expression for the radical exponents $\alpha, \beta, \gamma, \delta, \ldots\ldots \kappa$.

SECTION XCIII.

Prob. Find the sums of the powers of the roots of the equation $x^n - 1 = 0$.

Solution 1. If we compare this equation with the general one

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots\ldots + Px + Q = 0,$$

we find $A = 0$, $B = 0$, $C = 0$, &c.; $P = 0$, $Q = -1$. Consequently, by means of the equations in 7, § VIII, we obtain

$${}^{\backprime}[1] = 0,\ {}^{\backprime}[2] = 0,\ {}^{\backprime}[3] = 0, \ldots\ldots {}^{\backprime}[n-1] = 0$$

on the other hand ${}^{\backprime}[n] = n$. In like manner we find

$${}^{\backprime}[n+1]=0,\ {}^{\backprime}[n+2]=0,\ {}^{\backprime}[n+3]=0, \ldots\ldots {}^{\backprime}[2n-1]=0$$

on the other hand ${}^{\backprime}[\overline{2n}]=n$. Generally, all those sums of powers, whose radical exponents are divisible by n, are equal to n, all the remaining ones, on the contrary, $=0$.

2. If we put $x = \frac{1}{y}$, the equation $x^n - 1 = 0$, is transformed into $\frac{1}{y^n} - 1 = 0$, or $y^n - 1 = 0$, whose roots consequently are the reciprocals of the roots of the former equation (§ X). But since the equations $x^n - 1 = 0$, $y^n - 1 = 0$, are similar to one another, we also have

$${}^{\backprime}[-1]=0,\ {}^{\backprime}[-3]=0,\ \ldots\ldots\ {}^{\backprime}[-n+1]=0,\ {}^{\backprime}[-n]=n$$

and generally, all those negative sums of powers, whose exponents are divisible by n, $=n$, all the remaining ones, on the contrary, $= 0$.

SECTION XCIV.

Prob. Find the value of ${}^{\backprime}[\alpha\beta]$.

Solution. Since, generally, for every equation $[\alpha\beta] = [\alpha]\ [\beta] - [\alpha + \beta]$, and when $\alpha = \beta$, $2[\alpha^2] = [\alpha^2] - [\overline{2\alpha}]$, so also in particular for the equation $x^n - 1 = 0$.

$${}^{\backprime}[\alpha\beta] = {}^{\backprime}[\alpha]\ {}^{\backprime}[\beta] - {}^{\backprime}[\alpha + \beta]$$

$${}^{\backprime}[\alpha^2] = \tfrac{1}{2}\,{}^{\backprime}[\alpha]^2 - \tfrac{1}{2}\,{}^{\backprime}[\overline{2\alpha}]$$

In order to determine from hence the numerical value of ${}^{\backprime}[\alpha\beta]$, there must be three different cases.

1. When $\alpha+\beta$ is divided by n, but the radical exponents α, β, are not taken singly. In this case, according to the foregoing §, $`[\alpha] = `[\beta] = 0$, $`[\alpha + \beta] = n$; consequently

$$`[\alpha\beta] = -n,$$

and when $\beta = \alpha$

$$`[\alpha^2] = -\frac{n}{2}.$$

2. When $\alpha+\beta$ is divisible by n, but at the same time also, the radical exponents α, β, are taken singly. In this case $`[\alpha] = `[\beta] = n$, and $`[\alpha+\beta] = n$; $\therefore$

$$`[\alpha\beta] = n^2 - n$$

and when $\alpha = \beta$

$$`[\alpha^2] = \frac{n^2 - n}{2}.$$

3. When $\alpha+\beta$ is not divisible by n. In this case $`[\alpha + \beta] = 0$; but likewise the product $`[\alpha]\,`[\beta] = 0$, because then α and β cannot at the same time both be divisible by n; consequently always

$$`[\alpha\beta] = 0.$$

SECTION XCV.

PROB. Required to find the value of $`[\alpha\beta\gamma]$.

Solution. Now

$$`[\alpha\beta\gamma] = `[\alpha]\,`[\beta]\,`[\gamma] - `[\alpha]\,`[\beta+\gamma] - `[\beta]\,`[\alpha+\gamma] - `[\gamma]\,`[\alpha+\beta] + 1\,.\,2\,`[\alpha+\beta+\gamma].$$

Here, then, the following cases are to be considered:

1. When $\alpha + \beta + \gamma$ is not divisible by n, then the last term of the solution of ${}^{\backprime}[\alpha\beta\gamma]$ here given $= 0$; then also all the remaining terms of this solution $= 0$, because in each of these there must be at least one numerical expression with one radical exponent not divisible by n: consequently for this case ${}^{\backprime}[\alpha\beta\gamma] = 0$.

2. If each of the magnitudes α, β, γ, be divisible by n, then each of the numerical expressions ${}^{\backprime}[\alpha]$, ${}^{\backprime}[\beta]$, ${}^{\backprime}[\gamma]$, ${}^{\backprime}[\alpha + \beta]$, ${}^{\backprime}[\alpha + \gamma]$, ${}^{\backprime}[\beta + \gamma]$, $= n$, and consequently

$$ {}^{\backprime}[\alpha\beta\gamma] = n^3 - 3n^2 + 2n. $$

3. In every other case always

$$ {}^{\backprime}[\alpha\beta\gamma] = - n^2 + 2n. $$

SECTION XCVI.

Prob. Find the value of the general numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$.

Solution 1. The last term in the development of this numerical expression is, according to the second chapter,

$$ \mp \frac{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots\ldots\ldots\ldots\ldots \nu - 1}{1 \,.\, 2 \ldots \mathfrak{a} \times 1 \,.\, 2 \ldots \mathfrak{b} \times \ldots \times 1 \,.\, 2 \ldots \mathfrak{k}} \, {}^{\backprime}(\mathfrak{a}\alpha + \mathfrak{b}\beta + \ldots + \mathfrak{k}\kappa) $$

when $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \ldots\ldots + \mathfrak{k}$ is put $= \nu$; the upper sign when ν is even, and the under one when ν is odd.

2. Now, if n be not a divisor of $\mathfrak{a}\alpha + \mathfrak{b}\beta + \mathfrak{c}\gamma + \ldots + \mathfrak{k}\kappa$, then this last term $= 0$; but at the same time also, all the remaining terms vanish, because in each of these there must be at least one numerical expression, whose radical exponent is not divisible by n, for otherwise the sum of all

the radical exponents, contrary to the supposition, must be divisible by n.

3. But if $\mathfrak{a}\alpha+\mathfrak{b}\beta+\mathfrak{c}\gamma+\ldots\ldots+\mathfrak{k}\kappa$, be divisible by n, then this last term alone is

$$\mp\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,\nu-1}{1\,.\,2\ldots\mathfrak{a}\times1\,.\,2\ldots\mathfrak{b}\times\ldots\ldots\times1\,.\,2\ldots\mathfrak{k}}.n$$

4. In order to obtain the values of the remaining terms, we can proceed as follows. Analyze the expression $\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}$ in all possible ways into combinations of two, three, and so on magnitudes, the sums of whose numbers are divisible by n, and assign to each such combination the coefficient K in 7, § XXX, when all the combinations are different; on the other hand, the coefficient K in 10, § XXX, when some of them are equal, and determine the sign as in 13. Then, without any further reference to the combinations themselves, take the aggregate of all the coefficients, which arise from the divisions into two combinations; further, the aggregate of all the coefficients which arise from the divisions into three combinations, and so on. If, then, we denote these different aggregates in their order by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, then the value of all the terms besides the last

$$=\mathfrak{A}n^2+\mathfrak{B}n^3+\mathfrak{C}n^4+\mathfrak{D}n^5+\&c.$$

5. From 3 and 4 we obtain $\therefore$ the following value of the numerical expression $[\alpha^{\mathfrak{a}}\beta^{\mathfrak{b}}\gamma^{\mathfrak{c}}\ldots\ldots\kappa^{\mathfrak{k}}]$:

$$\mp\frac{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,\nu-1}{1\,.\,2\ldots\mathfrak{a}\times1\,.\,2\ldots\mathfrak{b}\times\ldots\ldots\times1\,.\,2\times\mathfrak{k}}.n$$
$$+\mathfrak{A}n^2+\mathfrak{B}n^3+\mathfrak{C}n^4+\mathfrak{D}n^5+\&c.$$

Example. Suppose it is wished to find the value of $'[1^2 2^3 5^2]$, when $n = 6$, we then proceed according to the following scheme:

Combination $1^2 2^3 5^2$

Divisions.	*Coefficients.*
$15, \quad 12^3 5$	$-\dfrac{1\,.\,2\,.\,3\,.\,4}{1\,.\,2\,.\,3} = -4$
$2^3, \quad 1^2 5^2$	$-\dfrac{1\,.\,2 \times 1\,.\,2\,.\,3}{1.2.3 \times 1.2 \times 1\,.\,2} = -\frac{1}{2}$
$1^2 2^2, \quad 25^2$	$-\dfrac{1\,.\,2\,.\,3 \times 1\,.\,2}{1\,.\,2 \times 1\,.\,2 \times 1\,.\,2} = -\frac{5}{2}$
$15, \quad 15, \quad 2^3$	$+\dfrac{1\,.\,2}{1\,.\,2 \times 1\,.\,2\,.\,3} = +\frac{1}{6}$

We have $\therefore\ \mathfrak{A} = -4 - \frac{1}{2} - \frac{5}{2} = -6$, $\mathfrak{B} = \frac{1}{6}$, $\therefore$ since also $\mathfrak{a} = 2$, $\mathfrak{b} = 3$, $\mathfrak{c} = 2$, consequently $\nu = 7$,

$$'[1^2 2^3 5^2] = \frac{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}{1\,.\,2 \times 1\,.\,2\,.\,3 \times 1\,.\,2}\,.\,6 - 6\,.\,6^2 + \tfrac{1}{6}\,.\,6^3 = 0.$$

SECTION XCVII.

The symmetrical functions of the roots of the equation $x^n - 1 = 0$ are principally of use in eliminating irrational magnitudes from equations. To eliminate irrational magnitudes from an equation, or to make it rational, implies no more than from this equation to derive another, which only contains rational magnitudes, and is such, that the roots of the former equation are also roots of the latter. In order that it may be seen how this is effected, I shall assume that we have the equation of the first degree $x - A = 0$, in which A denotes any irrational

expression, and that we wish to find for x an equation which is free from irrational magnitudes. Since each irrational magnitude in the expression A, has more values than one, consequently the value of x has various significations, and while nothing nearer than this is determined, this value is doubtful. An equation free from irrational magnitudes which has the expression A for a root, can, on account of this very doubtful signification, neither give this nor that value which we suppose it to have, but must, at the same time, necessarily give all those different values, which this equation contains, because otherwise there would be no sufficient reason why it should give exactly this one, and not every other value likewise. Now, since the values of x are functions of the coefficients of the equation, but the coefficients, according to the condition, are rational, consequently the difference of these values cannot arise from the coefficients, but must be founded in the degree of the equation. The degree of the equation must ∴ be equal to the number of the different values, which the expression A can have. But if the condition respecting the rationality of the coefficients be omitted, then certainly equations of lower degrees may be found, which have this expression for roots, because then the coefficients themselves are undetermined.

For instance, let $x = \sqrt{k}$. The irrational magnitude $\sqrt{k}$ has two roots, viz. $+\sqrt{k}$ and $-\sqrt{k}$. The required equation must ∴ have these two values for roots, and it consequently is $(x + \sqrt{k})\ (x - \sqrt{k}) = 0$, or, by actual multiplication, $x^2 - k = 0$; a rational equation, which we

should also have obtained, if we had squared both parts of the equation $x = \sqrt{k}$.

Further, let $x = \sqrt[3]{k}$. If we denote the three roots of the equation $x^3 - 1 = 0$ by α, β, γ, then the irrational magnitude $\sqrt[3]{k}$ has the three values $\alpha\sqrt[3]{k}$, $\beta\sqrt[3]{k}$, $\gamma\sqrt[3]{k}$, and these consequently must be the roots of the required equation. It is $\therefore$

$$(x-\alpha\sqrt[3]{k})\,(x-\beta\sqrt[3]{k})\,(x-\gamma\sqrt[3]{k}) = 0.$$

By actual multiplication, we obtain

$$x^3 - {}'[1]\sqrt[3]{k}\,.\,x^2 + {}'[1^2]\sqrt[3]{k^2}\,.\,x - {}'[1^3]\,k = 0\,;$$

or, since by the preceding §, $'[1] = 0$, $'[1^2] = 0$, $'[1^3] = \frac{1\,.\,2}{1\,.\,2\,.\,3}\cdot 3 = 1$,

$$x^3 - k = 0\,;$$

a rational equation, which we should also have found, by raising both parts of the equation $x = \sqrt[3]{k}$ to the third power.

I shall now put $x = \sqrt[4]{k}$. Since $+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, are the four roots of the equation $x^4 - 1 = 0$, then $+\sqrt[4]{k}$, $-\sqrt[4]{k}$, $+\sqrt{-1}\,.\,\sqrt[4]{k}$, $-\sqrt{-1}\,.\,\sqrt[4]{k}$, are the four values of $\sqrt[4]{k}$, and the required equation is $\therefore$

$$(x-\sqrt[4]{k})(x+\sqrt[4]{k})(x-\sqrt{-1}.\sqrt[4]{k})(x+\sqrt{-1}.\sqrt[4]{k}) = 0$$

or by actually performing the multiplication

$$x^4 - k = 0,$$

as was required.

If for $\sqrt[4]{k}$ we had only taken the two values $+\sqrt[4]{k}$,

$-\sqrt[4]{k}$, and from these had formed the equation $(x-\sqrt[4]{k})$ $(x+\sqrt[4]{k}=o)$ we might have foreseen, at once, that no rational equation could be found. And this is actually the case; thus we obtain $x^2-\sqrt{k}=o$. Each of the two values of x contains the irrational magnitude $\sqrt{k}$, and since this has a two-fold value, we consequently obtain the four values of x.

Hence we perceive, at least, how we are to proceed with equations of the first degree, in order to render them rational. Thus, if $x=A$ be an equation of this kind, and we denote by A', A'', A''', &c. the different values which the irrational expression A contains, by reason of the many significations which its irrational magnitudes have, then

$$(x-A')(x-A'')(x-A''')\dots\dots\dots = o$$

is always the equation free from irrational magnitudes, which was sought.

The following problems will throw more light on this subject.

SECTION XCVIII.

PROB. Make the equation $x=\sqrt{p}+\sqrt{q}$ rational.

Solution. The irrational expression $\sqrt{p}+\sqrt{q}$ may here have four different values, according as we give the roots $\sqrt{p}$, $\sqrt{q}$, the sign $+$ or $-$, and these values are

$$+\sqrt{p}+\sqrt{q},\ -\sqrt{p}-\sqrt{q},\ +\sqrt{p}-\sqrt{q},\ -\sqrt{p}+\sqrt{q}.$$

The equation free from irrational magnitudes is consequently

$$(x-\sqrt{p}-\sqrt{q})\ (x+\sqrt{p}+\sqrt{q})$$
$$(x-\sqrt{p}+\sqrt{q})\ (x+\sqrt{p}-\sqrt{q}) = 0,$$

or, when we multiply the first and second factors together, as also the third and fourth

$$(x^2-p-q-2\sqrt{pq})\ (x^2-p-q+2\sqrt{pq}) = 0,$$

or lastly, by completing the multiplication the second time

$$x^4 - 2(p+q)\,x^2 + (p-q)^2 = 0$$

SECTION XCIX.

Prob. Make the equation $x=a\sqrt{p}+\frac{b}{\sqrt{p}}$ rational.

Solution. Since in this equation there is only one irrational magnitude, and that the square root, viz. $\sqrt{p}$, consequently x can have no more than two values, and these are

$$a\sqrt{p}+\frac{b}{\sqrt{p}},\quad -a\sqrt{p}-\frac{b}{\sqrt{p}}.$$

The rational equation is ∴

$$(x-a\sqrt{p}-\frac{b}{\sqrt{p}})\ (x+a\sqrt{p}+\frac{b}{\sqrt{p}}) = 0$$

or $$x^2-a^2p-2ab-\frac{b^2}{p}=0$$

or likewise $px^2-(ap+b)^2=0$

SECTION C.

Prob. Make the equation $x=a\sqrt[3]{p}+b\sqrt[3]{p^2}$ rational.

Solution 1. The cubic irrational magnitude $\sqrt[3]{p}$, may have three values viz.

$$a\sqrt[3]{p},\ \beta\sqrt[3]{p},\ \gamma\sqrt[3]{p},$$

when a, β, γ denote the three roots of the equation $x^3-1=0$, unity included. These three values of $\sqrt[3]{p}$ correspond to the three following values of its square $\sqrt[3]{p^2}$;

$$a^2\sqrt[3]{p^2},\ \beta^2\sqrt[3]{p^2},\ \gamma^2\sqrt[3]{p^2}.$$

Consequently x can contain no more than these three values:

$$aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2},\ \beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2},\ \gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}.$$

The rational equation is $\therefore$ of the third degree.

2. It is represented by

$$x^3 - Px^2 + Qx - R = 0$$

then

$$\begin{aligned}
P &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})+(\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2}) \\
&\quad + (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}) \\
&= {}'[1]a\sqrt[3]{p}+{}'[2]b\sqrt[3]{p^2} \\
Q &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2})+ \\
&\quad (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2})+ \\
&\quad (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2})\ (\gamma a\sqrt[3]{p}+\gamma^2b\sqrt[3]{p^2}) \\
&= {}'[1^2]a^2\sqrt[3]{p^2}+{}'[12]abp+{}'[2^2]\,.\,b^2p\sqrt[3]{p} \\
R &= (aa\sqrt[3]{p}+a^2b\sqrt[3]{p^2})\ (\beta a\sqrt[3]{p}+\beta^2b\sqrt[3]{p^2}) \\
&\quad (\gamma a\sqrt[3]{p}+\gamma^2\sqrt[3]{p^2}) \\
&= {}'[1^3]a^3p+{}'[1^22]a^2bp\sqrt[3]{p}+{}'[12^2]ab^2p\sqrt[3]{p^2} \\
&\quad + {}'[2^3]\,b^3p^2
\end{aligned}$$

3. Here, then, the numerical expressions $'[1]$, $'[2]$, $'[1^2]$, $'[2^2]$, $'[1^2 2]$, $'[1 2^2]$, $'[12]$, $'[1^3]$, $[2^3]$ occur, the first six of which vanish, by 2, § XCV. Further, by the same §, since here $n=3$, $'[12]=-3$, $'[1^3]=1$, $'[2^3]=1$. By the substitution of these values in the expressions for P, Q, R, we obtain $P=0$, $Q=-3abp$, $R=a^3p+b^3p^2$, and hence the required rational equation

$$x^3 - 3abpx - a^3p - b^3p^2 = 0.$$

Remark. We could essentially have shortened the calculation for determining the values of P, Q, R, by omitting at once all those terms in which p is included under the radical sign, because it might have been foreseen that they vanish in the results, as the required equation must contain no irrational magnitudes.

SECTION CI.

The problem in the preceding § leads immediately to the solution of equations of the third degree. For since the supposed roots $\alpha a\sqrt[3]{p}+\alpha^2 b\sqrt[3]{p^2}$, $\beta a\sqrt[3]{p}+\beta^2 b\sqrt[3]{p^2}$, $\gamma a\sqrt[3]{p}+\gamma^2 b\sqrt[3]{p^2}$ led to the equation $x^3-3abpx-a^3p-b^3p^2=0$, it may be inferred conversely, that every equation of this form must have these three roots. If we put $p=1$, then this equation is transformed into

$$x^3 - 3abx - a^3 - b^3 = 0$$

and the three roots of this equation are consequently

$$\alpha a + \alpha^2 b,\ \beta a + \beta^2 b,\ \gamma a + \gamma^2 b.$$

Since one of the three roots α, β, γ must be equal to

unity, we may ∵. put $\gamma=1$; farther, if $\alpha^2=\beta$, $\beta^2=\alpha$; the three roots of the equation $x^3-3abx-a^3-b^3=0$ consequently assume the following form:

$$\alpha a+\beta b,\ \ \beta a+\alpha b,\ \ a+b.$$

We obtain also precisely the same result from Cardan's Formula, which, as is already known, tries to reduce the given equation to the form $x^3-3abx-a^3-b^3=0$.

SECTION CII.

The problem, § C, may also be solved by another method. Thus put $\sqrt[3]{p}=y$, then $\sqrt[3]{p^2}=y^2$, ∴ $x=ay+by^2$: the value of y determines that of x. But y has three values, viz. $\alpha\sqrt[3]{p}$, $\beta\sqrt[3]{p}$, $\gamma\sqrt[3]{p}$, which are all comprehended in the equation $y^3-p=0$; consequently the values of x must be the result of the elimination of y in the two equations

$$\text{I. } y^3-p=0$$
$$\text{II. } x-ay-by^2=0$$

In order to perform this elimination, we give, by § LXXVI, to the equation II. the form $1+(1)y+(2)y^2=0$, so that $(1)=-\frac{a}{x}$, $(2)=-\frac{b}{x}$, and we then obtain the following equation:

$$\begin{aligned}0=1+(1)[1]+(2)[2]+(12)[12]+(2^2)[2^2]\\ +(1^2)[1^2]+(1^3)[1^3]+(1^22)[1^22]\\ +(12^2)[12^2]+(2^3)[2^3]\end{aligned}$$

The numerical expressions may be taken from the annexed tables, if we put $A=0$, $B=0$, $C=p$. If after this we

substitute again for (1), (2), their values $-\frac{a}{x}$, $-\frac{b}{x}$, we then obtain the equation $x^3-3abpx-b^3p-a^3p^2=0$, as in § C.

SECTION CIII.

PROB. Make the equation $x=a\sqrt[3]{p}+b\sqrt[3]{q}$ rational.

Solution 1. The cubic irrational magnitude $\sqrt[3]{p}$ has three values, viz. $\alpha\sqrt[3]{p}$, $\beta\sqrt[3]{p}$, $\gamma\sqrt[3]{p}$. In like manner $\sqrt[3]{q}$ contains the values $\alpha\sqrt[3]{q}$, $\beta\sqrt[3]{q}$, $\gamma\sqrt[3]{q}$. Each of the first may be combined with each of the latter, and this gives nine values of x, viz.

$$\begin{array}{lll} \alpha a\sqrt[3]{p}+\alpha b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\alpha b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\alpha b\sqrt[3]{q} \\ \alpha a\sqrt[3]{p}+\beta b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\beta b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\beta b\sqrt[3]{q} \\ \alpha a\sqrt[3]{p}+\gamma b\sqrt[3]{q}, & \beta a\sqrt[3]{p}+\gamma b\sqrt[3]{q}, & \gamma a\sqrt[3]{p}+\gamma b\sqrt[3]{q} \end{array}$$

Hence we can find the rational equation in the usual way. But we can also attain this object by elimination as in the preceding §.

2. With this view, put $a\sqrt[3]{p}=y$, $b\sqrt[3]{q}=z$; then $x=y+z$. Now, since y has the values $\alpha a\sqrt[3]{p}$, $\beta a\sqrt[3]{p}$, $\gamma a\sqrt[3]{p}$, and z the values $\alpha b\sqrt[3]{q}$, $\beta b\sqrt[3]{q}$, $\gamma b\sqrt[3]{q}$, which are all included in the two equations $y^3-a^3p=0$, $z^3-b^3q=0$, it merely amounts to this, from the three equations

I. $x = y + z$
II. $y^3 - a^3p = 0$
III. $z^3 - b^3q = 0$

to eliminate the magnitudes y and z.

3. Raise the equation I. to the third power, and put for y^3, z^3, their values a^3p, b^3q from II. and III., also x for $y+z$, we then obtain

$$x^3 = a^3p + b^3q + 3yzx$$
$$\text{or } x^3 - a^3p - b^3q = 3yzx$$

4. Raise this equation again to the third power, then we get

$$(x^3 - a^3p - b^3q)^3 = 27y^3z^3x^3$$

and if for y^3, z^3 we substitute their values

$$(x^3 - a^3p - b^3q)^3 = 27a^3b^3pqx^3.$$

5. If this equation be solved and properly arranged, we obtain

$$x^9 - 3(a^3p+b^3q)x^6 + [3(a^3p + b^3q)^2 - 27a^3b^3pq]\,x^3 - (a^3p + b^3q)^3 = 0,$$

and since this is of the ninth degree, it consequently is, as appears from 1, the most simple rational equation which can be deduced from $x=a\sqrt[3]{p}+b\sqrt[3]{q}$.

SECTION CIV.

PROB. Make the equation $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$ rational.

Solution 1. Since the quadratic irrational magnitudes

may be assumed either positive or negative, their combination gives for x the following eight roots of the required equation:

$$\begin{array}{ll} \sqrt{p}+\sqrt{q}+\sqrt{r}, & -\sqrt{p}-\sqrt{q}-\sqrt{r} \\ \sqrt{p}+\sqrt{q}-\sqrt{r}, & -\sqrt{p}-\sqrt{q}+\sqrt{r} \\ \sqrt{p}-\sqrt{q}+\sqrt{r}, & -\sqrt{p}+\sqrt{q}-\sqrt{r} \\ \sqrt{p}-\sqrt{q}-\sqrt{r}, & -\sqrt{p}+\sqrt{q}+\sqrt{r} \end{array}$$

2. Since here every two roots which are opposite each other only differ in these signs, the equation can only contain even powers of x, and it has ∴, when we put $x^2=y$, the following form:

$$y^4 - Ay^3 + By^2 - Cy + D = 0$$

and the roots of this equation are

$$\begin{array}{ll} (\sqrt{p}+\sqrt{q}+\sqrt{r})^2, & (\sqrt{p}+\sqrt{q}-\sqrt{r})^2 \\ (\sqrt{p}-\sqrt{q}+\sqrt{r})^2, & (\sqrt{p}-\sqrt{q}-\sqrt{r})^2 \end{array}$$

3. In order to determine from hence the coefficients A, B, C, D, we only need take the sum of these roots, the sum of every two of them, and so on. The following treatment, which has been frequently made use of already in the preceding part of this work, leads to the object in a shorter way. Let $S1$, $S2$, $S3$, $S4$, denote the sum of these roots, the sum of their squares, cubes, and fourth powers; then, when in § IX, $-A$ and $-C$ are put for A and C, and the symbol S for the one [ ] there used,

$$A = S1$$

$$B = \frac{AS1 - S2}{2}$$

$$C = \frac{BS1 - AS2 + S3}{3}$$

$$D = \frac{CS1 - BS2 + AS3 - S4}{4}$$

4. The expressions $S1$, $S2$, $S3$, $S4$, must necessarily be rational, because otherwise the coefficients A, B, C, D, could not be rational. Consequently the irrational magnitudes in the solution must alternately be left out, and they may $\therefore$ be entirely omitted in the calculation. With reference to this remark, and since we treat the trinomial $\sqrt{p}+\sqrt{q}+\sqrt{r}$ as a binomial $\sqrt{p}+(\sqrt{q}+\sqrt{r})$, the calculation stands thus:

$$(\sqrt{p}+\sqrt{q}+\sqrt{r})^2=p+q+r+\&c.$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^4&=p^2+6p(\sqrt{q}+\sqrt{r})^2+(\sqrt{q}+\sqrt{r})^4\ \&c.\\&=p^2+6p(q+r)+q^2+6qr+r^2+\&c.\\&=p^2+q^2+r^2+6(pq+pr+qr)+\&c.\end{aligned}$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^6&=p^3+15p^2(\sqrt{q}+\sqrt{r})^2+15p(\sqrt{q}+\sqrt{r})^4+(\sqrt{q}+\sqrt{r})^6+\&c.\\&=p^3+15p^2(q+r)+15p(q^2+6qr+r^2)\\&\quad+q^3+15q^2r+15qr^2+r^3+\&c.\\&=p^3+q^3+r^3+15(p^2q+pq^2+p^2r+pr^2\\&\quad+q^2r+qr^2)+99pqr+\&c.\end{aligned}$$

$$\begin{aligned}(\sqrt{p}+\sqrt{q}+\sqrt{r})^8&=p^4+28p^3(\sqrt{q}+\sqrt{r})^2+70p^2(\sqrt{q}+\sqrt{r})^4+28p(\sqrt{q}+\sqrt{r})^6+(\sqrt{q}+\sqrt{r})^8+\&c.\\&=p^4+28p^3(q+r)+70p^2(q^2+6qr+r^2)+28p(q^3+15q^2r+15qr^2+r^3)\\&\quad+q^4+28q^3r+70q^2r^2+28qr^3+r^4\ \&c.\\&=p^4+q^4+r^4+28(p^3q+pq^3+p^3r\\&\quad+pr^3+q^3r+qr^3)+70(p^2q^2+p^2r^2\\&\quad+q^2r^2)+420(pqr^2+pq^2r+p^2qr)\\&\quad+\&c.\end{aligned}$$

5. It is easily seen, that if, instead of $\sqrt{p}+\sqrt{q}+\sqrt{r}$, we had raised every alternate one of the expressions $\sqrt{p}+\sqrt{q}-\sqrt{r}$, $\sqrt{p}-\sqrt{q}+\sqrt{r}$, $\sqrt{p}-\sqrt{q}-\sqrt{r}$ to the same powers, the rational parts would have been the same. Now, since in $S1$, $S2$, $S3$, $S4$, the rational terms must be left out, we obtain

$$S1=4\,(p+q+r)$$
$$S2=4\,[p^2+q^2+r^2+6(pq+pr+qr)]$$
$$S3=4\,[p^3+q^3+r^3+15(p^2q+pq^2+p^2r+pr^2+q^2r+qr^2)+90pqr]$$
$$S4=4\,[p^4+q^4+r^4+28\,(p^3q+pq^3+p^3r+pr^3+q^3r+qr^3)+70\,(p^2q^2+p^2r^2+q^2r^2)+420\,(pqr^2+pq^2r+p^2qr)]$$

or more briefly, when the brackets [ ] refer to the magnitudes p, q, r,

$$S1=4\,[1]$$
$$S2=4\,([2]+6\,[1^2])$$
$$S3=4\,([3]+15\,[12]+90\,[1^3])$$
$$S4=4\,([4]+28\,[13]+70\,[2^2]+420\,[1^2 2])$$

6. If these values be substituted in the equations in 3, we then obtain the coefficients A, B, C, D, expressed by the given magnitudes p, q, r.

SECTION CV.

Prob. Make the equation $x=\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}$ rational.

Solution 1. It may be shown by inferences, as is in 1 and 2 of the preceding §, that, when we put $x^2=y$, the rational equation is of the form

$$y^8 - Ay^7 + By^6 - Cy^5 + Dy^4 - Ey^3 + Fy^2 - Gy + H = 0$$

and has the following expressions for roots:

$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} - \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} + \sqrt{q} - \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} + \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} + \sqrt{r} - \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} - \sqrt{r} + \sqrt{s})^2$$
$$(\sqrt{p} - \sqrt{q} - \sqrt{r} - \sqrt{s})^2$$

2. Give the symbol S the meaning which it had in the preceding §, and first of all try to find the expressions $S1$, $S2$, $S3$, ... $S8$. Since the calculation is managed in the same way as in the preceding §, I shall not detain my readers with it, but only remind them, that in the involution the expression $\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s}$ may be considered as a binomial, whose two parts are $\sqrt{s}$, and $\sqrt{p} + \sqrt{q} + \sqrt{r}$.

3. Consequently we have

$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^2 =$$
$$s + (\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + \&c.$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^4 =$$
$$s^2 + 6s(\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + (\sqrt{p} + \sqrt{q} + \sqrt{r})^4 + \&c.$$
$$(\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s})^6 =$$
$$s^3 + 15s^2(\sqrt{p} + \sqrt{q} + \sqrt{r})^2 + 15s(\sqrt{p} + \sqrt{q} + \sqrt{r})^4 + (\sqrt{p} + \sqrt{q} + \sqrt{r})^6 + \&c.$$

&c.

or, when the developement of the powers of $\surd p + \surd q + \surd r$ in 4 of the preceding §, are used,

$$(\surd p + \surd q + \surd r + \surd s)^2 =$$
$$p + q + r + s + \&c.$$
$$(\surd p + \surd q + \surd r + \surd s)^4 =$$
$$p^2 + q^2 + r^2 + s^2 + 6(pq + pr + ps + qr + qs + rs) + \&c.$$
$$(\surd p + \surd q + \surd r + \surd s)^6 =$$
$$p^3 + q^3 + r^3 + s^3 + 15\,(p^2q + pq^2 + p^2r + pr^2 + p^2s + ps^2$$
$$+ q^2r + qr^2 + q^2s + qs^2 + r^2s + rs^2) + 90\,(pqr + pqs +$$
$$prs + qrs + \&c.$$
$$\&c.$$

in which only the irrational terms have been omitted.

4. For the same reasons as in 5 of the preceding §, we obtain from hence

$$S1 = 8\,[1]$$
$$S2 = 8\,([2] + 6\,[1^2])$$
$$S3 = 8\,([3] + 15\,[12] + 90\,[1^3])$$
$$\&c.$$

and the substitution of these values in the formulæ in 3 of the preceding §, which must be extended for this purpose, gives the coefficients A, B, C, &c.

SECTION CVI.

PROB. Make the following equation of the first degree, with an indeterminate number of quadratic irrational magnitudes, rational, viz.

$$x = \surd p + \surd q + \surd r + \surd s + \ldots\ldots + \surd w.$$

Solution 1. By the two preceding §§ it is easily

inferred, that when n is the number of the irrational magnitudes $\sqrt{p}$, $\sqrt{q}$, ... $\sqrt{w}$, the degree of the rational equation is equal to the power 2^n. But since the different values of x are such, that two of them are always similar, but with different signs, the equation consequently is only of the 2^{n-1}th degree, when we put $x^2 = y$.

2. The conclusions in the two preceding §§, when extended, give the following results:

$$S1 = 2^{n-1}\,[1]$$
$$S2 = 2^{n-1}\,([2] + 6\,[1^2])$$
$$S3 = 2^{n-1}\,([3] + 15\,[12] + 90\,[1^3])$$
$$S4 = 2^{n-1}\,([4] + 28\,[13] + 70\,[2^2] + 420\,[1^2 2] + 2520\,[1^4])$$
$$S5 = 2^{n-1}\,([5] + 45\,[14] + 210\,[23] + 1260\,[1^2 3] + 3150\,[12^2] + 18900\,[1^3 2] + 113400\,[1^5])$$

&c.

3. Hence the law of the formation is easily perceived. As an example, I will take $S5$. The number 5, and its divisions into combinations of two, three, &c, give the numerical expressions [5], [14], [23], $[1^2 3]$, $[12^2]$, $[1^3 2]$, $[1^5]$. The coefficients are no other than the number of transpositions of different things, whose repeating exponents are twice as great as the radical exponents of the numerical expression; consequently the coefficients of [5], [14], [23], $[1^2 3]$, $[12^2]$, $[1^3 2]$, $[1^5]$, the number of transpositions of the different things a^{10}, a^2b^8, a^4b^6, $a^2b^2c^6$, $a^2b^4c^4$, $a^2b^2c^2d^4$, $a^2b^2c^2d^2e^2$, or 1, 45, 210, 1260, 3150, 18900, 113400. Those of my readers, who understand the polynomial theorem, will not have the

least difficulty in comprehending the reason of this. For by the two preceding sections, the expressions $S1$, $S2$, $S3$, &c., when 2^{n-r} is left out, are no other than the developments of the second, fourth, sixth, eighth, &c. powers of $\sqrt{p} + \sqrt{q} + \sqrt{r} + \ldots\ldots + \sqrt{w}$, with the omission of all those terms which contain irrational magnitudes, or, which is the same, the developments of the even powers of $p+q+r+\ldots\ldots+w$ with the omission of all those terms in which there are odd exponents, and by dividing the exponents in the remaining ones by two.

4. Now, if we put $2^n = m$, then the required rational equation

$$x^m - Ax^{m-2} + Bx^{m-4} - Cx^{m-6} + \&c. = 0$$

and the coefficients A, B, C, &c. are determined by the following equations:

$$\begin{aligned} A &= S1 \\ 2B &= AS1 - S2 \\ 3C &= BS1 - AS2 + S3 \\ &\&c. \end{aligned}$$

Remark. To this belongs the celebrated problem which Fermat proposed to the analysts of his time, and to the solution of which he more particularly challenged Descartes. It is this from the equation

$$ab = \sqrt{(ab - a^2)} + \sqrt{(a^2 + ad + s^2)} + \sqrt{ma} + \sqrt{(d^2 - a^2)} - \sqrt{(ar + a^2)}$$

to take away the irrational magnitudes. It is only necessary to substitute x for ab, and for the compound magnitudes under the radical signs to put the monomials p, q, r, s, t; then it only remains to make the equation $x = \sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s} + \sqrt{t}$ rational, and in the

equation thus obtained, for x to substitute again its values p, q, r, s, t.

SECTION CVII.

PROB. Make the equation $x=a\sqrt[4]{p}+b\sqrt[4]{p^2}+c\sqrt[4]{p^3}$ rational.

Solution 1. Put $\sqrt[4]{p}=y$, then this equation is

$$x - ay - by^2 - cy^3 = 0.$$

Now, since y contains four values, viz. $+y$, $-y$, $+y\sqrt{-1}$, $-y\sqrt{-1}$, we get the four following equations, all of which obtain at the same time.

$$\begin{aligned}
&x - ay && - by^2 - cy^3 &&= 0\\
&x + ay && - by^2 + cy^3 &&= 0\\
&x - ay\sqrt{-1} &&+ by^2 + cy^3\sqrt{-1} &&= 0\\
&x + ay\sqrt{-1} &&+ by^2 - cy^3\sqrt{-1} &&= 0
\end{aligned}$$

or

$$\begin{aligned}
&(x - by^2) - (ay + cy^3) &&= 0\\
&(x - by^2) + (ay + cy^3) &&= 0\\
&(x + by^2) - (ay - cy^3)\sqrt{-1} &&= 0\\
&(x + by^2) + (ay - cy^3)\sqrt{-1} &&= 0
\end{aligned}$$

The equation sought must $\therefore$ be the product of these.

2. If the two first and the two last be multiplied together, we obtain

$$x^2 - 2by^2x + b^2y^4 - a^2y^2 - 2acy^4 - c^2y^6 = 0$$
$$x^2 + 2by^2x + b^2y^4 + a^2y^2 - 2acy^4 + c^2y^6 = 0$$

or if in these equations we substitute p for y^4,

$$[x^2 + (b^2 - 2ac)p] - (2bx + a^2 + c^2p)y^2 = 0$$
$$[x^2 + (b^2 - 2ac)p] + (2bx + a^2 + c^2p)y^2 = 0$$

3. If we multiply these equations, and then put p for y^4, we obtain the required rational equation of the fourth degree,

$$x^4 - 2(b^2 + 2ac)px^2 - 4(a^2 + c^2p)bpx$$
$$+ (b^2 - 2ac)^2p^2 - (a^2 + c^2p)^2p = 0$$

and the four roots of this equation are

$$a\sqrt[4]{p} \qquad + b\sqrt[4]{p^2} + c\sqrt[4]{p^3}$$
$$-a\sqrt[4]{p} \qquad + b\sqrt[4]{p^2} - c\sqrt[4]{p^3}$$
$$a\sqrt[4]{p}\,.\,\sqrt{-1} - b\sqrt[4]{p^2} - c\sqrt[4]{p^3}\,.\,\sqrt{-1}$$
$$-a\sqrt[4]{p}\,.\,\sqrt{-1} - b\sqrt[4]{p^2} + c\sqrt[4]{p^3}\,.\,\sqrt{-1}$$

Corollary When ∴ an equation of the fourth degree has the form just found, then its four roots may always be determined without any further calculation. I shall now show, that, presupposing the solution of cubic equations to be known, every equation of the fourth degree can have this form.

SECTION CVIII.

Let

$$x^4 - Ax^2 - Bx - C = 0$$

be the equation to be solved: it is general, because in every equation the second term, if there be such, may be omitted. If this equation be identical with that in 3

of the preceding §, we then must have

$$\text{I.} \quad 2p(b^2+2ac) = A$$
$$\text{II.} \quad 4bp(a^2+c^2p) = B$$
$$\text{III.} \quad (a^2+c^2p)^2p \ldots (b^2-2ac)^2p^2 = C$$

The two first equations give

$$(b^2+2ac)p = \frac{A}{2}$$

$$a^2 + c2p = \frac{B}{4bp}$$

and if we make use of these values in the equation III, after having previously given it the form

$$(a^2+c^2p)^2p - (b^2+2ac)^2p^2 + 8ab^2cp^2 = C$$

we then obtain

$$\frac{B^2}{16b^2p} - \frac{A^2}{4} + 8ab^2cp^2 = C$$

From the equation I we also obtain

$$\text{IV.} \quad 4acp = A - 2b^2p$$

and the substitution of this value in the equation just found, gives

$$\frac{B^2}{16b^2p} - \frac{A^2}{4} + 2Ab^2p - 4b^4p^2 = C$$

Since the three equations I, II, III, contain four indeterminate magnitudes a, b, c, p, we can ∴ assume any one of them. Put $b = 1$, then, after getting rid of the denominator,

$$B^2 - 4A^2p + 32Ap^2 - 64p^3 = 16Cp$$

or, when arranged according to p,

$$V.\ p^3 - \tfrac{1}{2}Ap^2 + \tfrac{1}{4}(C + \tfrac{1}{4}A^2)p - \tfrac{1}{64}B^2 = 0$$

an equation of the third degree, which merely contains the unknown magnitude p.

From the equations II and IV we obtain, when we put $b = 1$, and divide the latter by $2\sqrt{p}$,

$$a^2 + c^2 p = \frac{B}{4p},\ 2ac\sqrt{p} = \frac{A - 2p}{2\sqrt{p}}$$

and when we add the second to the first, and also subtract the one from the other, then again extract the square root from the sum and the remainder

$$a + c\sqrt{p} = \sqrt{\left(\frac{B}{4p} + \frac{A}{2\sqrt{p}} - \sqrt{p}\right)}$$

$$a - c\sqrt{p} = \sqrt{\left(\frac{B}{4p} - \frac{A}{2\sqrt{p}} + \sqrt{p}\right)}$$

But the four roots in 3 of the preceding §, when we put $b = 1$, has the following form

$$\sqrt{p} + (a + c\sqrt{p})\sqrt[4]{p}$$

$$\sqrt{p} - (a + c\sqrt{p})\sqrt[4]{p}$$

$$-\sqrt{p} + (a - c\sqrt{p})\sqrt[4]{p}\,.\,\sqrt{-1}$$

$$-\sqrt{p} - (a - c\sqrt{p})\sqrt[4]{p}\,.\,\sqrt{-1}$$

and when in this we substitute for $a + c\sqrt{p}$, $a - c\sqrt{p}$ their values, we obtain the following roots of the equation $x^4 - Ax^2 - Bx - C = 0$:

$$\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(B\sqrt{p} + 2Ap - 4p^2)}$$

$$\sqrt{p} - \frac{1}{2\sqrt{p}}\sqrt{(B\sqrt{p} + 2Ap - 4p^2)}$$

$$-\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(-\sqrt{p} + 2Ap - 4p^2)}$$

$$-\sqrt{p} + \frac{1}{2\sqrt{p}}\sqrt{(-\sqrt{p} + 2Ap - 4p^2)}$$

Having $\therefore$ already determined the value of p from the equation V, we also obtain the roots of the given equation. Besides, it is exactly the same which of the three values of p we make use of, because in each case we must necessarily always get the same roots.

SECTION CIX.

PROB. Make the equation

$$x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$$

rational

Solution 1. When we put $\sqrt[5]{p} = y$, $\therefore y^5 - p = 0$, then the equation is transformed into

$$x - ay - by^2 - cy^3 - dy^4 = 0.$$

Now y has five values, viz. αy, βy, γy, δy, εy, when α, β, γ, δ, ε, denote the five roots of the equation $y^5 - 1 = 0$ (unity included); we have $\therefore$ the five following distinct equations:

$$\begin{aligned}
x - \alpha ay - \alpha^2 by^2 - \alpha^3 cy^3 - \alpha^4 dy^4 &= 0 \\
x - \beta ay - \beta^2 by^2 - \beta^3 cy^3 - \beta^4 dy^4 &= 0 \\
x - \gamma ay - \gamma^2 by^2 - \gamma^3 cy^3 - \gamma^4 dy^4 &= 0 \\
x - \delta ay - \delta^2 by^2 - \delta^3 cy^3 - \delta^4 dy^4 &= 0 \\
x - \varepsilon ay - \varepsilon^2 by^2 - \varepsilon^3 cy^3 - \varepsilon^4 dy^4 &= 0
\end{aligned}$$

and their product will give the required rational equation, if we again put $\sqrt[5]{p}$ for y.

2. In fact, this implies no more than to eliminate x from the two equations $x - ay - by^2 - cy^3 - dy^4 = 0$, $y^5 - p = 0$, and consequently in this case all the methods of elimination in the preceding chapter are applicable.

If we make use of Cramer's method as the easiest, we shall arrive at numerical expressions, which exceed the limits of the annexed tables, and ∴ must be calculated. But we arrive at this object in a much shorter way by managing the calculation in such a way, that the numerical expressions refer only to the roots of unity, because these are more easily calculated.

3. For this purpose we only require from the two equations

$$\text{I.} \quad z^5 - 1 = 0$$

$$\text{II.} \quad x - ayz - by^2z^2 - cy^3z^3 - dy^4z^4 = 0$$

to eliminate the magnitude z; for if we substitute in II the five values $\alpha, \beta, \gamma, \delta, \varepsilon$, of z from the first, we then obtain the same equations as in 1.

4. In order to be able to apply Cramer's method of elimination (§ LXXVI), we give the equation II the form

$$1 + (1)\,z + (2)\,z^2 + (3)\,z^3 + (4)\,z^4 = 0$$

then $(1) = -\frac{ay}{x}$, $(2) = -\frac{by^2}{x}$, $(3) = -\frac{cy^3}{x}$, $(4) = -\frac{dy^4}{x}$,

5. Since the numerical expressions in the equation, § LVIII, LIX, (c), in the present case relate to the roots of the equation $z^5 - 1 = 0$, then all those in which the sum of the radical exponents is not divisible by five vanish, by 2, § XCV. With reference to this remark, we obtain the following final equation:

$$
\begin{aligned}
o = 1 &+ (14)'[14] + (24^2)'[24^2] + (34^3)'[34^3] + (4^5)'[4^5] \\
&+ (23)'[23] + (3^2 4)'[3^2 4] + (124^3)'[124^3] \\
&+ (1^2 3)'[1^2 3] + (1^2 4^2)'[1^2 4^2] + (13^2 4^2)'[13^2 4^2] \\
&+ (12^2)'[12^2] + (1234)'[1234] + (2^2 34^2)'[2^2 34^2] \\
&+ (1^3 2)'[1^3 2] + (13^3)'[13^3] + (23^3 4)'[23^3 4] \\
&+ (1^5)'[1^5] + (2^3 4)'[2^3 4] + (3^5)'[3^5] \\
& + (2^2 3^2)'[2^2 3^2] \\
& + (1^3 34)'[1^3 34] \\
& + (1^2 2^2 4)'[1^2 2^2 4] \\
& + (1^2 23^2)'[1^2 23^2] \\
& + (12^3 3)'[12^3 3] \\
& + (2^5)'[2^5]
\end{aligned}
$$

6. The numerical expressions in this equation may also be calculated by § XCV, which, indeed, is not difficult for the present case. If after this again, we put for the symbols (1), (2), (3), (4), their values from 4, likewise p for y^5, and multiply the equation by x^5, we then obtain

$$
o =
$$

$$
\begin{array}{llll}
x^5 \left.\begin{array}{l} -5ad \\ -5bc \end{array}\right) px^3 & \left.\begin{array}{l} -5bd^2 \\ -5c^2d \end{array}\right) p^2x & -5cd^3p^3x - d^5p^4 & \\
\left.\begin{array}{l} -5a^2c \\ -5ab^2 \end{array}\right) px^2 & \left.\begin{array}{l} -5a^2d^2 \\ -5abcd \\ -5ac^3 \\ -5b^3d \\ +5b^2c^2 \end{array}\right\} p^2x & \left.\begin{array}{l} +5abd^3 \\ -5ac^2d^2 \\ -5b^2cd^2 \\ +5bc^3d \\ -c^5 \end{array}\right\} p^3 & \\
-5a^3bpx & & & \\
-a^5p & & & \\
& \left.\begin{array}{l} +5a^3cd \\ -5a^2b^2d \\ -5a^2bc^2 \\ +5ab^3c \\ -b^5 \end{array}\right\} p^2 & &
\end{array}
$$

7. If ∴ the required rational equation be represented by

$$x^5 - Ax^3 - Bx^2 - Cx - D = 0$$

then

$$\begin{aligned}
A &= 5\,(ad\ + bc)\,p \\
B &= 5\,(a^2c + ab^2 + bd^2p + c^2dp)\,p \\
C &= 5\,(a^3b + b^3dp + ac^3p + cd^3p^2)\,\mathrm{p}\ - \\
&\qquad 5\,(a^2d^2 + b^2c^2)\,p^2 + 5\ abcdp^2 \\
D &= a^5p + b^5p^2 + c^5p^3 + d^5p^4 - \\
&\qquad 5\,(a^3cd + ab^3c + bc^3dp + abd^3p)\,p^2 + \\
&\qquad 5\,(a^2b^2d + a^2bc^2 + ac^2d^2p + b^2cd^2p)\,p^2
\end{aligned}$$

SECTION CX.

The equation $x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$ led to an equation of the fifth degree, of the form in 7 of the preceding §; and the five roots of this last equation are consequently

$$\begin{aligned}
x &= \alpha a\sqrt[5]{p} + \alpha^2b\sqrt[5]{p^2} + \alpha^3c\sqrt[5]{p^3} + \alpha^4d\sqrt[5]{p^4} \\
x &= \beta a\sqrt[5]{p} + \beta^2b\sqrt[5]{p^2} + \beta^3c\sqrt[5]{p^3} + \beta^4d\sqrt[5]{p^4} \\
x &= \gamma a\sqrt[5]{p} + \gamma^2b\sqrt[5]{p^2} + \gamma^3c\sqrt[5]{p^3} + \gamma^4d\sqrt[5]{p^4} \\
x &= \delta a\sqrt[5]{p} + \delta^2b\sqrt[5]{p^2} + \delta^3c\sqrt[5]{p^3} + \delta^4d\sqrt[5]{p^4} \\
x &= \varepsilon a\sqrt[5]{p} + \varepsilon^2b\sqrt[5]{p^2} + \varepsilon^3c\sqrt[5]{p^3} + \varepsilon^4d\sqrt[5]{p^4}
\end{aligned}$$

Therefore, conversely, if an equation of the fifth degree has the given form, we have its roots in its stead. If ∴ we could reduce every given equation of the fifth degree to this form, we should then have the general solution of equations of this degree. To effect this, it is indispen-

sably necessary, from the given coefficients A, B, C, D, by means of the equations in 7 of the preceding §, to be able to determine the magnitudes a, b, c, d, p, one of which is arbitrary, in a similar way with that in § CVIII in the case of equations of the fourth degree, and also in § C, where the transformed equation had Cardan's form. But all the endeavours of the greatest Analysts to attain this object have been fruitless, and we shall see in the sequel, why it must be the case. However, a treatise by Euler on the general solution of equations, and particularly those of the fifth degree, may always be read with pleasure and instruction; it is to be found in the ninth part of the new Petersburg Commentaries, and also in the third Part of Michelsen's Translation of Euler's Introduction.

SECTION CXI.

Although, however, we cannot obtain the general solution of equations of the fifth degree by the method in the preceding §, yet there are several particular equations, to which this solution is applicable, of which I shall, with Euler, only adduce those which do not lead to very complicated forms.

I. If in the equations in 7, § CIX, we put $c = 0$, $d = 0$, we then have

$$A = 0,\ B = 5ab^2p,\ C = 5a^3bp,$$
$$D = a^5p + b^5p^2.$$

From the second and third of these equations we obtain

$$a^5p = \frac{C^2}{5B},\ b^5p^2 = \frac{B^3}{25C}$$

hence

$$D = \frac{C^2}{5B} + \frac{B^3}{25C}$$

Also

$$a\sqrt[5]{p} = \sqrt[5]{\frac{C^2}{5B}},\ b\sqrt[5]{p^2} = \sqrt[5]{\frac{B^3}{25C}}$$

If $\therefore$ the equation

$$x^5 - Bx^2 - Cx - \frac{C^2}{5B} - \frac{B^3}{25C} = 0$$

be given, then

$$a\sqrt{\frac{C^2}{5B}} + a^2\sqrt{\frac{B^3}{25C}}$$

is one of its roots, and the remaining roots are obtained by substituting β, γ, δ, ϵ successively for a.

We should in like manner have found the same equation and the same roots, if we had put a and b, or a and c, or b and $d = 0$. Thus if we put $b = 0$, and $d = 0$, we have

$$A = 0,\ B = 5a^2cp,\ C = 5ac^3p^2$$
$$D = a^5p + c^5p^3.$$

From the second and third equation we obtain

$$a^5p = \frac{B^3}{25C},\ c^5p^3 = \frac{C^2}{5B}$$

and these give

$$D = \frac{B^3}{25C} + \frac{C^2}{5B}$$

$$a\sqrt[5]{p} = \sqrt[5]{\frac{B^3}{25C}},\ c\sqrt[5]{p^3} = \sqrt[5]{\frac{C^2}{5B}}$$

We have $\therefore$ again the equation

$$x^5 - Bx^2 - Cx - \frac{B^3}{25C} - \frac{C^2}{5B} = 0$$

and one of its roots

$$= a\alpha\sqrt[5]{p}+\alpha^3 c\sqrt[5]{p^3} = \alpha\sqrt[5]{\frac{B^3}{25C}}+\alpha^3\sqrt[5]{\frac{C^2}{5B}}$$

The remaining ones are obtained by substituting β, γ, δ, ε, successively for α. Moreover, that the five roots, which we find by these means, are not different from those already found, we may easily convince ourselves by putting α^2, α^3, α^4, α^5 ($= 1$) for β, γ, δ, ε (§ LXXXVII).

II. If in the equations in 7, § CVIII, we put $b=0$, and $c=0$, we obtain

$$A = 5\,a\,dp,\ B = 0,\ C = -\ 5\,a^2d^2p^2,$$
$$D = a^5p + d^5p^4$$

The first and third of these equations give

$$C = -\frac{A^2}{5}$$

Further, the fourth gives

$$D^2 = (a^5p + d^5p^4)^2 = (a^5p - d^5p^4)^2 + 4a^5d^5p^5$$
$$= (a^5p - d^5p^4)^2 + 4\left(\frac{A}{5}\right)^5$$

consequently

$$a^5p - d^5p^4 = \sqrt{\left[D^2 - 4\left(\frac{A}{5}\right)^5\right]}$$

Now since

$$a^5p + d^5p^4 = D$$

then

$$a^5p = \tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}$$

$$d^5p^4 = \tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}$$

and $\therefore$

$$a\sqrt[5]{p} = \sqrt[5]{\left[\tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]}$$

$$d\sqrt[5]{p^4} = \sqrt[5]{\left[\tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]}$$

Consequently, if the equation

$$x^5 - Ax^3 + \frac{A^2}{5}x - D = 0$$

then each of its roots is expressed by

$$a\sqrt[5]{\left[\tfrac{1}{2}D + \sqrt{\left[\tfrac{1}{4}D^2 - \left(\frac{A}{5}\right)^5\right]}\right]} +$$

$$a^4\sqrt[5]{\left[\tfrac{1}{2}D - \sqrt{\left[\tfrac{1}{4}D - \left(\frac{A}{5}\right)^5\right]}\right]}$$

This root resembles very much, as we see, that which Cardan's formula gives for equations of the third degree. Besides this equation belongs to a peculiar class of particular equations of all degrees, the solution of which was first taught by Moivre, and of which we shall treat hereafter.

SECTION CXII.

In the same way as in § CX the equation

$$x = a\sqrt[5]{p} + b\sqrt[5]{p^2} + c\sqrt[5]{p^3} + d\sqrt[5]{p^4}$$

was made rational, every other equation of the form

$$x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \dots\dots\dots + k\sqrt[n]{p^{n-1}}$$

may generally be made rational, and the degree of the rational equation will always be equal to the radical index. In this there is no other difficulty than the trouble of the calculation. Hauber has omitted this operation when $n=6$

[See the Second Collection of Combination Analytical Treatises, p. 248]. It would be desirable, if it could also be done with other values of n, because from them, as has been already shown by a few examples, the solutions of a great number of particular equations might be derived, which are so much the more worthy of observation, because they cannot be analyzed; for otherwise there could not be in the roots any radicals of the same degree as the equations themselves. Yet a great number of particular cases of the same kind may be found without such complicated calculations, by omitting at the very beginning, several terms in the general irrational expression $a\sqrt[n]{p}+b\sqrt[n]{p^2}+c\sqrt[n]{p^3}+\ldots\ldots+k\sqrt[n]{p^{n-1}}$; as the following problems will show.

SECTION CXIII.

PROB. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$ rational.

Solution. Here the two cases, in which n is an even, and the other where n is an odd number, must be distinguished.

First Case.

1. Let $x = a\sqrt[2m]{p} + b\sqrt[2m]{p^2}$; further, let α, β, γ, δ, &c. be the roots of the equation $x^{2m} - 1 = 0$. If y be substituted for $\sqrt[2m]{p}$, then the different values, which x has in relation to these irrational magnitudes, are

$$\begin{array}{l} \alpha a y + \alpha^2 b y^2 \\ \beta a y + \beta^2 b y^2 \\ \gamma a y + \gamma^2 b y^2 \\ \quad \&c. \end{array}$$

as far as $2m$. The required rational equation is represented by

$$x^{2m} - \overset{1}{A}x^{2m-1} + \overset{2}{A}x^{2m-2} - \overset{3}{A}x^{2m-3} + \ldots\ldots$$
$$\ldots \pm \overset{m}{A}x^m \mp \overset{m+1}{A}x^{m-1} \pm \overset{m+2}{A}x^{m-2} \mp \ldots\ldots\ldots\ldots$$
$$\ldots - \overset{2m-1}{A}x + \overset{2m}{A} = 0$$

the upper signs obtain when m is even, and the lower when m is odd. The coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, &c. are then the sums of the former values of x, taken singly, two and two, three and three, &c.

2. The developement of these combinations gives

$$\overset{1}{A} = `[1]ay + `[2]by^2$$
$$\overset{2}{A} = `[1^2]a^2y^2 + `[12]aby^3 + `[2^2]b^2y^4$$
$$\overset{3}{A} = `[1^3]a^3y^3 + `[1^2 2]a^2by^4 + `[12^2]ab^2y^5 + `[2^3]b^3y^5$$
$$\&c.$$

3. But it is evident from 2, § XCVI, that all the coefficients $\overset{1}{A}$ $\overset{2}{A}$ $\overset{3}{A}$, as far as $\overset{m}{A}$ &c. vanish, because the sum of the radical exponents in each numerical expression, is always $< 2m$, and consequently cannot be divisible by $2m$; this was evident before, from this consideration, that in the required equation there are only such powers of y, as are divisible by $2m$, for otherwise

it could not be rational. It only remains to find the coefficients $\overset{m}{A}$ $\overset{m+1}{A}$, $\overset{m+2}{A}$, ... $\overset{2m}{A}$.

4. When in the terms of which these coefficients consist, we omit all those, which by 2, § XCVI, = o, we then find

$$\overset{m}{A} = {}^{\backprime}[2^m]\, b^m y^{2m}$$
$$\overset{m+1}{A} = {}^{\backprime}[1^2 2^{m-1}]\, a^2 b^{m-1} y^{2m}$$
$$\overset{m+2}{A} = {}^{\backprime}[1^4 2^{m-2}]\, a^4 b^{m-2} y^{2m}$$
$$\overset{m+3}{A} = {}^{\backprime}[1^6 2^{m-3}]\, a^6 b^{m-3} y^{2m}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\overset{2m-1}{A} = {}^{\backprime}[1^{2m-2} 2]\, a^{2m-2} b y^{2m}$$
$$\overset{2m}{B} = {}^{\backprime}[1^{2m}]\, a^{2m} y^{2m} + [2^{2m}]\, b^{2m} y^{2m}$$

5. But (§ XCVI).

$${}^{\backprime}[2^m] = \mp \frac{1 \,.\, 2 \,.\, 3 \,......\, m-1}{1 \,.\, 2 \,.\, 3 \,......\, m} \,.\, 2m = \mp 2$$

$${}^{\backprime}[1^2 2^{m-1}] = \pm \frac{1 \,.\, 2 \,.\, 3 \,...............\, m}{1 \,.\, 2 \,.\, 3 \,...\, m-1 \times 1 \,.\, 2} \,.\, 2m = \pm \frac{m}{1 \,.\, 2} \,.\, 2m$$

$${}^{\backprime}[1^4 2^{m-2}] = \mp \frac{1 \,.\, 2 \,.\, 3 \,............\, m+1}{1 \,.\, 2 \,...\, m-2 \times 1 \,.\, 2 \,.\, 3 \,.\, 4} = \mp \frac{m \,.\, m^2-1}{1 \,.\, 2 \,.\, 3 \,.\, 4} \,.\, 2m$$

$${}^{\backprime}[1^6 2^{m-3}] = \pm \frac{1 \,.\, 2 \,.\, 3 \,..................\, m+2}{1 \,.\, 2 \,.\, 3 \,...\, m-3 \times 1 \,.\, 2 \,...\, 6} \,.\, 2m =$$
$$\pm \frac{m \,.\, m^2-1 \,.\, m^2-4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} \,.\, 2m$$

$${}^{\backprime}[1^8 2^{m-4}] = \mp \frac{1 \,.\, 2 \,.\, 3 \,..............\, m+3}{1 \,.\, 3 \,.\, 3 \,...\, m-4 \times 1 \,.\, 2 \,...\, 8} \,.\, 2m =$$
$$\mp \frac{m \,.\, m^2-1 \,.\, m^2-4 \,.\, m^2-9}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8} \,.\, 2m$$

. . . .

$$`[1^{2m-2}2] = +\frac{1\,.\,2\,.\,3\;\ldots\ldots\ldots\ldots\;2m-2}{1\times1\,.\,2\,.\,3\;\ldots\ldots\;2m-2}.\;2m$$

$$= +\frac{m\,.\,m^2-1\,.\,m^2-4\ldots\ldots m^2-\overline{m-2}^2}{1\,.\,2\,.\,3\,.\,4\;\ldots\ldots\ldots\ldots\;2m-2}.\;2m$$

$$`[1^{2m}] \quad = -\frac{1\,.\,2\,.\,3\;\ldots\ldots\ldots\ldots\;2m-1}{1\,.\,2\,.\,3\;\ldots\ldots\ldots\ldots\ldots\;2m}.\;2m = -1$$

$$`[2^{2m}] = +\frac{(1.2.3\ldots m-1)^2}{(1.2\ldots m)^2\times1.2}(2m)^2 - \frac{1.2.3\ldots2m-1}{1.2.3\ldots2m}\,.\;2m$$

$$= +2-1 = +1$$

in which the upper signs obtain when m is an even, and the lower when n is an odd number.

6. If we substitute these values in the expressions for $\overset{m}{A}, \overset{m+1}{A}, \overset{m+2}{A}$, &c., and then again put p for y^{2m}, we obtain the required rational equation

$$x^{2m} - 2b^m px^m - \frac{m}{1.2}.\;2m\,.\,a^2b^{m-1}px^{m-1}$$

$$-\frac{m\,.\,m^2-1}{1\,.\,2\,.\,3\,.\,4}.\;2m\,.\,a^4b^{m-2}px^{m-2}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}.\;2m\,.\,a^6b^{m-3}px^{m-3}$$

$$-\frac{m\,.\,m^2-1\,.\,m^2-4\,.\,m^2-9}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8.}.\;2m\,.\,a^8b^{m-4}px^{m-8}$$

. . . .

$$-\frac{m.m^2-1\,.\,m^2-4\,.\,m^2-9\ldots m^2-\overline{m-2}^2}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\;\ldots\ldots\;2m-2}.\;2m\,.\,a^{2m-2}bpx$$

$$-a^{2m}p + b^{2m}p^2 = 0$$

Second Case.

7. Now, let $x=a\sqrt[2m+1]{p}+b\sqrt[2m+1]{p^2}$, or y be substituted for $\sqrt[2m+1]{p}$, $x=ay+by^2$. Further, let

$$x^{2m+1}-\overset{1}{A}x^{2m}+\overset{2}{A}x^{2m-1}-\overset{3}{A}x^{2m-2}+\ldots\ldots$$
$$\ldots\ldots\mp\overset{m+1}{A}x^{m}\pm\overset{m+2}{A}x^{m-1}\mp\overset{m+3}{A}x^{m-2}\pm\ldots\ldots$$
$$\ldots\ldots+\overset{2m}{A}x-\overset{2m+1}{A}=0$$

be the required equation, in which the upper signs obtain, when m is even, and the lower when it is odd. Then, as before, all the coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, &c. as far as $\overset{m+1}{A}$ vanish, and we have the numerical expressions referred to the equation $x^{2m+1}-1=0$,

$$\overset{m+1}{A} = `[12^m]ab^my^{2m+1}$$
$$\overset{m+2}{A} = `[1^32^{m-1}]a^3b^{m-1}y^{2m+1}$$
$$\overset{m+3}{A} = `[1^52^{m-2}]a^5b^{m-2}y^{2m+1}$$
$$\overset{m+4}{A} = `[1^72^{m-3}]a^7b^{m-3}y^{2m+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\overset{2m}{A} = `[1^{2m-1}2]\,a^{2m-1}by^{2m+1}$$
$$\overset{2m+1}{A} = `[1^{2m+1}]a^{2m+1}y^{2m+1}+`[2^{2m+1}]b^{2m+1}y^{2m+1}$$

8. But

$$`[12^m] = \pm\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\,m}{1\,.\,2\,.\,3\,\ldots\ldots\,m\times 1}.\,2m+1=\pm 2m+1$$

$$`[1^32^{m-1}]=\mp\frac{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,m+1}{1\,.\,2\,.\,3\,\ldots\ldots\,m-1\times 1.2.3}.\,2m+1$$

$$=\mp\frac{m\,.\,m+1}{1\,.\,2\,.\,3}.\,2m+1$$

$$'[1^3 2^{m-1}] = \pm \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots\, m+2}{1 \,.\, 2 \,.\, 3 \ldots\ldots m-2 \times 1.2.3.4.5} \,.\, 2m+1$$

$$= \pm \frac{m \,.\, m^2-1 \,.\, m+2}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \,.\, 2m+1$$

$$'[1^7 2^{m-3}] = \mp \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\ldots\ldots\, m+3}{1.2.3 \ldots m-3 \times 1.2.3.4.5.6.7} \,.\, 2m+1$$

$$= \mp \frac{m \,.\, m^2-1 \,.\, m^2-4 \,.\, m+3}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} \,.\, 2m+1$$

. . . .

$$'[1^{2m-1} 2] = - \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\ldots\, 2m-1}{1 \times 1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\, 2m-1} \,.\, 2m+1$$

$$= - \frac{m \,.\, m^2-1 \,.\, m^2-4 \,.\, m^2-9 \ldots m^2-\overline{m-2}^2 \,.\, 2m-1}{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots\ldots\ldots\ldots\ldots\, 2m-1} \,.\, 2m+1$$

$$'[1^{2m+1}] = + \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\ldots\, 2m}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\, 2m+1} \,.\, 2m+1 = +1$$

$$'[2^{2m+1}] = + \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\ldots\, 2m}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\, 2m+1} \,.\, 2m+1 = +1$$

9. If these values be regularly substituted, we then obtain the required equation

$$x^{2m+1} - (2m+1)\, ab^m p x^m$$

$$- \frac{m \,.\, m+1}{1 \,.\, 2 \,.\, 3} \,.\, 2m+1 \,.\, a^3 b^{m-1} p x^{m-1}$$

$$- \frac{m \,.\, m^2-1 \,.\, m+2}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \,.\, 2m+1 \,.\, a^5 b^{m-2} p x^{m-2}$$

$$- \frac{m \,.\, m^2-1 \,.\, m^2-4 \,.\, m+3}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7} \,.\, 2m+1 . a^7 b^{m-3} p x^{m-3}$$

$$- \frac{m \,.\, m^2-1 \,.\, m^2-4 \,.\, m^2-9 \,.\, m+4}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8 \,.\, 9} \,.\, 2m+1 . a^9 b^{m-4} p x^{m-4}$$

. . . .

$$- \frac{m.m^2-1.m^2-4.m^2-9 \ldots m^2-\overline{m-2}^2.2m-1}{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\, 2m-1} . 2m+1 . a^{2m-1} bpx$$

$$- a^{2m+1} p - b^{2m+1} p^2 = 0$$

SECTION CXIV.

Conversely, if equations of the form in 6 and 9 of the preceding § be given, we may always find their roots. As examples, and for the sake of their use, I shall here give a few equations of this kind.

I. When $m=2$, we obtain from 9 of the preceding §, the equation

$$x^5 - 5ab^2px^2 - 5a^3bpx - a^5p - b^5p^2 = 0$$

and each of its roots

$$\alpha a\sqrt[5]{p} + \alpha^2 b\sqrt[5]{p^2}$$

when α is a root of the equation $x^5 \ldots 1 = 0$. Moreover, this equation is the same as that in I, § CXI, which was derived from the general equation of the fifth degree in § CX.

II. When $m=3$, we obtain from 6 of the preceding § the equation

$$x^6 - 2b^3px^3 - 9a^2b^2px^2 - 6a^4bpx - a^6p + b^6p^2 = 0$$

the roots of which are expressed by

$$\alpha a\sqrt[6]{p} + \alpha b\sqrt[6]{p^2}$$

when α denotes a root of the equation $x^6 - 1 = 0$.

Further, for the same value of m we obtain from 9 of the preceding §, the equation

$$x^7 - 7ab^3px^3 - 14a^3b^2px^2 - 7a^5bpx$$
$$- a^7p - b^7p^2 = 0$$

and each of its roots

$$\alpha a\sqrt[7]{p} + \alpha^2 b\sqrt[7]{p^2}$$

when α denotes a root of the equation $x^7 - 1 = 0$.

III. When $m=4$, we obtain from 6 of the preceding § the equation

$$x^8 - 2b^4px^4 - 16a^2b^3px^3 - 20a^4b^2px^2$$
$$- 8a^6bpx - a^8p + b^8p^2 = 0$$

and for each of its roots

$$\alpha a\sqrt[8]{p} + \alpha^2 b\sqrt[8]{p^2}$$

Further, for the same value of m, we obtain from 9 the equation

$$x^9 - 9ab^4px^4 - 30a^3b^3px^3 - 27a^5b^2px^2$$
$$- 9a^7bpx - a^9p - b^9p^2 = 0$$

and for each of its roots

$$\alpha a\sqrt[9]{p} + \alpha^2 b\sqrt[9]{p^2}$$

and so on.

SECTION CXV.

Prob. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$ rational.

Solution 1. If we denote the roots of the equation $x^n - 1 = 0$ by α, β, γ, δ, &c. then the roots of the required rational equation are

$$\alpha a\sqrt[n]{p} + \alpha^{n-1}b\sqrt[n]{p^{n-1}}$$
$$\beta a\sqrt[n]{p} + \beta^{n-1}b\sqrt[n]{p^{n-1}}$$
$$\gamma a\sqrt[n]{p} + \gamma^{n-1}b\sqrt[n]{p^{n-1}}$$
&c.

or since $\alpha^n = \beta^n = \gamma^n = \&c. = 1$,

$$\alpha a\sqrt[n]{p} + \alpha^{-1}b\sqrt[n]{p^{n-1}}$$
$$\beta a\sqrt[n]{p} + \beta^{-1}b\sqrt[n]{p^{n-1}}$$
$$\gamma a\sqrt[n]{p} + \gamma^{-1}b\sqrt[n]{p^{n-1}}$$
&c.

Hence we could derive this equation in the same way as in § CXIII; the following method, however, which has been often used already, leads to the object in a shorter way.

2. Denote the sum of the first, second, third, and so on, powers of these roots by $S1$, $S2$, $S3$, &c.; then

$$S1 = {}'[1]a\sqrt[n]{p} + {}'[-1]b\sqrt[n]{p^{n-1}}$$
$$S2 = {}'[2]a^2\sqrt[n]{p^2} + 2{}'[0]abp + {}'[-2]b^2p^2\sqrt[n]{p^{-2}}$$
$$S3 = {}'[3]a^3\sqrt[n]{p^3} + 3{}'[1]a^2bp\sqrt[n]{p} + 3{}'[-1]ab^2p^2\sqrt[n]{p^{-1}}$$
$$+ {}'[-3]b^3p^3\sqrt[n]{p^{-3}}$$
$$S4 = {}'[4]a^4\sqrt[n]{p^4} + 4{}'[2]a^3bp\sqrt[n]{p^2} + 6{}'[0]a^2b^2p^2$$
$$+ 4{}'[-2[ab^3p^3\sqrt[n]{p^{-2}} + {}'[-4]b^4p^4\sqrt[n]{p^{-4}}$$
&c.

3. From the form of these values, and from § XCVI, it follows, that the expressions $S1$, $S3$, $S5$, &c. $=$ o, and that generally each expression $S\mu = $ o, when μ is an odd number and less than n. Further, since ${}'[\text{o}] = n$, we have

$$S2 = \frac{2}{1}nabp$$

$$S4 = \frac{4 \,.\, 3}{1 \,.\, 2} na^2 b^2 p^2$$

$$S6 = \frac{6 \,.\, 5 \,.\, 4}{1 \,.\, 2 \,.\, 3} na^3 b^3 p^3$$

$$S8 = \frac{8 \,.\, 7 \,.\, 6 \,.\, 5}{1 \,.\, 2 \,.\, 3 \,.\, 4} na^4 b^4 p^4$$

and generally

$$S2\mu = \frac{2\mu \,.\, 2\mu - 1 \,.\, 2\mu - 2 \ldots\ldots \mu + 1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots \mu} na^\mu b^\mu p^\mu$$

whilst $2\mu < n$. The expression Sn, when n is an even number, besides the term

$$\frac{n \,.\, n-1 \,.\, n-2 \ldots\ldots \frac{n}{2} + 1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots \frac{n}{2}} na^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}}$$

which does not vanish, and which is derived from the values of $S2\mu$, when $2\mu = n$, contains also the two terms $[n]a^n p$, $[-n]b^n p^{n-1}$, which do not vanish, or $na^n p$, $nb^n p^{n-1}$. On the other hand, when n is an odd number, then the expression Sn only contains the two last-mentioned terms. Accordingly, in the case where n is an odd number, we have

$$Sn = na^n p + nb^n p^{n-1}$$

and in the case where n is an even number

$$Sn = \frac{n \,.\, n-1 \ldots\ldots \frac{n}{2} + 1}{1 \,.\, 2 \ldots\ldots\ldots\ldots \frac{n}{2}} na^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} + na^n p + nb^n p^{n-1}$$

4. From the values of $S1$, $S2$, $S3$, ... Sn, already found, we are now enabled, by means of the formulæ in

§ IX (which serve for determining the coefficients of an equation from the known values of the sums of the powers of its roots), to find the coefficients of the required rational equation. Thus, if this equation be represented by

$$x^n + \overset{1}{A}x^{n-1} + \overset{2}{A}x^{n-2} + \overset{3}{A}x^{n-3} + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots + \overset{m}{A}x^{n-m} + \ldots + \overset{n-1}{A}x + \overset{n}{A} = 0$$

we then obtain the following values of the assumed coefficients $\overset{1}{A}$, $\overset{2}{A}$, $\overset{3}{A}$, ... $\overset{n}{A}$, in which those, whose index is odd, are omitted, because all of them (the last $\overset{n}{A}$ excepted) $= 0$:

$$\overset{2}{A} = -\frac{S2}{2} = -nabp$$

$$\overset{4}{A} = -\frac{\overset{2}{A}S2 + S4}{4} = +\frac{n\,.\,n-3}{1\,.\,2}a^2b^2p^2$$

$$\overset{6}{A} = -\frac{\overset{4}{A}S2 + \overset{2}{A}S4 + S6}{6} =$$

$$-\frac{n\,.\,n-4\,.\,n-5}{1\,.\,2\,.\,3}a^3b^3p^3$$

$$\overset{8}{A} = -\frac{\overset{6}{A}S2 + \overset{4}{A}S4 + \overset{2}{A}S6 + S8}{8} =$$

$$+\frac{n\,.\,n-5\,.\,n-6\,.\,n-7}{1\,.\,2\,.\,3\,.\,4}a^4b^4p^4$$

&c.

Hence the law of the progression is easily known; thus the general term is

$$\overset{2\lambda}{A} = \pm\frac{n\,.\,n-\lambda-1\,.\,n-\lambda-2\ldots\ldots n-2\lambda+1}{1\,.\,2\,\quad 3\,\ldots\ldots\ldots\ldots\ldots\,\lambda}a^\lambda b^\lambda p^\lambda$$

The last coefficient $\overset{n}{A}$, when n is odd,

$$= -a^n p - b^n p^{n-1}$$

and when n is even is

$$\pm \frac{n \,.\, \frac{n}{2} - 1 \,.\, \frac{n}{2} - 2 \,\ldots\ldots\ldots\, 2 \,.\, 1}{1 \,.\, 2 \,.\, 3 \ldots\ldots \frac{n}{2} - 1 \,.\, \frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^{n-1}$$

or, more briefly, $\pm\ 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^n p^{n-1}$
the upper sign of the two $\pm$ obtains, when n is of the form $4m$, and the lower, when n is of the form $4m + 2$.

5. If these values be substituted in the assumed equation, we then obtain, when n is even, the required rational equation

$$x^n - \frac{n}{1} a b p x^{n-2} + \frac{n \,.\, n-3}{1 \,.\, 2} a^2 b^2 p^2 x^{n-4}$$

$$- \frac{n \,.\, n-4 \,.\, n-5}{1 \,.\, 2 \,.\, 3} a^3 b^3 p^3 x^{n-6}$$

$$+ \frac{n \,.\, n-5 \,.\, n-6 \,.\, n-7}{1 \,.\, 2 \,.\, 3 \,.\, 4} a^4 b^4 p^4 x^{n-8}$$

$$- \frac{n \,.\, n-6 \,.\, n-7 \,.\, n-8 \,.\, n-9}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} a^5 b^5 p^5 x^{n-10}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mp\ 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^{\frac{n}{2}} - a^n p - b^n p^{n-1}$$

$$= 0$$

the upper sign of the two $\mp$ obtains, when $n = 4m + 2$, the lower, when $n = 4m$. When n is odd, the same

equation obtains, only in the last term the magnitude $2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$ must be omitted.

EXAMPLE. When $n=5$, the equation is

$$x^5 - 5abpx^3 + 5a^2b^2p^2x - a^5p - b^5p^4$$

and each of its roots is

$$\alpha a\sqrt[5]{p} + \alpha^4 b\sqrt[5]{p^4}$$

Moreover, this is the same equation which was found in II, § CXI. the b in the present equation corresponding to d in the former.

II. When $n=6$, the equation is

$$x^6 - 6abpx^4 + 9a^2b^2p^2x^2 - 2a^3b^3p^3 - a^6p - b^6p^5 = 0$$

and each of its roots is

$$\alpha a\sqrt[6]{p} + \alpha^5 b\sqrt[6]{p^5}$$

REMARK. Compare Michelson's Translation of Euler's Introduction, third Book, pp. 10, 11, with this §. Euler finds the same equation, in a shorter, but less analytical way; what in his method is $\sqrt[n]{\beta}$ and α, in mine is abp and $a^np + b^np^{n-1}$. Compare also with it Huguenin's Mathematical Contributions for the further Improvement of the young Geometrician, p. 181, and so on.

If we put

$$nabp = A,\ a^np + b^np^{n-1} = T$$

then the above general equation, when n is an odd number,

is transformed into

$$x^n - Ax^{n-2} + \frac{n \cdot n-3}{1 \cdot 2} \quad \frac{A^2}{n^2} x^{n-4} -$$

$$\frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \cdot \frac{A^3}{n^3} x^{n-6} + \ldots = T$$

the first part of this equation being continued till we come to a coefficient = o. But from the two equations $nabp = A$, $a^n p + b^n p^{n-1} = T$, we obtain

$$(a^n p - b^n p^{n-1})^2 = (a^n p + b^n p^{n-1})^2 - 4a^n b^n p^n$$

$$= T^2 - \frac{4A^n}{n^n}$$

consequently

$$a^n p - b^n p^{n-1} = \surd(T^2 - \frac{4A^n}{n^n})$$

If we combine this equation with the one $a^n p + b^n p^{n-1} = T$, we obtain, by addition and subtraction,

$$a^n p = \tfrac{1}{2} T + \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})$$

$$b p^{n-1} = \tfrac{1}{2} T - \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})$$

and when we extract the nth root

$$a\sqrt[n]{p} = \sqrt[n]{[\tfrac{1}{2}T + \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})]}$$

$$b\sqrt[n]{p^{n-1}} = \sqrt[n]{[\tfrac{1}{2}T - \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})]}$$

Therefore

$$a\sqrt[n]{[\tfrac{1}{2}T + \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})]} + \frac{1}{a}\sqrt[n]{[\tfrac{1}{2}T - \tfrac{1}{2} \surd (T^2 - \frac{4A^n}{n^n})]}$$

is the general expression for every root of the above equation $x^n - Ax^{n-2} + \&c. = o$.

From the resemblance of this formula to Cardan's, it follows, that the equation of Moivre is only an extension of Cardan's, and that both may be deduced in the same way, as is actually shown in the two above-mentioned works.

SECTION CXVI.

PROB. Make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$ rational, in the case in which n is an odd number, and not divisible by three.

Solution 1. Let

$$x^n - \overset{1}{A}x^{n-1} + \overset{2}{A}x^{n-2} + \overset{3}{A}x^{n-3} + \ldots\ldots - \overset{n}{A} = 0$$

be the required equation, whose roots consequently are

$$\alpha a\sqrt[n]{p} + \alpha^{n-2}b\sqrt[n]{p^{n-2}}, \quad \text{or} \quad \alpha a\sqrt[n]{p} + \frac{1}{\alpha^2}bp\sqrt[n]{\frac{1}{p^2}}$$

$$\beta a\sqrt[n]{p} + \beta^{n-2}b\sqrt[n]{p^{n-2}}, \quad \text{or} \quad \beta a\sqrt[n]{p} + \frac{1}{\beta^2}bp\sqrt[n]{\frac{1}{p^2}}$$

&c. &c.

Further, let the symbols $S1$, $S2$, $S3$, &c. have the same signification in reference to these roots, as in the preceding §.

2. Any undetermined power k of the first of the above roots contains, when α only is considered, the following terms:

$$\alpha^k, \ \alpha^{k-3}, \ \alpha^{k-6}, \ \alpha^{k-9}, \ \ldots\ldots \ \alpha^{-(2k-3)}, \ \alpha^{-2k}$$

the same terms contain also, with reference to β, and so on, the power k of the second root. Consequently Sk, when

α, β, γ, &c. only are considered, consists of the following terms:

$$'[k],\ '[k-3],\ '[k-6],\ \ldots\ldots\ '[-2k+3],\ '[-2k]$$

3. Now, if $k < n$, then amongst these there are no expressions but $'[o]$ and $'[-n]$, whose radical exponents are divisible by n; and indeed the first only obtains, when k is divisible by 3, but the second only when $2k-n$ is positive, and divisible by 3. But both can never occur at the same time in the same numerical expression, for otherwise, contrary to the supposition, n must be divisible by 3. Hence it follows immediately, first, that when $'[o]$ occurs in Sk, k must be of the form 3μ; secondly, when $'[-n]$ occurs in Sk, the least value which k can have, is $\frac{n+3}{2}$, for which I shall substitute m; and thirdly, that then every other value of k must have the form $m+3\nu$. Consequently Sk always vanishes, when k has not one of the two forms 3μ and $m+3\nu$.

4. Now, if we seek the binomial coefficients of $'[o]$ and $'[-n]$, we obtain, when for these numerical expressions their value n is substituted,

$$S3\mu = \frac{3\mu \,.\, 3\mu-1 \,.\, 3\mu-2 \ldots\ldots 2\mu+1}{1 \,.\, 2 \,.\, 3 \qquad\qquad \mu} na^{2\mu} b^{\mu} p^{\mu}$$

$$S\overline{m+3\nu} = \frac{m+3\nu . m+3\nu-1 \ldots m+\nu}{1 \,.\, 2 \ldots\ldots 2\nu+1} na^{2\nu+1} b^{m+\nu-1} p^{m+\nu-3}$$

From the first formula we obtain

$$S3 = \frac{3}{1} na^2 bp$$

$$S6 = \frac{6\,.\,5}{1\,.\,2}\,na^4b^2p^2$$

$$S9 = \frac{9\,.\,8\,.\,7}{1\,.\,2\,.\,3}\,na^6b^3p^3$$

&c.

and from the second

$$Sm = \frac{m}{1}nab^{m-1}p^{m-2} = \tfrac{1}{2}\cdot\frac{n+3}{1}nab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}$$

$$S\overline{m+3} = \frac{m+3\,.\,m+2\,.\,m+1}{1\,.\quad 2\,.\quad 3}\,na^3b^mp^{m-1}$$

$$= \frac{1}{2^3}\cdot\frac{n+9\,.\,n+7\,.\,n+5}{1\,.\quad 2\,.\quad 3}\,na^3b^{\frac{n+3}{2}}p^{\frac{n+1}{2}}$$

$$S\overline{m+6} = \frac{m+6\,.\,m+5\,.\,m+4\,.\,m+3\,.\,m+2}{1\,.\quad 2\,.\quad 3\,.\quad 4\,.\quad 5}\,na^5b^{m+1}p^m$$

$$= \frac{1}{2^5}\cdot\frac{n+15\,.\,n+13\,.\,n+11\,.\,n+9\,.\,n+7}{1\,.\quad 2\,.\quad 3\,.\quad 4\,.\quad 5}\,na^5b^{\frac{n+5}{2}}b^{\frac{n+3}{2}}$$

&c.

Every other Sk, which is not included amongst these, $=$ o.

5. Hence the coefficients of the required equation may be determined by means of the general formula

$$\pi\overset{\pi}{A} = \overset{\pi-1}{A}S1 - \overset{\pi-2}{A}S2 + \overset{\pi-3}{A}S3 - \overset{\pi-4}{A}S4 + \ldots\ldots$$

$$\ldots\ldots\ldots \mp \overset{1}{A}S\overline{\pi-1} \pm S\pi$$

When in this we substitute 1, 2, 3, 4, 5, &c. successively for π, it will immediately be shown, that all the coefficients, those excepted which are under the forms $\overset{3\mu}{A}$, $\overset{m+3\mu}{A}$, vanish, because in the products, of which this formula is compounded, either a coefficient, or a numerical expression, or even both at once vanish. Further, we find

$$\overset{3}{A} = \frac{S3}{3} = na^2bp$$

$$\overset{6}{A} = \frac{\overset{6}{A}S3 - S6}{6} = \frac{n\,.\,n-5}{1\,.\,2}a^4b^2p^2$$

$$\overset{9}{A} = \frac{\overset{6}{A}S3 - \overset{3}{A}S6 + S9}{9} = \frac{n\,.\,n-7\,.\,n-8}{1\,.\,2\,.\,3}a^6b^3p^3$$

&c.

$$\overset{m}{A} = \mp\frac{Sm}{m} = \mp\, nab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}$$

$$\overset{m+3}{A} = \frac{\overset{m}{A}S3 \mp \overset{3}{A}Sm}{m+3} = \frac{1}{2^2}\cdot\frac{n\,.\,n-5\,.\,n-7}{1\,.\,2\,.\,3}a^3b^{\frac{n+3}{2}}p^{\frac{n+1}{2}}$$

$$\overset{m+6}{A} = \frac{\overset{m+3}{A}S3 - \overset{m}{A}S6 \mp \overset{6}{A}Sm \pm \overset{3}{A}S\overline{m+3} \mp S\overline{m+6}}{m+6}$$

$$= \mp\frac{1}{2^4}\cdot\frac{n\,.\,n-7\,.\,n-9\,.\,n-11\,.\,n-13}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^5b^{\frac{n+5}{2}}p^{\frac{n+3}{2}}$$

the upper signs obtain when m is even, the lower when n is odd.

6. The last term $\overset{n}{A}$ can neither be included in the form $\overset{3\mu}{A}$, nor in the form $\overset{m+3\nu}{A}$, because otherwise n must be divisible by 3. But this term can be very easily found by taking the product of all the roots in 1, which product is here reducible merely to the sums of the products of all its first and all its second parts, and consequently $= a^np + b^np^{n-2}$.

7. If we substitute these values of the coefficients in the assumed equation in 1, we then obtain the required rational equation

$$x^n - \frac{n}{1}a^2bpx^{n-3} + \frac{n\,.\,n-5}{1\,.\,2}a^4b^2p^2x^{n-6}$$

$$-\frac{n\,.\,n-7\,.\,n-8}{1\,.\,2\,.\,3}a^6b^3p^3x^{n-9}$$

$$+\frac{n\,.\,n-9\,.\,n-10\,.\,n-11}{1\,.\,2\,.\,3\,.\,4}a^8b^4p^4x^{n-12}$$

$$-\frac{n\,.\,n-11\,.\,n-12\,.\,n-13\,.\,n-14}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{10}b^5p^5x^{n-15}$$

$$+\frac{n\,.\,n-13\,.\,n-14\,.\,n-15\,.\,n-16\,.\,n-17}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}a^{12}b^6p^6x^{n-18}$$

&c.

$$-\frac{n}{1}ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}x^{\frac{n-3}{2}}$$

$$-\frac{1}{2^2}\cdot\frac{n\,.\,n-5\,.\,n-7}{1\,.\,2\,.\,3}a^3b^{\frac{n+3}{2}}p^{\frac{n+1}{2}}x^{\frac{n-9}{2}}$$

$$-\frac{1}{2^4}\cdot\frac{n\,.\,n-7\,.\,n-9\,.\,n-11\,.\,n-13}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^5b^{\frac{n+5}{2}}p^{\frac{n+3}{2}}x^{\frac{n-15}{2}}$$

$$-\frac{1}{2^6}\cdot\frac{n\,.\,n-9\,.\,n-11\,.\,n-13\,.\,n-15\,.\,n-17\,.\,n-19}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}a^7b^{\frac{n+7}{2}}p^{\frac{n+5}{2}}x^{\frac{n-21}{2}}$$

&c.

$$-a^np - b^np^{n-2}$$

$$=0.$$

The two series in this equation are continued till we arrive at the negative exponents of x.

EXAMPLE I. When $n=5$, we find the equation

$$x^5 - 5a^2bpx^2 - 5ab^3p^2x - a^5p - b^5p^3 = 0$$

and for each of its roots

$$a\alpha\sqrt[5]{p} + \alpha^3b\sqrt[5]{p^3}$$

If we put $5a^2bp=B$, $5ab^3p^2=C$, then this equation is

transformed into the following one:

$$x^5 - Bx^2 - Cx - \frac{B^3}{25C} - \frac{C^2}{5B} = 0$$

the same equation, which was found in I, § CXI.

EXAMPLE II. When $n = 7$, we obtain the equation

$$x^7 - 7a^2bpx^4 - 7ab^4p^3x^2 + 7a^4b^2p^2x$$
$$- a^7p - b^7p^5 = 0$$

and each of its roots is

$$\alpha a\sqrt[7]{p} + \alpha^5 b\sqrt[7]{p^5}$$

To this belong all equations of the seventh degree of the form

$$x^7 - 7Ax^4 - 7Bx^2 + 7A^2x - \frac{A^4}{B} - \frac{B^2}{A} = 0$$

as may be easily found, by putting $a^2bp = A$, $ab^4p^3 = B$.

SECTION CXVII.

In the same way as in § CXIII, § CXV, and § CXVI, we can derive innumerable other general forms of solvible equations from the binomial $\sqrt[n]{p^\mu} + \sqrt[n]{p^\nu}$ by eliminating the irrational magnitudes. However, since otherwise this subject possesses no interest, I shall here content myself with giving an equation, which Waring, one of the most celebrated Analysts that England ever possessed, gives in a treatise on the general solution of equations, by assuming $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$ (Philosophical Transactions for the year 1779, p. 92). When n is odd, this equation is

$$x^n - p\,[na^{n-3}bx^2 + \frac{n\,.\,\overline{n-5}}{1\,.\,2}a^{n-6}b^2x^4 +$$

$$\frac{n\,.\,\overline{n-7}\,.\,\overline{n-8}}{1\,.\,2\,.\,3}a^{n-9}b^3x^6 + \frac{n.\overline{n-9}.\overline{n-10}.\overline{n-11}}{1\,.\,2\,.\,3\,.\,4}a^{n-12}b^3x^8$$

$$+ \frac{n\,.\,\overline{n-11}\,.\,\overline{n-12}\,.\,\overline{n-13}\,.\,\overline{n-14}}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{n-15}b^5x^{10} + \&c.]$$

$$\pm\, p^2[na^{\frac{n-3}{2}}b^{\frac{n+1}{2}}x - \frac{1}{2^2}\cdot\frac{n\,.\,\overline{n-5}\,.\,\overline{n-7}}{1\,.\,2\,.\,3}a^{\frac{n-1}{2}}b^{\frac{n+3}{2}}x^3$$

$$+ \frac{1}{2^4}\cdot\frac{n\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\,.\,\overline{n-13}}{1\,.\,2\,.\,3\,.\,4\,.\,5}a^{\frac{n-15}{2}}b^{\frac{n+5}{2}}x^5 -$$

$$\frac{1}{2^6}\,\frac{n.\overline{n-9}.\overline{n-11}.\overline{n-13}.\overline{n-15}.\overline{n-17}.\overline{n-19}}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}a^{\frac{n-21}{2}}b^{\frac{n+7}{2}}x^7 + \&c.]$$

$$- a^np - b^np^3 = 0.$$

The factor p^2 has the sign $+$, when $\frac{n-3}{4}$ is a whole number, but in other cases the sign $-$.

SECTION CXVIII.

The method by which equations of the first degree are made rational, which hitherto we have chiefly used for finding solvable equations, can also be used with advantage when we merely wish to clear the unknown magnitudes in an equation of the radical sign. I shall elucidate this by two examples.

Let the equation

$$p + \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \sqrt[\rho]{t} \ldots\ldots = 0$$

be given, and let p, q, r, s, &c. be rational functions of the unknown magnitudes y, z, &c.: it is required to make this equation rational.

Put $-p = x$; then we have in reference to x the equation of the first degree, viz.

$$x = \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \sqrt[\rho]{t} \ldots\ldots$$

We try to make this equation rational, and then again put $-p$ for x, we then have the required equation clear of irrational magnitudes.

If the equation

$$\sqrt[\mu]{p} + \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} \ldots\ldots = 0$$

be given, in which there is not even one rational term, we put $-\sqrt[\mu]{p} = x$, and then make the equation $x = \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} + \ldots\ldots$ rational. Having done this, we substitute in the obtained equation for x successively its values $-\alpha\sqrt[\mu]{p}$, $-\beta\sqrt[\mu]{p}$, $-\gamma\sqrt[\mu]{p}$, &c.; then there arise μ equations. If we multiply these together, we then obtain the required equation clear of irrational magnitudes.

On the subject of clearing equations of irrational magnitudes, there is a very able treatise by Fischer, in the Hindenburg Archives of Mathematics, Number VIII, written with that clearness and perspicuity peculiar to the author.

The rational equation which is obtained from the equation $x = \sqrt[\mu]{p} + \sqrt[\nu]{q} + \sqrt[\pi]{r} + \sqrt[\rho]{s} + \ldots$ so long as the irrational magnitudes $\sqrt[\mu]{p}$, $\sqrt[\nu]{q}$, $\sqrt[\pi]{r}$, $\sqrt[\rho]{s}$, &c. have no particular relation to each other, always rises to the degree $\mu\nu\pi\rho\ldots$, because the μ values of $\sqrt[\mu]{p}$, the ν vales of $\sqrt[\nu]{q}$, the π values of $\sqrt[\pi]{r}$, and so on, may be combined

together exactly this number of times. But if the above irrational magnitudes have any relation to each other, so that if one or other is determined, the remaining ones are either all, or only in part determined, then the rational equation is always of a lower degree. Here follow a few examples by way of elucidation.

The rational equation for $x=\sqrt[n]{p}+\sqrt[n]{p^2}+\sqrt[n]{p^3}+\sqrt[n]{p^4}+\dots+\sqrt[n]{p^n}$ only rises to the nth degree, because $\sqrt[n]{p^2}=(\sqrt[n]{p})^2$, $\sqrt[n]{p^3}=(\sqrt[n]{p})^3$, $\sqrt[n]{p^4}=(\sqrt[n]{p})^4$, &c. and consequently x contains no more values than the irrational magnitudes $\sqrt[n]{p}$.

The rational equation for $x=\sqrt[12]{p}+\sqrt[3]{p}+\sqrt[5]{q}$ only rises, but necessarily, to the sixtieth degree, because $\sqrt[3]{p}=(\sqrt[12]{p})^4$, and the twelve values of $\sqrt[12]{p}$, combined with the five values of $\sqrt[5]{q}$, gives sixty different values of x.

The rational equation for $x=\sqrt[12]{p^5}+\sqrt[9]{p^7}+\sqrt[8]{p^3}=p^{\frac{5}{12}}+p^{\frac{7}{9}}+p^{\frac{3}{8}}$ only rises to the seventy-second degree. For if we reduce the small fractional exponents to the least common denominator, we have $x=p^{\frac{30}{72}}+p^{\frac{56}{72}}+p^{\frac{27}{72}}$. If $\therefore\ \alpha$ be a root of the equation $y^{72}-1=0$, then is $\alpha^{30}p^{\frac{30}{72}}+\alpha^{56}p^{\frac{56}{72}}+\alpha^{27}p^{\frac{27}{72}}$ or $\alpha^{30}\sqrt[12]{p^5}+\alpha^{56}\sqrt[9]{p^7}+\alpha^{27}\sqrt[8]{p^5}$ the corresponding value of x, and there are seventy-two of these values.

SECTION CXIX.

PROB. Find a factor, by which the given irrational expression $p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots\ldots$ must be multiplied, in order to make it rational.

Solution. Let a, b, c, d, &c. denote the different values which this expression has, when the irrational magnitudes are taken in all possible ways. Now, if we form the equation $x=p+\sqrt[\mu]{q}+\sqrt[\nu]{r}+\sqrt[\pi]{s}+\ldots\ldots$, then the rational equation derived from it, is

$$(x-a)\ (x-b)\ (x-c)\ (x-d)\ \ldots\ldots=0$$

and its last term $=\pm\, a\,b\,c\,d\ldots\ldots$ Now, since this product must be rational, it follows that the given expression is rational, when we multiply it by the product of all the remaining expressions which its different values give; and this product consequently is the factor sought.

EXAMPLE I. Let the expression $p+\sqrt[3]{q}$ be given. If then 1, α, β, be the three roots of the equation $y^3-1=0$, then the required factor

$$=(p+\alpha\sqrt[3]{q})\ (p+\beta\sqrt[3]{q})$$

or, since $\alpha+\beta=-1$, $\alpha\beta=1$,

$$=p^2-p\sqrt[3]{q}+\sqrt[3]{q^2}$$

EXAMPLE II. For the expression $p+\sqrt{q}+\sqrt{r}$, the required factor

$= (p + \sqrt{q} - \sqrt{r})\ (p - \sqrt{q} + \sqrt{r})\ (p - \sqrt{q} - \sqrt{r})$
$= p^3 - pq - pr - (p^2 - q + r)\sqrt{q} -$
$(p^2 + q - r)\sqrt{r} + 2p\sqrt{qr}$
which is already known.

EXAMPLE III. For the expression $p + \sqrt[3]{q} - \sqrt[4]{r}$, when $1, \alpha, \beta$, are the roots of the equation $y^3 - 1 = 0$, we obtain the following factor:

$$(p + \sqrt[3]{q} + \sqrt[4]{r})\ (p + \alpha\sqrt[3]{q} + \sqrt[4]{r})\ (p + \beta\sqrt[3]{q} + \sqrt[4]{r})$$
$$(p + \sqrt[3]{q} + \sqrt[4]{r} \,.\, \sqrt{-1})\ (p + \alpha\sqrt[3]{q} + \sqrt[4]{r} \,.\, \sqrt{-1})$$
$$(p + \beta\sqrt[3]{q} + \sqrt[4]{r} \,.\, \sqrt{-1})\ (p + \sqrt[3]{q} - \sqrt[4]{r} \,.\, \sqrt{-1})$$
$$(p + \alpha\sqrt[3]{q} - \sqrt[4]{r} \,.\, \sqrt{-1})\ (p + \beta\sqrt[3]{q} - \sqrt[4]{r} \,.\, \sqrt{-1})$$
$$(p + \alpha\sqrt[3]{q} - \sqrt[4]{r})\ (p + \beta\sqrt[3]{q} - \sqrt[4]{r})$$

Remark. When $p = 0$, a more simple factor may often be found, by which the object to make the given expression rational may be attained. Thus, if the expression $\sqrt[3]{q} + \sqrt[3]{r}$ be already rational, when we multiply it only by the factor $(\sqrt[3]{q} + \alpha\sqrt[3]{r})\ (\sqrt[3]{q} + \beta\sqrt[3]{r}) = \sqrt[3]{q^2} - \sqrt[3]{qr} + \sqrt[3]{r^2}$; and the expression $\sqrt{q} + \sqrt{r} + \sqrt{s}$ is so by multiplying by the factor $(\sqrt{q} + \sqrt{r} - \sqrt{s})$ $(\sqrt{q} - \sqrt{r} + \sqrt{s})\ (\sqrt{q} - \sqrt{r} - \sqrt{s})$. This is always the case, when the indices of the roots have a common divisor, or the irrational magnitudes have a certain relation to each other.

Corollary. From what has been already said, it follows, that it is always possible, by the multiplication of the

numerator and denominator of a given fraction by a proper factor, to clear the denominator of irrational magnitudes. Consequently also an equation of the form

$$x = \frac{p + \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \ldots}{p' + \sqrt[\mu']{q'} + \sqrt[\nu']{r'} + \sqrt[\pi']{s'} + \ldots}$$ may always be reduced to an equation of the form $x = p + \sqrt[\mu]{q} + \sqrt[\nu]{r} + \sqrt[\pi]{s} + \ldots$ and as one of this kind can always be made rational by the preceding §, so in like manner the former may always be made rational.

SECTION CXX.

Prob. Make the equation $x = \sqrt{p} + \sqrt{q}$ rational, when the magnitudes p and q are not immediately given, but only assumed to be the roots of an equation of the second degree $y^2 - Ay + B = 0$.

Solution. The values which x has by the different determination of its irrational magnitudes, are

$$\sqrt{p} + \sqrt{q}, \quad -\sqrt{p} + \sqrt{q}$$
$$\sqrt{p} - \sqrt{q}, \quad -\sqrt{p} - \sqrt{q}$$

The mere inspection of these values shows, that when we substitute p and q for each other, they undergo no further change, than that one is transformed into the other. Consequently in the rational equation derived from $x = \sqrt{p} + \sqrt{q}$ there is no change, when we substitute p for q, and it must ∴ necessarily be a symmetrical function of these magnitudes; and consequently may be expressed rationally by the coefficients A, B. If ∴ we eliminate

p and q by means of the given coefficients A, B, we obtain the required rational equation.

But from $x = \sqrt{p} + \sqrt{q}$ we obtain the equation (§ XCVIII.)

$$x^4 - 2(p+q)x^2 + (p+q)^2 = 0$$

or

$$x^4 - 2(p+q)x^2 + (p+q)^2 - 4pq = 0$$

If in the latter equation we substitute for $p+q$ and pq their values A and B, we obtain the required equation

$$x^4 - 2Ax^2 + A^2 - 4B = 0$$

SECTION CXXI.

PROB. Make the equation

$$x = \sqrt[n]{p} + \sqrt[n]{q} + \sqrt[n]{r} + \sqrt[n]{s} + \ldots.$$

rational; when the m magnitudes p, q, r, s, &c. are not immediately given, but only assumed to be the roots of a given equation of the mth degree

$$y^m - Ay^{m-1} + By^{m-2} - Cy^{m-3} + \&c. = 0$$

Solution. If we endeavour to find all the possible values of x, which arise from the different combinations of the values of the roots, and then in these put the magnitudes p, q, r, s, &c. for one another in any way, but let it be the same one in all the values, the consequence will only be this, that these values either undergo no change, or merely that one is transformed into another. For let α, β, γ, δ, &c. be the roots of the equation $x^n - 1 = 0$, and $\alpha\sqrt[n]{p} + \beta\sqrt[n]{q} + \gamma\sqrt[n]{r} + \delta\sqrt[n]{s} + \&c.$ any value of x. Now, if it be possible, that from this expres-

sion, by any substitution of the magnitudes p, q, r, s, &c. for one another, viz. by that of p for q, another expression $\alpha\sqrt[n]{q}+\beta\sqrt[n]{p}+\gamma\sqrt[n]{r}+\delta\sqrt[n]{s}+$ &c. is generated, which does not belong to the values of x; then there must be, contrary to the supposition, a combination of the values of the roots, which is not included in the values of x. Since $\therefore$ the values of x remain the same, however the magnitudes p, q, r, s, &c. are subtituted for one another, then must the rational equation derived from $x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+$ &c. be a symmetrical function of these magnitudes, and consequently may be expressed rationally by A, B, C, &c. the coefficients of the given equation. If $\therefore$ we eliminate the magnitudes p, q, r, s, &c. by means of these coefficients, we obtain the required equation.

Corollary. The equation, which we obtain under the condition of the problem, is consequently always of the same degree as the equation derived from $x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+$ &c. when the magnitudes p, q, r, s, &c. are independent of each other. In the latter case, however, the rational equation is of the degree $n\,n\,n\,n\ldots=n^m$, when m denotes the number of the magnitudes p, q, r, s, &c. (§ CXVIII), consequently also in the former.

SECTION CXXII.

RULE The rational equation for

$$x=\sqrt[n]{p}+\sqrt[n]{q}+\sqrt[n]{r}+\sqrt[n]{s}+\text{\&c.}$$

in the case in which the m magnitudes p, q, r, s, &c. are either wholly independent of each other, or the roots of an equation of the mth degree, can only contain such powers of x, whose exponents are divisible by n.

Proof. Let

$$x^k + \overset{1}{A}x^{k-1} + \overset{2}{A}x^{k-2} + \ldots\ldots + \overset{\mu}{A}x^{k-\mu} + \ldots = 0$$

be the equation, which arises from the multiplication of all the positive distinct equations of the form

$$x - \alpha\sqrt[n]{p} - \beta\sqrt[n]{q} - \gamma\sqrt[n]{r} - \delta\sqrt[n]{s} - \ldots\ldots = 0$$

where $\alpha, \beta, \gamma, \delta$, &c. denote the roots of the equation $x^n - 1 = 0$; now it is evident, that the undetermined coefficient $\overset{\mu}{A}$ can contain no other numerical expressions of these roots, but those in which the sum of the radical exponents $= \mu$. Now, since these always vanish, when μ is not divisible by n (§ XCVI. 2), then likewise must the term $\overset{\mu}{A}x^{k-\mu}$ always vanish. But when μ is not divisible by n, then also $k-\mu$ cannot be divisible by n, because $k = n^m$; $\therefore$ all those terms vanish which contain $k-\mu$, consequently the exponent of x is not divisible by n. Therefore the rational equation contains those terms only, in which the exponent of x is divisible by n. *Q. E. D.*

SECTION CXXIII.

Prob. Let the m magnitudes y, z, t, u, &c. be given by the m equations

$$y^\mu + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = o$$
$$z^\nu + A'z^{\nu-1} + B'z^{\nu-2} + C'z^{\nu-3} + \&c. = o$$
$$t^\pi + A''t^{\pi-1} + B''t^{\pi-2} + C''t^{\pi-3} + \&c. = o$$
$$\&c.$$

consequently irrational: required to make the equation

$$x = y + z + t + u + \&c.$$

rational.

Solution. Since y is given by an equation of the μth degree, z by an equation of the νth degree, and so on, we have μ values for y, ν values for z, and so on. If we combine all these values in as many ways as possible, to the number $y + z + t + u +$ &c., we find all the values of x, and consequently by the multiplication of all the distinct equations of the form $x - (y+z+t+u+\&c.) = o$ the required equation, which necessarily is of the degree $\mu\nu\pi\ldots$, because the different values of y, z, t, u, &c. may be combined in this number of ways. Now, I assert, that with respect to the different values of y, which may be y', y'', y''', &c. this equation is symmetrical. For since the values of x in the substitution of these magnitudes, undergo no further change, than that one is transformed into another, consequently, also the equation itself must be such, that it undergoes no change in the substitution of the magnitudes, y', y'', y''', &c. Therefore, the magnitudes y', y'', y''', &c. may be eliminated by means of the coefficients A, B, C, &c, What has been said here of y and its different values y', y'', y''', &c. applies also to z, t, &c. and its values z', z'', z''', &c. t', t'', t''', &c. &c.; and consequently these

magnitudes may also be eliminated by means of the coefficients A', B', C', &c. A'', B'', C'', &c. &c. In this way we likewise obtain a rational equation for x, which only contains known magnitudes; and this is the equation sought.

Example. Let $x = y + z$; let the magnitudes y and z be given by the two equations

$$y^3 - Ay^2 + By - C = 0$$
$$z^2 - A'z + B' = 0$$

required to find the rational equation for x.

The different values of x are

$$\begin{array}{l} y' + z',\ y' + z'' \\ y'' + z',\ y'' + z' \\ y''' + z',\ y''' + z'' \end{array}$$

Denote the sum of the first, second, third, &c. powers of these roots by $S1$, $S2$, $S3$, &c. we then have

$$\begin{aligned} S1 &= 2(y' + y'' + y''') + 3(z' + z'') = 2A + 3A' \\ S2 &= 2(y'^2 + y''^2 + y'''^2) + 2(y' + y'' + y''')(z' + z'') \\ &\quad + 3(z'^2 + z''^2) \\ &= 2(A^2 - 2B) + 2AA' + 3(A'^2 - 2B') \\ S3 &= 2(y'^3 + y''^3 + y'''^3) + 3(y'^2 + y''^2 + y'''^2)(z' + z'') \\ &\quad + 3(y' + y'' + y''')(z'^2 + z''^2) + 3(z'^3 + z''^3) \\ &= 2(A^3 - 3AB + 3C) + 3(A^2 - 2B)A' + \\ &\quad 3A(A'^2 - 2B') + (3A'^3 - 3A'B') \end{aligned}$$

&c.

Thus having calculated the values of $S1$, $S2$, $S3$, &c. when the required equation is represented by

$$x^6 - \overset{1}{A}x^5 + \overset{2}{A}x^4 - \overset{3}{A}x^3 + \overset{4}{A}x^2 - \overset{5}{A}x + \overset{6}{A} = 0$$

we then obtain its coefficients, by means of the equations

$$\overset{1}{A} = S1$$

$$\overset{2}{A} = \frac{\overset{1}{A}S1 - S2}{2}$$

&c.

SECTION CXXIV.

PROB. Let $f:\ (y)\ (z)\ (t)\ (u) \ldots\ldots$ denote any rational function of the magnitudes y, z, t, u, &c., which are represented by the same number of equations

$$y^{\mu} + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0$$
$$z^{\nu} + Bz^{\nu-1} + Bz^{\nu-2} + Cz^{\nu-3} + \&c. = 0$$
$$t^{\pi} + At^{\pi-1} + Bt^{\pi-2} + Ct^{\pi-3} + \&c. = 0$$

find an equation for the values of this function.

Solution. Put $x = f:\ (y)\ (z)\ (t)\ (u) \ldots$, find all the possible values of this function, and from these form the equation for x: then eliminate the values y', y'', y''', &c. of y, by means of the coefficients A, B, C, &c. of that equation by which this magnitude is given; which may always be done, because the equation for x must necessarily be a symmetrical function of the magnitudes y', y'', y''', &c. If we proceed with z, t, u, &c. in the same way as we did with y, we obtain an equation for x, whose coefficients are all known; and this is the required equation.

Corollary. In order to find all the values of x, we must combine the μ values of y, the ν values of z, and

so on, in all possible ways in the function $f:(y)(z)(t)(u)\ldots$ But it is evident, that the number of these combinations $= \mu\nu\pi\ldots$; consequently also this product gives the number of the values of x, and $\therefore$ the degree of the transformed equation.

If the function only contains the magnitude y, or if $x = f:(y)$, then the equation for y is only of the μth degree, consequently in this case, the transformed equation is of the same degree, as the equation by which y is given.

SECTION CXXV.

PROB. The unknown magnitude x is given by the equation of the nth degree, viz.

$$x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \&c. = 0$$

the coefficients P, Q, &c. however, are not given immediately, but merely all assumed to be known functions of a magnitude y, which depends on an equation of the nth degree

$$y^\mu + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0$$

required to find an equation for x, which only contains known magnitudes.

Solution. Denote the roots of the equation $y^\mu + Ay^{\mu-1} + \&c. = 0$ by y', y'', y''', &c., and introduce these values into the functions P, Q, &c. Now if we denote that, into which these functions are transformed, when we substitute in them y', y'', y''', &c. successively for y, by P', Q', &c., P'', Q'', &c. P''', Q''', &c. &c.; we then

obtain the following μ equations, all of which must obtain at the same time:

$$x^n + P'x^{n-1} + Q'x^{n-2} + R'x^{n-3} + \&c. = 0$$
$$x^n + P''x^{n-1} + Q''x^{n-2} + R''x^{n-3} + \&c. = 0$$
$$x^n + P'''x^{n-1} + Q'''x^{n-2} + R'''x^{n-3} + \&c. = 0$$
&c.

The product of these ∴ gives the required equation.

But since these equations are such, that in the transposition of the magnitudes y', y'', y''', &c. no other change takes place, but that one is merely transformed to another; consequently their product suffers no change in their transposition, and ∴ with reference to y', y'', y''', &c. they are symmetrical. Therefore these magnitudes may always be eliminated by means of the given coefficients A, B, C, &c.

Corollary. But conversely, every equation of the $n\mu$th degree, which can be considered as arising from the elimination of y in the two equations $x^n + Px^{n-1} + \&c. = 0$, $y^\mu + Ay^{\mu-1} + \&c. = 0$, may always be solved, if the solution of equations of the nth and μth degrees be presupposed; for the second equation gives the value of y, and when we substitute this value in the first equation, then the latter gives the value of x.

SECTION CXXVI.

Prob. The unknown magnitude x is given by the equation of the nth degree, viz.

$$x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} + \&c. = 0$$

It is assumed, that the coefficients P, Q, R, &c. are functions of the magnitudes y, z, t, u, &c., and that these magnitudes are given by the equations

$$\begin{aligned} &y^{\mu} + Ay^{\mu-1} + By^{\mu-2} + Cy^{\mu-3} + \&c. = 0 \\ &z^{\nu} + A'z^{\nu-1} + B'z^{\nu-2} + C'z^{\nu-3} + \&c. = 0 \\ &t^{\pi} + A''t^{\pi-1} + B''t^{\pi-2} + C''t^{\pi-3} + \&c. = 0 \\ &\qquad\qquad \&c. \end{aligned}$$

Find an equation for x, which contains known magnitudes only.

Solution. In the first place we consider P, Q, R, &c. merely as functions of y, and eliminate these magnitudes by the method in the preceding §; then we obtain an equation for x, of the $n\mu$th degree, which only contains the unknown magnitudes z, t, u, &c. If in the same way we also eliminate the magnitudes z, t, u, &c. successively, we shall at length get an equation of the $n\mu\nu\pi$...th degree, which only contains x and the known magnitudes A, B, C, &c. A', B', C', &c., A'', B'', C'', &c. &c. and which consequently is the equation sought.

VI.—General inquiries respecting the transformation of equations.

SECTION CXXVII.

AT the end of the fourth chapter the transformation of equations and Tschirnhausen's method were mentioned, and its application to the solution of equations of the third and fourth degrees. This is now the proper place to give some deeper inquiries respecting it, in order that we may ascertain, what may be expected of this method in its application to equations of higher degrees.

SECTION CXXVIII.

Prob. Let

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

be the given equation, and

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots + kx + l = y$$

the auxiliary one; consequently the transformed equation for y is of the nth degree (§ LXXXI), viz.

$$y^n + \mathfrak{A}y^{n-1} + \mathfrak{B}y^{n-2} + \mathfrak{C}y^{n-3} + \&c. = 0$$

Show in what dimension the coefficients a, b, c, &c. of the assumed equation enter into the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.

Solution. If x', x'', x''', &c. denote the roots of the given equation, then the transformed equation $y^n + \mathfrak{A}y^{n-1} + \text{\&c.} = 0$ has the following roots:

$$
\begin{aligned}
y' &= x'^m + ax'^{m-1} + bx'^{m-2} + \ldots\ldots + kx' + l \\
y'' &= x''^m + ax''^{m-1} + bx''^{m-2} + \ldots\ldots + kx'' + l \\
y''' &= x'''^m + ax'''^{m-1} + bx'''^{m-2} + \ldots\ldots + kx''' + l \\
&\text{\&c.}
\end{aligned}
$$

Now, since the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. are the sums of these roots taken singly, two and two, three and three, and so on, we may ∴ conclude, that the letters a, b, c, &c. occur in $\mathfrak{A}$ in the first dimension, in $\mathfrak{B}$ in the second, in $\mathfrak{C}$ in the third, and, generally, in the pth coefficient in the pth power.

SECTION CXXIX.

Prob. Required to find of what degree the auxiliary equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots + kx + l = y$$

must be, when it is possible to transform the general equation of the nth degree

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{\&c.} = 0$$

into an equation of two terms of the form

$$y^n - V = 0$$

Solution. In the equation $y^n - V = 0$, $n-1$ terms are wanting; ∴ the auxiliary equation contains as many undetermined magnitudes a, b, c, &c. by determining which, we are enabled to eliminate these terms; it must consequently be of the $n-1$th degree, and ∴ $m = n - 1$.

SECTION CXXX.

Prob. To transform the given equation $x^3 - Ax^2 + Bx - C = 0$ into one of two terms $y^3 - V = 0$, we must assume the auxiliary equation $x^2 + ax + b = y$ (preceding §): determining, a priori, the degree of the equations, on which the coefficients a, b, depend.

Solution 1. If we denote one of the two primitive roots of the equation $y^3 - 1 = 0$ by α, and put $\sqrt[3]{V} = y'$, then y', $\alpha y'$, $\alpha^2 y'$ are the three roots of the equation $y^3 - V = 0$. Each of the three roots x', x'', x''', of the given equation corresponds to one of the values of y: which? It remains undetermined.

2. If we combine the values of x in all possible ways with the values of y, we then obtain the six following combinations:

$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x', & x'', & x''' \end{pmatrix}; \quad \begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x', & x''', & x'' \end{pmatrix}$$
$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x'', & x', & x''' \end{pmatrix}; \quad \begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x'', & x''', & x' \end{pmatrix}$$
$$\begin{pmatrix} y', & \alpha y', & \alpha^2 y' \\ x''', & x', & x'' \end{pmatrix}; \quad \begin{pmatrix} y', & y', & \alpha^2 y' \\ x''', & x'', & x' \end{pmatrix}$$

3. If the values of x and y, which correspond to each other in the first combination, be substituted in the auxiliary equation, we obtain the three equations:

$$\begin{aligned} x'^2 + ax' + b &= y' \\ x''^2 + ax'' + b &= \alpha y' \\ x'''^2 + ax''' + b &= \alpha^2 y' \end{aligned}$$

and by means of these equations the values of a and b may be determined from x', x'', x'''. In the first place, in order to determine a, multiply the second equation by α, and the third by α^2, and then add it to the first; thus, since $\alpha^4 = \alpha^3 . \alpha = \alpha$, and $1 + \alpha + \alpha^2 = \sqrt[3]{1} = 0$,

$$x'^2 + \alpha x''^2 + \alpha^2 x'''^2 + a\,(x' + \alpha x'' + \alpha^2 x''') = 0$$

we obtain

$$a = -\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}$$

4. From the remaining five combinations in 2, there may be found five other values of a. To effect this, however, it is not necessary to begin the calculation anew: for since the above combinations only differ in this, that the roots x', x'', x''' are transposed, we can even take the transposition found in the expression for a. By these means we obtain the six following values.

$$\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}, \quad \frac{x'^2 + \alpha x'''^2 + \alpha x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$\frac{x''^2 + \alpha x'''^2 + \alpha^2 x'^2}{x'' + \alpha x''' + \alpha^2 x'}, \quad \frac{x'''^2 + \alpha x''^2 + \alpha^2 x'^2}{x''' + \alpha x'' + \alpha^2 x'}$$

$$\frac{x'''^2 + \alpha x'^2 + \alpha^2 x''^2}{x''' + \alpha x' + \alpha^2 x''}, \quad \frac{x''^2 + \alpha x'^2 + \alpha^2 x'''^2}{x'' + \alpha x' + \alpha^2 x'''}$$

5. But of these six values, only the two first, which are opposite to each other, are actually different. For if in these two values we multiply both the numerator and denominator by α^2, we then obtain the next two; and from these again the two last, by multiplying the numerator and denominator by α^2

6. The coefficient a .·. has only the two different values

$$\frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''}, \quad \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

and consequently it depends merely on an equation of the second degree.

7. Further, if we add the three equations in 3 together, we obtain

$$x'^2 + x''^2 + x'''^2 + a\,(x' + x'' + x''') + 3b = 0$$

or

$$[2] + a\,[1] + 3b = 0,$$

and when in this we substitute for a its two values, we then also obtain for b two values, and consequently this coefficient also depends on an equation of the second degree.

Corollary. If we wish actually to find the equation for a, then let

$$a^2 - Pa + Q = 0.$$

The roots of this equation, then, are the two which were found in 6, and we have .·.

$$P = \frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''} + \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$= \frac{2\,[3] + (\alpha + \alpha^2)\,[12]}{[2] + (\alpha + \alpha^2)\,[1^2]}$$

$$Q = \frac{x'^2 + \alpha x''^2 + \alpha^2 x'''^2}{x' + \alpha x'' + \alpha^2 x'''} \times \frac{x'^2 + \alpha x'''^2 + \alpha^2 x''^2}{x' + \alpha x''' + \alpha^2 x''}$$

$$= \frac{[4] + (\alpha + \alpha^2)\,[2^2]}{[2] + (\alpha + \alpha^2)\,[1^2]}$$

If in these we put for the numerical expressions their values taken from the annexed tables, we then obtain, since $a + a^2 = -1$,

$$P = \frac{2A^3 - 7AB + 9C}{A^2 - 3B}$$

$$Q = \frac{4^4 - 4A^2B + B^2 + 6AC}{A^2 - 3B}$$

If we substitute these values of P and Q in the equation $a^2 - Pa + Q = 0$, we then obtain by its solution the two values of a, expressed by the coefficients A, B, C of the given equation.

Having, however, found the two values of a, we then obtain from them the two values of b, by means of the equation (7).

$$b = -\frac{[2] + a\,[1]}{3}$$

SECTION CXXXI.

Prob. To reduce the given equation of the fourth degree $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ to an equation of the form $y^4 - Vy^2 + Z = 0$, we must make use of the auxiliary equation $x^2 + ax + b = y$: required to determine, a priori, the degrees of the equations, on which the coefficients a, b, depend.

Solution 1. Since in the transformed equation there are only even powers of y, then two and two are equal and opposite each other. If ∴ we denote these by y', $-y'$, y'', $-y''$, and the roots of the given equation by x', x'', x''', x^{IV}, we then have, when these values of x and y are made use of in the auxiliary equation,

$$\begin{aligned} x'^2 + ax' + b &= y' \\ x''^2 + ax'' + b &= -y' \\ x'''^2 + ax''' + b &= y'' \\ x'^{v2} + ax'^{v} + b &= -y'' \end{aligned}$$

If we add the two first and the two last of these equations together, we obtain

$$\begin{aligned} x'^2 + x''^2 + a\,(x' + x'') + 2\,b &= 0 \\ x'''^2 + x'^{v2} + a\,(x''' + x'^{v}) + 2\,b &= 0 \end{aligned}$$

and hence

$$a = -\frac{x'^2 + x''^2 - x'''^2 - x'^{v2}}{x' + x'' - x''' - x'^{v}}$$

2. Since it matters not, which values of x and y are considered as belonging to each other, consequently a has not only this single value, but all the values at once, which can arise from all the possible transpositions of the roots. But the function which was found for a, belongs, as may be easily seen, to that class of functions, for which

$$f\colon (x')\,(x'')\,(x''')\,(x'^{v}) = f\colon (x'')\,(x')\,(x''')\,(x'^{v}) = \\ f\colon (x')\,(x'')\,(x'^{v})\,(x''') = f\colon (x''')\,(x'^{v})\,(x')\,(x'')$$

consequently to the same class, which we have treated of in the second example, § LV. This function ∴, as was there found, has no more than three different values, which are given by the types $f\colon (x')\,(x'')\,(x''')\,(x'^{v})$, $f\colon (x''')\,(x')\,(x'^{v})\,(x'')$, $f\colon (x'')\,(x''')\,(x'^{v})\,(x')$. If for these symbols we put what is denoted by them, we then obtain the following three values for a:

$$-\frac{x'^2 + x''^2 - x'''^2 - x'^{v2}}{x' + x'' - x''' - x'^{v}}$$

$$-\frac{x^{\prime\prime\prime 2} + x^{\prime 2} - x^{\prime \mathrm{v} 2} - x^{\prime\prime 2}}{x^{\prime\prime\prime} + x^{\prime} - x^{\prime \mathrm{v}} - x^{\prime\prime}}$$

$$-\frac{x^{\prime\prime 2} + x^{\prime\prime\prime 2} - x^{\prime \mathrm{v} 2} - x^{\prime 2}}{x^{\prime\prime} + x^{\prime\prime\prime} - x^{\prime \mathrm{v}} - x^{\prime}}$$

and consequently the coefficient a depends on an equation of the third degree.

3. Further, if we add the four equations in 1, we obtain

$$x^{\prime 2} + x^{\prime\prime 2} + x^{\prime\prime\prime 2} + x^{\prime \mathrm{v} 2} + a(x^{\prime} + x^{\prime\prime} + x^{\prime\prime\prime} + x^{\prime \mathrm{v}}) + 4b = 0$$

or

$$[2] + a\,[1] + 4b = 0$$

$\therefore$

$$b = -\frac{[2] + a\,[1]}{4}$$

Now, since a has three values, b has also three, and consequently b depends on an equation of the third degree.

Remark. Hence, if it were required, we could find the equations for a and b, by the method sufficiently well known from the third chapter; but since this subject possesses no difficulty, I shall not detain my readers any longer with it.

SECTION CXXXII.

Prob. To reduce the equation of the fourth degree $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ to one of two terms, $y^4 - V = 0$, we must assume an auxiliary equation with three unknown magnitudes, because three terms must vanish. Let $x^3 + ax^2 + bx + c = y$ be this auxiliary

equation. Required to find the degree of the equations, on which the coefficients a, b, c depend.

Solution 1. The roots of the equation $y^4 - V = o$ are, when we put $\sqrt[4]{V} = y'$, y', $-y'$, $+ y'\sqrt{-1}$ $- y'\sqrt{-1}$. The roots of the given equation x', x'', x''', x^{IV}, may correspond to these values of y in $1 \cdot 2 \cdot 3 \cdot 4 = 24$ different ways; viz. as often as these values can be transposed. For while these are undetermined, it is quite immaterial, which values of x and y are considered as belonging to one another. We can, however, as was also done in the two preceding sections, combine the values of x and y together in any way we please, and then introduce in the results the above-mentioned transpositions.

2. The substitution of the values of x and y in the auxiliary equation gives

$$(\phi)\quad \begin{aligned} x'^3 + ax'^2 + bx' + c &= y' \\ x''^3 + ax''^2 + bx'' + c &= -y' \\ x'''^3 + ax'''^2 + bx''' + c &= y'\sqrt{-1} \\ x^{IV3} + ax^{IV2} + bx^{IV} + c &= -y'\sqrt{-1} \end{aligned}$$

3. If the two first and the two last equations be added together, we obtain

$$x'^3 + x''^3 + a(x'^2 + x''^2) + b(x' + x'') + 2c = o$$
$$x'''^3 + x^{IV3} + a(x'''^2 + x^{IV2}) + b(x''' + x^{IV}) + 2c = o$$

and when these again are subtracted from one another

$$x'^3 + x''^3 - x'''^3 - x^{IV3} + a(x'^2 + x''^2\, x'''^2 - x^{IV2})$$
$$+ b(x' + x'' - x''' - x^{IV}) = o.$$

4. But if we subtract the two first and the two last of the equations (ϕ) from one another, we obtain

$$x'^3 - x''^3 + a\,(x'^2 - x''^2) + b\,(x' - x'') = 2y'$$
$$x'''^3 - x^{\prime\mathrm{v}3} + a\,(x'''^2 - x^{\prime\mathrm{v}2}) + b\,(x''' - x^{\prime\mathrm{v}}) = 2y'\sqrt{-1}$$

and if we multiply the second of these equations by $\sqrt{-1}$, and then add it to the first,

$$\begin{aligned} & x'^3 - x''^3 + (x'''^3 - x^{\prime\mathrm{v}3})\sqrt{-1} \\ & + a\,[x'^2 - x''^2 + (x'''^2 - x^{\prime\mathrm{v}2})\sqrt{-1}] \\ & + b\,[x' - x'' + (x''' - x^{\prime\mathrm{v}})\sqrt{-1}] = 0 \end{aligned}$$

5. Since this equation, as also the one found in 3, only contains a and b, $\therefore$ they can conjointly serve to determine these two coefficients. Thus if we eliminate b, and, for the sake of brevity, put

$$\left(\begin{array}{l} (x'^3 + x''^3 - x'''^3 - x^{\prime\mathrm{v}3})\,(x' - x'') \\ -(x'^3 - x''^3)\,(x' + x'' - x''' - x^{\prime\mathrm{v}}) \end{array}\right) = M$$

$$\left(\begin{array}{l} (x'^3 + x''^3 - x'''^3 - x^{\prime\mathrm{v}3})(x''' - x^{\prime\mathrm{v}}) \\ -(x'''^3 - x^{\prime\mathrm{v}3})\,(x' + x'' - x''' - x^{\prime\mathrm{v}}) \end{array}\right) = N$$

$$\left(\begin{array}{l} (x'^2 + x''^2 - x'''^2 - x^{\prime\mathrm{v}2})\,(x' - x'') \\ -(x'^2 - x''^2)\,(x' + x'' - x''' - x^{\prime\mathrm{v}}) \end{array}\right) = P$$

$$\left(\begin{array}{l} (x'^2 + x''^2 - x'''^2 - x^{\prime\mathrm{v}2})\,(x''' - x^{\prime\mathrm{v}}) \\ -(x'''^2 - x^{\prime\mathrm{v}2})\,(x' + x'' - x''' - x^{\prime\mathrm{v}}) \end{array}\right) = Q$$

we then obtain the equation

$$M + N\sqrt{-1} + (P + Q\sqrt{-1})\,a = 0$$

and hence

$$a = -\frac{M + N\sqrt{-1}}{P + Q\sqrt{-1}}$$

6. If in the functions denoted by M, N, P, Q, we substitute x''' for x', $x^{\prime\mathrm{v}}$ for x'', x'' for x''', and x' for

$x^{\prime v}$, then M is transformed into $-N$, N into M, P into $-Q$, and Q into P. Therefore the expression found for a, by this substitution, is transformed into $-\frac{-N+M\sqrt{-1}}{-Q+P\sqrt{-1}}$, or, if we multiply the numerator and denominator by $-\sqrt{-1}$, into $-\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}}$, consequently the same as before. The expression for a belongs $\therefore$ to that class of functions for which

$$f: (x') (x'') (x''') (x^{\prime v}) = f: (x''') (x^{\prime v}) (x'') (x').$$

7. From this equation we obtain (§ LIV) the following period of equal types:

$$\begin{array}{l} f: (x') (x'') (x''') (x^{\prime v}) \\ f: (x''') (x^{\prime v}) (x'') (x') \\ f: (x'') (x') (x^{\prime v}) (x''') \\ f: (x^{\prime v}) (x''') (x') (x'') \end{array}$$

Amongst the $1 . 2 . 3 . 4 = 24$ values, which the coefficient a contains by the transposition of the roots x', x'', x''', $x^{\prime v}$, there are 6 times 4 equal values, consequently only 6 different ones, which are denoted by the following types:

$$\begin{array}{l} f: (x') (x'') (x''') (x^{\prime v}) \\ f: (x') (x'') (x^{\prime v}) (x''') \\ f: (x') (x''') (x'') (x^{\prime v}) \\ f: (x') (x''') (x^{\prime v}) (x'') \\ f: (x') (x^{\prime v}) (x'') (x''') \\ f: (x') (x^{\prime v}) (x''') (x'') \end{array}$$

and which are obtained merely by the transposition of the three roots x'', x''', $x^{\prime v}$. Since $\therefore$ a has six different

values, these magnitudes consequently depend on an equation of the sixth degree.

8. In the same way it may be proved, that b likewise depends on an equation of the sixth degree. For if we eliminate a instead of b in the equations found in 3 and 4, we obtain, when for the sake of brevity we put

$$\left(\begin{array}{l} (x'^3+x''^3-x'''^3-x'^{v3})\ (x'^2-x''^2) \\ -(x'^3-x''^3)\ (x'^2+x''^2-x'''^2-x'^{v2}) \end{array}\right) = M$$

$$\left(\begin{array}{l} (x'^3+x''^3-x'''^3-x'^{v3})\ (x'''^2-x'^{v2}) \\ -(x'''^3-x'^{v3})\ (x'^2+x''^2-x'''^2-x'^{v2}) \end{array}\right) = N$$

$$\left(\begin{array}{l} (x'+x''-x'''-x'^{v})\ \ (x'^2-x''^2) \\ -(x'-x'')\ (x'^2+x''^2-x'''^2-x'^{v2}) \end{array}\right) = P$$

$$\left(\begin{array}{l} (x'+x''-x'''+x'^{v})\ (x'''^2-x'^{v2}) \\ -(x'''-x'^{v})\ (x'^2+x''^2-x'''^2-x'^{v2} \end{array}\right) = Q$$

the equation

$$M+N\sqrt{-1}+b\ (P+Q\sqrt{-1})=0$$

and hence

$$b=-\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}}$$

If we substitute the roots x', x'', x''', x'^{v}, for the roots x''', x'^{v}, x'', x', respectively, then the expressions M, N, P, Q, are respectively transformed into $-N$, M, $-Q$, P; and ∴ it may be shewn, as in 6, that the expression found for b is of the form $f: (x')\ (x'')\ (x''')\ (x'^{v}) = f: (x''')\ (x'^{v})\ (x'')\ (x')$, and consequently has no more than six different values. Consequently b also depends on an equation of the sixth degree.

9. If the four equations in 2 be added together, we obtain

$$[3] + a\,[2] + b\,[1] + 4\,c = 0$$

$\therefore$

$$c = -\frac{[3] + a\,[2] + b\,[1]}{4}$$

consequently if the coefficients a and b are found, we then also have c.

10. Therefore the magnitudes a, b, depend on equations of the sixth degree, and consequently the transformation required in the Problem is not practicable, when these equations are not reducible to equations of the second or third degree. But of the possibility or impossibility of such a reduction, we can convince ourselves syllogistically without completing the calculation, as the following § will show. Moreover, I must again remind my readers, that the magnitudes a and b, are homogeneous functions of the roots x', x'', x''', x^{v}, (§ L); and it will be seen from the following chapter, that in this case it is quite sufficient, merely to have found one of these two magnitudes, because then the other may always be derived directly from it, without its being first necessary to solve a new equation. I shall $\therefore$ merely confine myself to inquiries respecting the equation for a.

SECTION CXXXIII.

Prob. Required to find, whether the equation of the sixth degree, on which the magnitude a of the preced-

ing § depends, is reducible to equations of a lower degree.

Solution 1. In the preceding § we have seen that the six roots of the equation for a, are expressed by the following types:

$$f: (x')\,(x'')\,(x''')\,(x^{iv}),\quad f: (x')\,(x'')\,(x^{iv})\,(x''')$$
$$f: (x')\,(x''')\,(x'')\,(x^{iv}),\quad f: (x')\,(x''')\,(x^{iv})\,(x'')$$
$$f: (x')\,(x^{iv})\,(x'')\,(x'''),\quad f: (x')\,(x^{iv})\,(x''')\,(x'')$$

I shall now assume, that the values of a, which correspond to the two first types, are the roots of an equation of the second degree,

$$a^2 - pa + q = o$$

then, according to the nature of equations,

$$p = f: (x')\,(x'')\,(x''')\,(x^{iv}) + f: (x')\,(x'')\,(x^{iv})\,(x''')$$
$$q = f: (x')\,(x'')\,(x''')\,(x^{iv}) \times f: (x')\,(x'')\,(x^{iv})\,(x''')$$

2. However, from this compound form, it immediately follows that p and q are functions of the form

$$\phi: (x')\,(x'')\,(x''')\,(x^{iv}) = \phi: (x')\,(x'')\,(x^{iv})\,(x''')$$

because in the transformation of the roots x''', x^{iv}, $f: (x')\,(x'')\,(x''')\,(x^{iv})$ into $f: (x')\,(x'')\,(x^{iv})\,(x''')$, and conversely, when the last is transformed into the first, no change arises in the expressions for p and q.

3. But by the preceding § the function denoted by f is of the form

$$f: (x')\,(x'')\,(x''')\,(x^{iv}) = f: (x''')\,(x^{iv})\,(x'')\,(x')$$

consequently also p and q are functions of this form. Therefore p and q are functions of the form

$$\phi : (x') (x'') (x''') (x^{\prime v}) = \phi : (x''') (x^{\prime v}) (x'') (x')$$
$$= \phi : (x') (x'') (x^{\prime v}) (x''')$$

and this form gives no more than three different values, viz.

$$\phi : (x') (x'') (x''') (x^{\prime v})$$
$$\phi : (x') (x''') (x'') (x^{\prime v})$$
$$\phi : (x') (x^{\prime v}) (x'') (x''')$$

4. Therefore the coefficients p, q depend only on two cubic equations

$$p^3 - A'p^2 + B'p + C' = 0$$
$$q^3 - A''q^2 + B''q - C'' = 0$$

If, therefore, we denote the three roots of the first equation by p', p'', p''', and the three roots of the second by q', q'', q''', we then obtain the three following equations of the second degree:

$$a^2 - p'a + q' = 0$$
$$a^2 - p''a + q'' = 0$$
$$a^2 - p'''a + q''' = 0$$

into which ∴ the equation of the sixth degree for a may be analyzed.

Corollary. If we substitute the roots x''', $x^{\prime v}$, for one another in the expressions M, N, P, Q, in 5 of the preceding §, the expressions M and P remain the same; the expressions N and Q, on the contrary, are transformed into $-N$, $-Q$. If ∴ we put

$$f : (x') (x'') (x''') (x^{\prime v}) = - \frac{M + N\sqrt{-1}}{P + Q\sqrt{-1}}$$

then

$$f:\ (x')\,(x'')\,(x^{\prime v})\,(x''') = -\frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

Therefore

$$p = -\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}} - \frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

$$q = -\frac{M+N\sqrt{-1}}{P+Q\sqrt{-1}} \times -\frac{M-N\sqrt{-1}}{P-Q\sqrt{-1}}$$

or

$$p = -\frac{2(MP+NQ)}{P^2+Q^2},\quad q = \frac{M^2+N^2}{P^2+Q^2}$$

From these functions the equations for p and q may be actually found by the method given in the third chapter.

SECTION CXXXIV.

The results in the two foregoing Sections may also be immediately derived from considering the equations (ϕ) in 2, § CXXXII. Thus, in § CXXXII, in order to combine the four values of x with the four values of y in all possible ways, instead of pre-supposing a transformation of the former, as was there done, we assume a transformation of the latter. The equations (ϕ), by the first as well as the other transformation, undergo twenty-four changes; and since each such change gives a value of a, we obtain twenty-four values of a, of which, as we have already seen, no more than six are different.

This conclusion, however, might have been foreseen, à priori, without knowing the value of a. Amongst the twenty-four combinations of the roots x', x'', x''', $x^{\prime v}$ with y', $-y'$, $y'\sqrt{-1}$, $-y'\sqrt{-1}$, there are also the following

$$x'^3 + ax'^2 + bx' + c = \ \ y'\sqrt{-1}$$
$$x''^3 + ax''^2 + bx'' + c = -y'\sqrt{-1}$$
$$x'''^3 + ax'''^2 + bx''' + c = -y'$$
$$x^{\prime v 3} + ax^{\prime v 2} + bx^{\prime v} + c = \ \ y'$$

and these four equations might have been obtained from those in 2, § CXXXII, by substituting throughout $y'\sqrt{-1}$ for y'. However, by such a substitution as this, the value of a can undergo no change; for after we have eliminated y, it matters not in the least what we substitute for it. Hence it follows, that these equations must give the same value of a, as the former; and since the former might also have been obtained from the latter by substituting the roots x', x'', x''', $x^{\prime v}$, for x''', $x^{\prime v}$, x'', x', consequently it follows, that the expression for a must be such, that it suffers no change by the above substitution; it must ∴ necessarily be of the form

$$f: (x')\,(x'')\,(x''')\,(x^{\prime v}) = f: (x''')\,(x^{\prime v})\,(x'')\,(x')$$

which coincides with 6, § CXXXII.

If in the equations (ϕ) in 2, § CXXXII, we substitute $-\sqrt{-1}$, for $\sqrt{-1}$, we again obtain a new set of equations, which only differ from the equations (ϕ) in this, that in the former $x^{\prime v}$ is combined with $y\sqrt{-1}$, and x''' with $-y'\sqrt{-1}$; whereas in the latter the reverse of this is the case; we might ∴ have obtained them also merely by substituting x''' for $x^{\prime v}$. But hence it follows, that the expression for a must be such, that we obtain $f: (x')\,(x'')\,(x^{\prime v})\,(x''')$ from $f: (x')\,(x'')\,(x''')\,(x^{\prime v})$ when we merely put $-\sqrt{-1}$ for $\sqrt{-1}$, which agrees with § CXXXIII, Corollary.

Further, since the functions p, q of the preceding §,

as sum and product of the two functions $f:(x')\ (x'')\ (x''')\ (x^{\text{v}})$, $f:(x')\ (x'')\ (x^{\text{v}})\ (x''')$, by the substitution of $-\sqrt{-1}$ for $\sqrt{-1}$ suffer no change, so likewise they undergo no change by the substitution of x''' for x'; and they are consequently functions of the form

$$\phi:(x')\ (x'')\ (x''')\ (x^{\text{v}}) = \phi:(x')\ (x'')\ (x^{\text{v}})\ (x''')$$

and since they are also functions of the form

$$\phi:(x')\ (x'')\ (x''')\ (x^{\text{v}}) = \phi:(x''')\ (x^{\text{v}})\ (x'')\ (x')$$

because they are compounded of these; it follows then, as in the foregoing §, that they can have three different values only, and they consequently depend on equations of the third degree.

SECTION CXXXV.

PROB. To transform the equation of the fifth degree

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

into one of two terms $y^5 - V = 0$, assume the auxiliary equation:

$$x^4 + ax^3 + bx^2 + cx + d = y$$

required to find the degree of the equations, which must be solved, in order to determine the coefficients a, b, c, d.

Solution 1. If α denote one of the imaginary roots of the equation $y^5 - 1 = 0$, and we put $\sqrt[5]{V} = y'$, then y', $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, $\alpha^4 y'$, as we know already, are the five roots of the equation $y^5 - V = 0$. If we introduce these values of y, together with the values of x, into the

auxiliary equation, we then obtain the five following equations.

$$\begin{aligned}
x'^4 + ax'^3 + bx'^2 + cx' + d &= y' \\
x''^4 + ax''^3 + bx''^2 + cx'' + d &= \alpha y' \\
x'''^4 + ax'''^3 + bx'''^2 + cx''' + d &= \alpha^2 y' \\
x'^{v4} + ax'^{v3} + bx'^{v2} + cx'^{v} + d &= \alpha^3 y' \\
x^{v4} + ax^{v3} + bx^{v2} + cx^{v} + d &= \alpha^4 y'
\end{aligned}$$

2. By these equations, which, with reference to a, b, c, d, are only of the first degree, we can express each of these coefficients by the roots x', x'', x''', x'^{v}, x^{v}, and then, as was done in the foregoing § in equations of the third and fourth degrees, permute in the expressions thus found the above roots as often as possible; in the present instance $1 . 2 . 3 . 4 . 5 = 120$ times; the number of different values, which we obtain by these means, will then determine the degree of the equations on which the magnitudes a, b, c, d depend. But let it not be supposed, that the elimination, and also the comparison of the 120 results, are very laborious in themselves: another particular inquiry will always be required, if the equations obtained last cannot be reduced lower. We wish ∴ to try whether, from the nature of the above equations themselves, there are not certain indications by which we may more easily attain the desired object.

3. Since the number of the values of each of the coefficients a, b, c, d, say a, (the same conclusions may be made for the others) has its foundation merely in this, that the values of x and y may be combined in more ways

than one, it only remains to examine the results of these different combinations. But we obtain all the possible combinations of the values of x and y, when in the above five equations we either permute the values of x, or the values of y in every way. If we make choice of the first method, and, according to Hindenburg's Rule of Permutation, let x' retain its place in the first equation, when we transpose the roots x'', x''', x^{iv}, x^{v}, we then obtain twenty-four sets, each consisting of five equations. If after this we introduce the roots x'', x''', x^{iv}, x^{v}, successively into the first equation, when we permute the four remaining roots, we then obtain 120 sets in all, each consisting of five equations, and consequently all the combinations of the values of x with the values of y. Each such set gives a value of a; $\therefore$ collectively 120 values of a.

4. Now I assert, that amongst these 120 values of a, there are no more than twenty-four different ones, and that these different values are obtained from the first twenty-four sets. For, in the first place, it is easily seen, that it is quite immaterial whether we introduce the roots x'', x''', x^{iv}, x^{v}, successively in the first equation, and permute the other roots, or whether we substitute in all the five equations $\alpha y'$, $\alpha^2 y'$ $\alpha^3 y'$, $\alpha^4 y'$, successively for y', and after each such substitution permute the roots x'', x''', x^{iv}, x^{v}, when x' retains its place in the first equation. Now, since y' occurs only once in the value of a, because it was eliminated at the very beginning in the above equations, so with respect to this value it matters not what we substitute for y'; and $\therefore$, by the above-men-

tioned substitutions, we shall find no other values than those which we obtain, when we let x' retain its place, and merely permute the roots x'', x''', x^{IV}, x^{V}.

5. Now, we are certain that the expression for a, which we obtain by actual calculation, is such, that amongst the 120 values which arise from the transposition of all the five roots, x', x'', x''', x^{IV}, x^{V}, there are no more than twenty-four different ones, and that these last values are those which are obtained exclusively from the transposition of the four roots x'', x''', x^{IV}, x^{V}. The equation for a ∴ rises no higher than the twenty-fourth degree. We shall now see whether this equation cannot be reduced to others of lower degrees.

6. Since we do not know the expression for a, we shall ∴ immediately assume the transposition of the roots x'', x''', x^{IV}, x^{V}, in the above five equations, while in this we set the first completely aside, because x' may be considered as a constant magnitude. For this purpose, according to Hindenburg's Rule of Transposition, we again let x'' retain its place, and only permute the roots x''', x^{IV}, x^{V}, we then obtain six sets, each consisting of five equations. If, after this, we put x''', x^{IV}, x^{V}, successively for x'' in the second equation, and after every such substitution permute the other three roots, we then obtain in all the twenty-four sets of equations, which give the twenty-four different values of a.

7. Instead of the above method, we can also make use of the following one. First put in the above five equa-

tions α^2, α^3, α^4 successively for α, we then obtain, since $\alpha^6=\alpha$, $\alpha^7=\alpha^2$, $\alpha^8=\alpha^3$, &c., the four following combinations of the values of x and y:

$$\begin{array}{ccccc}
x', & x'', & x''', & x^{\text{IV}}, & x^{\text{V}} \\ \hline
y', & \alpha y', & \alpha^2 y', & \alpha^3 y', & \alpha^4 y' \\
y', & \alpha^2 y', & \alpha^4 y', & \alpha y', & \alpha^3 y' \\
y', & \alpha^3 y', & \alpha y', & \alpha^4 y', & \alpha^2 y' \\
y', & \alpha^4 y', & \alpha^3 y', & \alpha^2 y', & \alpha y'
\end{array}$$

and if we permute in each of these combinations the three roots x''', x^{IV}, x^{V}, we then obtain the twenty-four combinations of the values of x and y, which are possible only under the condition that x' continues to be combined with y'.

8. If ∴ we had expressed the coefficient a, from the five equations in 1, by the roots x', x'', x''', x^{IV}, x^{V}, it would only have been necessary, in order to find its twenty-four different values, to have transformed α into α^2, α^3, α^4, in each of these four values merely to have permuted the three roots x''', x^{IV}, x^{V}.

9. Now, if we assume that the values of a, which give the four combinations in 7, are the roots of an equation of the fourth degree

$$a^4 - pa^3 + qa^2 - ra + s = 0$$

and if we denote these roots by a', a'', a''', a^{IV}, then

$$\begin{aligned}
p &= a' + a'' + a''' + a^{\text{IV}} \\
q &= a'a'' + a'a''' + a'a^{\text{IV}} + a''a''' + \&c. \\
&\qquad \&c.
\end{aligned}$$

Now, since the functions a', a'', a''', $a^{\prime v}$, are such, that by the substitution of α for α^2, α^3, α^4, they merely are transformed into one another, consequently p, q, r, s, are symmetrical functions of the roots of the equation $y^5-1=0$, and $\therefore$ they cannot contain α. The functions p, q, r, s, consequently contain neither y' nor α; and $\therefore$ in all the transpositions of the roots x', x'', x''', $x^{\prime v}$, x^{v}, have no more than six different values, viz. only those which arise from the permutation of the roots x''', $x^{\prime v}$, x^{v}.

10. Consequently the coefficients p, q, r, s, all depend on equations of the sixth degree; and if we do not mind the trouble, these equations may be actually found by the method given in the third chapter. But whether they are capable of any further reduction or not, will be seen in the sequel. I shall now only observe, that it is quite sufficient to solve one of these equations, viz, that for p, because then, as will be seen in the following chapter, the coefficients q, r, s, may be found directly.

SECTION CXXXVI.

PROB. To transform the given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. \pm 0$$

into one of two terms, viz. $y^n - V = 0$, we assume the auxiliary equation

$$x^{n-1} + ax^{n-1} + bx^{n-2} + \ldots\ldots + kx + l = y$$

with $n-1$ undetermined coefficients $a, b, c, \ldots\ldots k, l$: on the supposition that n is a prime number, required to determine the degrees of the equations, which must be solved, in order to find the assumed coefficients.

Solution 1. If we put $\sqrt[n]{V} = y'$, and α denote an imaginary root of the equation $y^n - 1 = 0$, then y', $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$, are the n roots of the equation $y^n - V = 0$. If we combine these values with the roots of the given equation, we obtain the following n equations:

$$x'^{n-1} + ax'^{n-2} + bx'^{n-3} + \ldots\ldots + kx' + l = y'$$
$$x''^{n-1} + ax''^{n-2} + bx''^{n-3} + \ldots\ldots + kx'' + l = \alpha y'$$
$$x'''^{n-1} + ax'''^{n-2} + bx'''^{n-3} + \ldots\ldots + kx''' + l = \alpha^2 y'$$

&c.

from which we first eliminate y', and from the $n - 1$ equations obtained by this elimination, the values a, b, c, d, &c. may be determined by the roots x', x'', x''', x^{iv}, &c.

2. The coefficient a (and this, as well as what follows, obtains also of b, c, d, &c.) in general contains as many values as the n values of x may be combined with the n values of y, in sets of n equations, as these in 1, on the condition, that every such set is different from the others in the same set. Of these combinations, however, there are exactly as many as there are transpositions in the n roots in the above set of n equations; consequently the number of the values which the coefficient a can have $= 1 . 2 . 3$ ...... n. Consequently also the equation on which a depends, must be of the $1 . 2 . 3$ ...... nth degree, supposing that amongst these values there are no equal ones.

3. When in the above equations we substitute $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$ successively for y', we then

obtain n sets of equations, in which the values of x and y are arranged according to the following scheme:

$$\begin{array}{cccccc}
x', & x'', & x''', & x^{IV}, & x^{V}, \ldots\ldots\ldots & x^{(n)} \\
\hline
y', & ay', & a^2y', & a^3y', & a^4y', \ldots\ldots\ldots & a^{n-1}y' \\
ay', & a^2y', & a^3y', & a^4y', & a^5y', \ldots\ldots\ldots & y' \\
a^2y', & a^3y', & a^4y', & a^5y', & a^6y', \ldots\ldots\ldots & ay' \\
a^3y', & a^4y', & a^5y', & a^6y', & a^7y', \ldots\ldots\ldots & a^2y' \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a^{n-2}y', & a^{n-1}y', & y', & ay', & a^2y', \ldots\ldots\ldots & a^{n-3}y' \\
a^{n-1}y', & y', & ay', & a^2y', & a^3y', \ldots\ldots\ldots & a^{n-2}y'.
\end{array}$$

Here we find, as may be seen on inspection, x' always united with another value of y, while at the same time the remaining $n-1$ values of x are combined with the remaining n—1 values of y. If now, in each of these n combinations we permute the roots x'', x''', x^{IV} ...$x^{(n)}$, we then always obtain $1 \,.\, 2 \,.\, 3 \ldots\ldots n-1$ combinations, and consequently from the whole together all the $1 \,.\, 2 \,.\, 3 \ldots\ldots n$ combinations, in which the values of x can enter with the values of y.

4. In order $\therefore$ to find the results of all the possible combinations of the values of x and y, it is only necessary, in the above equations, to substitute ay', a^2y', a^3y',...$a^{n-1}y'$ successively for y', then transpose the roots x'', x''', x^{IV}, ... $x^{(n)}$ in all possible ways, and from each set thus obtained find a value of a; or, which is here the same, find first the expression for a, and then make use of the above substitution and transposition. But since y' has totally vanished in the expression for a (1), so by the substitution of ay', a^2y', a^3y', ... $a^{n-1}y'$ for y', this expression

suffers no change, and $\therefore$ there are no more different values of this magnitude, than arise from the transposition of the $n-1$ roots x'', x''', x^{IV}, ... $x^{(n)}$.

5. Consequently the magnitude a depends on an equation of the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$th degree, and its roots are the values of the expression for this magnitude, which arises from the transposition of the roots x'', x''', ... $x^{(n)}$. Hence, then, this equation may actually be found, by the method given in the third chapter, and may be expressed in known magnitudes.

6. Since the unequal values of a belong exclusively to the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ sets, which arise from the transposition of the $n-1$ roots x'', x''', x^{IV}, ... $x^{(n)}$, in the above equations, it is allowable to consider the first equation, together with its roots x', y', as constant and invariable, and consequently we only require to take into consideration the $n-1$ equations

$$x''^{n-1} + ax''^{n-2} + \ldots\ldots\ldots + kx'' + l = \alpha y'$$
$$x'''^{n-1} + ax'''^{n-2} + \ldots\ldots\ldots + kx''' + l = \alpha^2 y'$$
&c.

If in these equations we transpose the roots x'', x''', ...$x^{(n)}$, in all possible ways, we obtain all the combinations between these roots and the roots $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$. But these combinations may also be found as follows.

7. In § LXXXVII, Cor., it was shown that, when in the series of roots α, α^2, α^3, ... α^{n-1} we successively substi-

tute α^2, α^3, α^4, ... α^{n-1} for α, the same roots, but in a different order, always present themselves. Hence it follows, that when in the equations in 6, we substitute successively α^2, α^3, α^4, ... α^{n-1} for α, the $n-1$ sets of equations thus obtained, only differ from one another in the combination of the values of x with the values of y. Now since also every set has also another of the magnitudes $\alpha y'$, $\alpha^2 y'$, $\alpha^3 y'$, ... $\alpha^{n-1} y'$ in the first place, it follows, that we obtain all the combinations of the values of x with the values of y, when in each set we permute the $n-2$ roots x''', $x^{\prime v}$, ... $x^{(n)}$ in the $n-2$ last equations in all possible ways.

8. The changes which we have effected with the equations themselves, we can also effect with their result, the expression for a. Thus, if by the equations in 1 we have expressed the magnitude a by x', x'', x''',...$x^{(n)}$, we then substitute in it successively α^2, α^3, α^4,......α^{n-1} for α, and permute in each of the values thus obtained, merely the roots x''', $x^{\prime v}$, x^{v},......$x^{(n)}$, while we let x' and x'' retain their places.

9. Now, if we denote the $n-1$ values of a, which arise from the substitution of α^2, α^3, α^4, ... α^{n-1}, for α, by a', a'', a''',......$a^{(n)}$, and assume that they are the roots of the following equation of the $n-1$th degree:

$$a^{n-1} - pa^{n-2} + qa^{n-3} - ra^{n-4} + \&c. = 0$$

then the coefficients p, q, r, &c. merely as functions of a', a'', a''', ... $a^{(n-1)}$, in like manner can have no more different values than the $1 \,.\, 2 \,.\, 3......n-1$, which

arise from the substitution of α for α^2, α^3, α^4,......α^{n-1}, and from the transposition of the roots x''', x^{iv}, x^{v},...$x^{(n)}$. But since these coefficients are also at the same time symmetrical functions of α, α^2, α^3,......$\alpha^{(n-1)}$, consequently of the roots of the equation $y^n - 1 = o$; $\therefore$, by the foregoing chapter, they must be rational, and consequently do not contain α. Wherefore these coefficients have no more than the $1 . 2 . 3 ... n-2$ different values, which arise from the transposition of the roots x''', x^{iv}, x^{v}, ...$x^{(n)}$, and $\therefore$ they all depend on equations of the $1 . 2 . 3 ...... n-2$th degree.

10. Let $1 . 2 . 3 ...... n-1 = \mu$, and let

$$p^{\mu} + A'p^{\mu-1} + B'p^{\mu-2} + C'p^{\mu-3} + \&c. = o$$

be the equation for p; then, as is already known from the third chapter, the coefficients A', B', C', &c. may always be found, and expressed rationally by the coefficients A, B, C, &c. of the given equation $x^n + Ax^{n-1} + \&c. = o$. If, then, we can solve this equation, and from it determine the values of p, then also we may, as will be shown in the following chapter, find the coefficients q, r, s, &c. directly, and without solving any other equation. Now, if we denote the values of p, q, r, &c. which we thus obtain, by p', q', r', &c., p'', q'', r'', &c, p''', q''', r''', &c. &c. we then obtain the following $1 . 2 . 3 ... n-2$ equations:

$$\begin{aligned}
&\alpha^{n-1} + p'\alpha^{n-2} + q'\alpha^{n-3} + r'\alpha^{n-4} + \&c. = o \\
&\alpha^{n-1} + p''\alpha^{n-2} + q''\alpha^{n-3} + r''\alpha^{n-4} + \&c. = o \\
&\alpha^{n-1} + p'''\alpha^{n-2} + q'''\alpha^{n-3} + r'''\alpha^{n-4} + \&c. = o \\
&\qquad\qquad \&c.
\end{aligned}$$

into which the equation for a in 5 may be analyzed. But

whether or not the equation for p is capable of any reduction, this is not yet the proper place to inquire.

Remark. Hence it follows, that an equation of the nth degree, when n is a prime number, leads, according to this mode of transformation, to an equation of the $1.2.3...n-2$th degree; consequently an equation of the fifth degree to an equation of the sixth degree, and an equation of the seventh degree leads even to one of the 120th degree; and so on.

SECTION CXXXVII.

Prob. All that has been said in the problem in the foregoing § obtains, only n is a compound number: determine the degree of the equations on which the assumed coefficients a, b, c, d, &c. depend.

Solution 1. When we suppose α, not to be, as in the preceding §, any arbitrary imaginary root of the equation $y^n-1=0$, but only a primitive root of it, then indeed all the conclusions made in 1, 2, 3, 4, 5, 6, of the preceding § are applicable to this case; on the other hand, the following solutions, (7, 8, 9, 10), on account of this very circumstance, must undergo some alterations. Thus, if we substitute here, as in 7 of the preceding §, in the series of roots α, α^2, α^3,......α^{n-1} the powers α^2, α^3, α^4,......α^{n-1} indiscriminately for α, we shall not always find again the same roots, but this will only be the case for those powers amongst them, α^ν, α^π, α^ρ, &c. whose exponents ν, π, ρ, &c. have no common measure with n,

because these only are primitive roots of the equation $y^n-1=0$ (§ XCI). Thus, if α be a primitive root of the equation $y^6-1=0$, when in α, α^2, α^3, α^4, α^5, we successively substitute α^2, α^3, α^4, α^5, for α, we obtain the following results: α^2, α^4, 1, α^2, α^4; α^3, 1, α^3, 1, α^3; α^4, α^2, 1, α^4, α^2; α^5, α^4, α^3, α^2, α, of which only the last contains again the same roots.

2. Hence it follows, that what has been said in the preceding § in 7, and the following solutions, respecting the substitution of the roots α^2, α^3, α^4,......α^{n-1} for α, must be limited to α^ν, α^π, α^ρ, &c. whose exponents ν, π, ρ, &c. have no common measure with n. If ∴ we assume, that λ is the number of the primitives α^ν, α^π, α^ρ, &c. and that the λ values of a, which we obtain by the substitution of these roots for α, are the roots of the following equations:

$$a^\lambda + pa^{\lambda-1} + qa^{\lambda-2} + ra^{\lambda-3} + \&c. = 0$$

consequently p (and the same obtains also of q, r, &c.) is such a function, as by the substitution of the root α for α^ν, α^π, α^ρ, &c., or, which is the same, by the substitution of the root x'' for $x^{(\nu)}$, $x^{(\pi)}$, $x^{(\rho)}$, &c. it remains unchanged. Now, since by these means all the $1.2.3 \ldots n-1$ values of p, taken λ and λ together, are equal, it follows, that this magnitude depends on an equation, whose degree

$$=\frac{1.2.3\ldots\ldots n-1}{\lambda}$$

Moreover, in order actually to find the equation $a^\lambda + pa^{\lambda-1} + \&c. = 0$, it is only required in the expression for a, which we obtain from 1 of the foregoing §, to eliminate

the root α, by means of that equation which only contains the primitive roots, the method to find which was given in § LXXXIX.

REMARK. Therefore the equation of the nth degree, when n is a compound number, leads to an equation of the $\frac{1\,.\,2\,.\,3\ldots\ldots n-1}{\lambda}$ th degree, in which λ denotes the number of the primitive roots of the equation $x^n - 1 = 0$; consequently an equation of the fourth degree leads to an equation of the third degree, because the equation $x^4 - 1 = 0$ has two primitive roots, α and α^3; an equation of the sixth degree to an equation of the sixtieth degree, because the equation $x^6 - 1 = 0$ has also two primitive roots, viz. α and α^5; an equation of the eighth degree to an equation of the 1260th degree, because the equation $x^8 - 1 = 0$ has four primitive roots, viz. α, α^3, α^5, α^7; and so on.

From this and the foregoing §, it follows, that the reduction of the equation $x^n + Ax^{n-1} + \&c. = 0$ to one of the form $y^n - V = 0$, always leads to a higher equation than the given one itself, whenever the given equation exceeds the fourth degree. However, I shall not enter here into the proof of other methods of transformations, because the whole of this subject will be considered hereafter in a higher point of view, to which this is only preparatory.

SECTION CXXXVIII.

When we make the equation $x = a\sqrt[n]{p} + b\sqrt[n]{p^2}$

$+ c\sqrt[n]{p^3} + \ldots\ldots + k\sqrt[n]{p^{n-1}}$, or more generally the equation $x = a + b\sqrt[n]{p} + c\sqrt[n]{p^2} + d\sqrt[n]{p^3} + \ldots\ldots + k\sqrt[n]{p^{n-1}}$ rational, we arrive, as we know from the foregoing chapter, at an equation of the nth degree, which I shall represent by $x^n + \mathfrak{A}x^{n-1} + \mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \&c. = 0$, in which the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. are certain rational functions of the magnitudes a, b, c, d, &c. and p. Conversely, if an equation of the nth degree $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$ be given, and we assume that the roots have the above form, we then have, for the determination of a, b, c, d, &c. the n conditional equations $\mathfrak{A} = A$, $\mathfrak{B} = B$, $\mathfrak{C} = C$, &c. If we solve these equations, and from them determine a, b, c, &c. we then have at once the n roots of the given equation, when we substitute successively for $\sqrt[n]{p}$ its n values.

Consequently all depends on the solution of these conditional equations. How difficult and troublesome this solution must be for equations of rather high degrees, may be perceived from the form of those equations, which were found for the fifth degree in 7, § CIX. Waring and Euler thought, that in this way we must arrive at the general solution of equations, by properly arranging and finishing the calculation, without regard to the trouble. But this trouble may be spared by subjecting à priori (as M. Lagrange does in the third volume of the New Memoirs of the Berlin Academy) the method to a preliminary proof.

SECTION CXXXIX.

PROB. Let it be assumed that

$$x=a+b\sqrt[n]{p}+c\sqrt[n]{p^2}+d\sqrt[n]{p^3}+\ldots\ldots+k\sqrt[n]{p^{n-1}}$$

is a root of the given equation

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\&c.=0$$

on the supposition that n is a prime number, required to determine the degrees of the equations on which the coefficients a, b, c, d, &c. depend.

Solution 1. Put $\sqrt[n]{p}=y$; then y, αy, βy, γy, δy, &c. are the n values of $\sqrt[n]{p}$, when 1, α, β, γ, δ, &c. are the n roots of the equation $x^n-1=0$. If $\therefore$ we denote the roots of the given equation by x', x'', x''', &c. we have the following n equations:

$$\begin{aligned}
x' &= a+by+cy^2+dy^3+\ldots\ldots+ky^{n-1}\\
x'' &= a+\alpha by+\alpha^2cy^2+\alpha^3dy^3+\ldots\ldots+\alpha^{n-1}ky^{n-1}\\
x''' &= a+\beta by+\beta^2cy^2+\beta^3dy^3+\ldots\ldots+\beta^{n-1}ky^{n-1}\\
x^{IV} &= a+\gamma by+\gamma^2cy^2+\gamma^3dy^3+\ldots\ldots+\gamma^{n-1}ky^{n-1}\\
&\&c.
\end{aligned}$$

from which the n unknown magnitudes a, b, c, d, ... k, must now be determined.

2. If we multiply these equations, in the order in which they stand vertically, first by the powers 1, α^n, β^n, γ^n, &c., then by the powers 1, α^{n-1}, β^{n-1}, γ^{n-1}, &c., after this by the powers 1, α^{n-2}, β^{n-2}, γ^{n-2}, &c., and so on; lastly by 1, α, β, γ, &c., and add the n results obtained by every such multiplication together; we then

get, since $'[n] = n$, and every numerical expression $'[p]$, whose radical exponent p is not divisible by $n = 0$,

$$
\begin{aligned}
na &= x' + x'' + x''' + x^{IV} + \&c.\\
nyb &= x' + \alpha^{n-1}x'' + \beta^{n-1}x''' + \gamma^{n-1}x^{IV} + \&c.\\
ny^2c &= x' + \alpha^{n-2}x'' + \beta^{n-2}x''' + \gamma^{n-2}x^{IV} + \&c.\\
ny^3d &= x' + \alpha^{n-3}x'' + \beta^{n-3}x''' + \gamma^{n-3}x^{IV} + \&c.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
ny^{n-2}i &= x' + \alpha^2x'' + \beta^2x''' + \gamma^2x^{IV} + \&c.\\
ny^{n-1}k &= x' + \alpha x'' + \beta x''' + \gamma x^{IV} + \&c.
\end{aligned}
$$

3. Hence now we may immediately determine a; for, since $x' + x'' + x''' + x^{IV} + \&c. = [1] = A$, we then obtain from the first equation

$$a = \frac{A}{n}$$

Consequently a has only one single value; on the other hand, all the remaining magnitudes b, c, d, ... i, k, have more values than one, which may be obtained from the $n-1$ following equations, by transposing their roots x', x'', x''', ... $x^{(n)}$ in all possible ways. The equation on which each of them depends, is ∴ generally assumed to be of the 1 . 2 . 3 ...... nth degree. But if amongst these values there should be any equal ones, or any relation amongst them, then the degree of the equations can be reduced.

4. Since one of the $n+1$ magnitudes $a, b, c, d, \ldots k, p$, may be assumed to be arbitrary, we shall put $p = 1$; ∴ also $y = 1$. At the same time, in order to make the formulæ more simple, we shall put

$$k = \frac{a'}{n},\ i = \frac{a''}{n},\ h = \frac{a'''}{n},\ \ldots\ldots\ldots\ b = \frac{a^{(n-1)}}{n}$$

By these means, if we omit the first, and set down the others in an inverted order, the equations in 2 are transformed into the following ones:

$$\begin{aligned}
a' &= x' + \alpha x'' + \beta x''' + \gamma x^{\prime v} + \&c.\\
a'' &= x' + \alpha^2 x'' + \beta^2 x''' + \gamma^2 x^{\prime v} + \&c.\\
a''' &= x' + \alpha^3 x'' + \beta^3 x''' + \gamma^3 x^{\prime v} + \&c.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
a^{(n-1)} &= x' + \alpha^{n-1} x'' + \beta^{n-1} x''' + \gamma^{n-1} x^{\prime v} + \&c.
\end{aligned}$$

5. Since the roots of the equation $x^n - 1 = o$ (because by the hypothesis n is a prime number) may be expressed by $1, \alpha, \alpha^2, \alpha^3, \ldots\ldots \alpha^{n-1}$, let α denote any imaginary root whatever; then we can put $\alpha^2, \alpha^3, \alpha^4$, &c. for β, γ, δ, &c.; by these means we obtain the first equation

$$a' = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \ldots\ldots + \alpha^{n-1} x^{(n)}$$

and the values of $a'', a''', a^{\prime v}, \ldots\ldots a^{(n-1)}$ are derived from this value of a', when we substitute $\alpha^2, \alpha^3, \alpha^4, \ldots\ldots \alpha^{n-1}$ for α, consequently when we substitute for α every imaginary root of the equation $y^n - 1 = o$. If .˙. undetermined t denote each of the magnitudes $a', a'', a''', \ldots\ldots a^{(n-1)}$, then

$$t = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \ldots\ldots + \alpha^{n-1} x^{(n)}$$

6. Now, in order to find all the values of which t is capable, it is only necessary to permute the roots $x', x'', x''', \ldots\ldots x^{(n)}$, in all possible ways. For our purpose, however, it is more convenient, in this case, to proceed as follows: 1st, we only transpose the $n - 2$ roots $x''', x^{\prime v}$,

x^{v}, ... $x^{(n)}$, while we let x', x'' retain their places, consequently we obtain $1 \cdot 2 \cdot 3 \ldots\ldots n-2$ results; 2ndly, we put in each of the results thus obtained, first α^2, afterwards α^3, then α^4, and so on, and lastly α^{n-1} for α; then generally we have $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ results, which we should also have obtained, if we had let x' retain its place, and merely transposed the roots x'', x''', x^{iv} .... $x^{(n)}$; 3rdly and lastly, we multiply the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ results so found, successively by $\alpha, \alpha^2, \alpha^3, \ldots\ldots \alpha^{n-1}$, and we obtain, together with the former, the $1 \cdot 2 \cdot 3 \ldots\ldots n$ results, which arise from the transposition of all the roots $x', x'', x''', \ldots\ldots x^{(n)}$ and at the same time also all the values of t.

7. If ∴ we denote the $1 \cdot 2 \cdot 3 \ldots\ldots n-1$ values of t, which are obtained by the two first operations, by t', t'', t''', t^{iv}, &c., then all the $1 \cdot 2 \cdot 3 \ldots\ldots n$ values of t may be expressed in the following way:

$$t',\ \alpha t',\ \alpha^2 t',\ \alpha^3 t',\ \alpha^4 t',\ \ldots\ldots\ldots\ \alpha^{n-1} t'$$
$$t'',\ \alpha t'',\ \alpha^2 t'',\ \alpha^3 t'',\ \alpha^4 t'',\ \ldots\ldots\ldots\ \alpha^{n-2} t''$$
$$t''',\ \alpha t''',\ \alpha^2 t''',\ \alpha^3 t''',\ \alpha^4 t''',\ \ldots\ldots\ldots\ \alpha^{n-1} t'''$$

&c.

If we put $t'^n = \theta'$, $t''^n = \theta''$, $t'''^n = \theta'''$ &c., then the values in the first horizontal series are the roots of the equation $t^n - \theta' = 0$, those in the second series are the roots of the equation $t^n - \theta'' = 0$, &c.; consequently the equation for t is the product of the equations

$$t^n - \theta' = 0,\ t^n - \theta'' = 0,\ t^n - \theta''' = 0$$

&c.

8. Hence it follows, that the equation for t only contains such powers as are divisible by n. If $\therefore$ we put $t^n = \theta$, so that

$$\theta = (x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\prime v} + \ldots\ldots + \alpha^{n-1} x^{(n)})^n$$

we then obtain an equation for θ of the $1 \cdot 2 \cdot 3 \ldots n-1$th degree, whose roots are θ', θ'', θ''', &c. which we obtain from θ, by transposing the $n-1$, roots x'', x''' $x^{\prime v}$, ... $x^{(n)}$ and allowing x to retain its place.

9. Instead of transposing the roots x'', x''', $x^{\prime v}$ ...... $x^{(n)}$ in the expression θ, it will be quite sufficient, as in t, merely to permute the $n-2$ roots x''', $x^{\prime v}$, x^{v}, ... $x^{(n)}$, then substitute α^2, α^3, α^4, ... α^{n-1} for α, and in the $n-1$ values of θ thus obtained, transpose the roots x''', $x^{\prime v}$, x^{v}, ... $x^{(n)}$.

10. We now assume, that the $n-1$ values of θ, which arise from the substitution of α^2, α^3, α^4, ... α^{n-1} for α, are roots of the following equation:

$$(A) \ldots\ldots \theta^{n-1} - p\theta^{n-2} + q\theta^{n-3} - r\theta^{n-4} + \&c. = 0;$$

then the coefficients p, q, r, &c. are functions of these values, and can $\therefore$, as is the case also with these last, suffer no change by the transposition of x'. But since they are also symmetrical with respect to these values, consequently they can likewise undergo no change by the substitution of α^2, α^3, α^4, ... α^{n-1} for α, because the only consequence arising from this substitution is, that of the $n-1$ values of θ, one is merely transformed into the other. Hence, however, it follows, that these coefficients can have no more unequal values, than those

which arise exclusively from the transposition of the $n-2$ roots $x''', x^{\prime v}, x^{v}, \ldots\ldots x^{(n)}$, and that consequently they all depend on equations of the $1 \,.\, 2 \,.\, 3 \ldots\ldots n-2$th degree.

11. Therefore the equation for θ, which, as we have seen, is of the $1 \,.\, 2 \,.\, 3 \ldots\ldots n-1$th degree, may always, when n is a prime number, be analyzed into $1 \,.\, 2 \,.\, 3 \ldots \ldots n-2$ equations of the $n-1$th degree. Now, if (A) be one of these equations, and $\theta', \theta'', \theta''' \ldots\ldots \theta^{(n-1)}$ its $n-1$ roots, then $\sqrt[n]{\theta'}, \sqrt[n]{\theta''}, \sqrt[n]{\theta'''}, \ldots\ldots \sqrt[n]{\theta^{(n-1)}}$, are the corresponding values of t, or of $a', a'', a''', \ldots\ldots \ldots a^{(n-1)}$, and if we substitute these values in any order in the expressions in 4, we then obtain

$$b = \frac{\sqrt[n]{\theta'}}{n},\ c = \frac{\sqrt[n]{\theta''}}{n},\ d = \frac{\sqrt[n]{\theta'''}}{n},\ \&c.$$

consequently, since we have put $p = 1$,

$$x = \frac{A}{n} + \frac{1}{n} (\sqrt[n]{\theta'} + \sqrt[n]{\theta''} + \sqrt[n]{\theta'''} + \ldots + \sqrt[n]{\theta^{(n-1)}})$$

But that for θ only those of the $1 \,.\, 2 \,.\, 3 \ldots\ldots n-1$ values may be assumed, which belong to one and the same equation, it matters not which, $\theta^{n-1} - p\,\theta^{n-2} + \&c. = 0$, appears from this circumstance, that the $n-1$ values of t, consequently the corresponding values of θ must so depend upon one another, that we obtain them all, when in one of them we substitute $a^2, a^3, a^4, \ldots\ldots a^{n-1}$ for a, 5.

12. If we wish actually to find the equation (A), first of all we solve the function

$$\theta = (x' + \alpha x'' + \alpha^2 x''' + \ldots\ldots + \alpha^{n-1} x^{(n)})^n$$

according to the powers of α, which may very easily be done by means of the polynomial theorem. But since α^n, α^{n+1}, α^{n+2}, &c. are no other than 1, α, α^2, &c. consequently this development, after the proper reduction, takes the following form:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{iv}\alpha^3 + \ldots\ldots + \xi^{(n)}\alpha^{n-1}$$

and ξ', ξ'', ξ''', ... $\xi^{(n-1)}$ are mere functions of x', x'', x''', ... $x^{(n-1)}$ without α. If in this value of θ we put α^2, α^3, α^4, ... α^{n-1} successively for α, or, which is the same, β, γ, δ, &c. for α, we then obtain the values of θ', θ'', θ''', ... $\theta^{(n-1)}$, viz.:

$$\begin{aligned}
\theta' &= \xi' + \xi''\alpha + \xi'''\alpha^2 + \ldots\ldots + \xi^{(n)}\alpha^{n-1} \\
\theta'' &= \xi' + \xi''\beta + \xi'''\beta^2 + \ldots\ldots + \xi^{(n)}\beta^{n-1} \\
\theta''' &= \xi' + \xi''\gamma + \xi'''\gamma^2 + \ldots\ldots + \xi^{(n)}\gamma^{n-1}
\end{aligned}$$

&c.

from which the equation (A) may be compounded in the usual way. The coefficients p, q, r, &c. are then still functions of the roots x', x'', x''', ... $x^{(n)}$, but such, that they only change when the roots x''', x^{iv}, x^{v}, ... $x^{(n)}$ are transposed. The equations for these functions may be found by the method given in the third chapter. Moreover it is quite sufficient, as will be shown in the following chapter, to find one of these equations, for instance, that for p, because from the known value of p, the values of q, r, &c. may be found directly, and without the solution of any other equation. It is likewise sufficient, as follows from 11, to find only a single value for each of the coefficients p, q, r, &c.

13. From all that has hitherto been said, it follows, that the solution of an equation of the nth degree, when n is a prime number, depends on the solution of an equation for p of the $1 \cdot 2 \cdot 3 \ldots n-2$th degree; and this result coincides with that which has been found by another method in § CXXXVI. With the view to further elucidation, I shall now apply it to an equation of the fifth degree.

REMARK. When n is a compound number, then the conclusions drawn with respect to the substitution of the root α for $\alpha^2, \alpha^3, \alpha^4, \ldots \ldots \alpha^{n-1}$, suffer the same changes, as the conclusions, § CXXXVI, § CXXXVII, must undergo in the same case, and we shall then find, as we did there, that an equation of the nth degree leads to an equation of the $\frac{1 \cdot 2 \cdot 3 \ldots n-1}{\lambda}$th degree, when λ denotes the number of the primitive roots of the equation $y^n - 1 = 0$.

SECTION CXL.

PROB. From the given equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

find actually the reduced equation for the magnitude p of the foregoing §.

Solution 1. The equation (A) in 10 of the foregoing §, when $n=5$, is here

$$\theta^4 - p\theta^3 + q\theta^2 - r\theta + s = 0$$

and the roots of this equation are, when α, β, γ, δ, denote the imaginary roots of the equation $y^5-1=0$,

$$\theta' = \xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \xi^{V}\alpha^4$$
$$\theta'' = \xi' + \xi''\beta + \xi'''\beta^2 + \xi^{IV}\beta^3 + \xi^{V}\beta^4$$
$$\theta''' = \xi' + \xi''\gamma + \xi'''\gamma^2 + \xi^{IV}\gamma^3 + \xi^{V}\gamma^4$$
$$\theta^{V} = \xi' + \xi''\delta + \xi'''\delta^2 + \xi^{IV}\delta^3 + \xi^{V}\delta^4$$

2. If we add these equations together, we then obtain, when $\theta'+\theta''+\theta'''+\theta^{V}=p$, and $\alpha+\beta+\gamma+\delta='[1]-1=-1$, $\alpha^2+\beta^2+\gamma^2+\delta^2='[2]-1=-1$, $\alpha^3+\beta^3+\gamma^3+\delta^3='[3]-1=-1$, $\alpha^4+\beta^4+\gamma^4+\delta^4='[4]-1=-1$;

$$p = 4\xi' - (\xi'' + \xi''' + \xi^{IV} + \xi^{V})$$
$$= 5\xi' - (\xi' + \xi'' + \xi''' + \xi^{IV} + \xi^{V})$$

3. The second part of the expression for p, viz. $\xi' + \xi'' + \xi''' + \xi^{IV} + \xi^{V}$, may be found immediately. For from 12 of the foregoing § it follows, that the development of

$$(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \alpha^4 x^{V})^5$$

assumes the following forms:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \xi^{V}\alpha^4;$$

and this form of the development is always correct, whichever root of the equation $x^5-1=0$ we substitute for α; consequently also, when we put $\alpha=1$. If this is actually done, we find

$$\xi' + \xi'' + \xi''' + \xi^{IV} + \xi^{V} =$$
$$(x' + x'' + x''' + x^{IV} + x^{V})^5$$
$$= [1]^5 = A^5$$

We have ∴ also

$$p = 5\xi' - A^5.$$

4. In order ∴ to determine p, it is only necessary to find ξ', consequently that term in the development of $(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v} + \alpha^4 x^{v})^5$, which does not contain α. For this purpose we can give this expression the following form:

$$\alpha^{-5}\,(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x'^{v} + \alpha^5 x^{v})^5$$

or, since $\alpha^{-5} = 1$, the following one

$$(\alpha x + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x'^{v} + \alpha^5 x^{v})^5$$

from which we derive this advantage, that the dashes over x coincide with the exponents of α. For now we have nothing more to do, as is shown in the polynomial theorem, but to combine the roots x', x'', x''', x'^{v}, x^{v}, in all possible ways, in such a way that the sum of the dashes $=5$, $=10$, $=15$, $=20$, $=25$, because $\alpha^5 = \alpha^{10} = \alpha^{15} = \alpha^{20} = \alpha^{25} = 1$. In this way we find, when $[5]$, $[1^5]$, are substituted for $x'^5 + x''^5 + x'''^5 + x'^{v5} + x^{v5}$, $x'x''x'''x'^{v}x^{v}$,

$$\xi' = [5] + [1^5] +$$

$$20\left\{\begin{matrix} x'^3x''x^{v} + x'^3x'''x'^{v} + x'x''^3x''' + x'x''x'^{v3} + \\ x'x'''^3x^{v} + x''^3x'^{v}x^{v} + x''x'''^3x'^{v} + x'x'^{v}x^{v3} \\ + x''x'''x^{v3} + x'''x'^{v3}x^{v} \end{matrix}\right\}$$

$$+30\left\{\begin{matrix} x'^2x''^2x'^{v} + x'^2x''x'''^2 + x'^2x'''x^{v2} + x'^2x'^{v2}x^{v} + \\ x'x''^2x^{v2} + x'x'''^2x'^{v2} + x''^2x'''^2x^{v} + x''^2x'''x'^{v2} \\ + x''x'^{v2}x^{v2} + x'''^2x'^{v}x^{v2} \end{matrix}\right\}$$

or when, for shortness' sake, in the value of ξ', we denote by ζ that which is not to be found in the crotchets we get

$$\xi' = [5] + [1^5] + \zeta$$

∴

$$p = 5\zeta + 5[5] + 5[1^5] - A^5$$

5. Amongst the 120 values, which the function ζ contains by the transposition of the roots x', x'', x''', x'^{v}, x^{v},

we find no more than six unequal ones, and they will be exactly those which arise exclusively from the transposition of the three roots x''', x^{IV}, x^{V}. If we denote these values by ζ', ζ'', ζ''', ξ^{IV}, ζ^{V}, ζ^{VI}, and the corresponding values of p by p', p'', p''', p^{IV}, p^{V}, p^{VI}, we then obtain

$$
\begin{aligned}
p' &= 5\,\zeta' + 5\,[5] + 5\,[1^5] - A^5 \\
p'' &= 5\,\zeta'' + 5\,[5] + 5\,[1^5] - A^5 \\
p''' &= 5\,\zeta''' + 5\,[5] + 5\,[1^5] - A^5 \\
p^{IV} &= 5\,\zeta^{IV} + 5\,[5] + 5\,[1^5] - A^5 \\
p^{V} &= 5\,\zeta^{V} + 5\,[5] + 5\,[1^5] - A^5 \\
p^{VI} &= 5\,\zeta^{VI} + 5\,[5] + 5\,[1^5] - A^5
\end{aligned}
$$

and these six values of p are the roots of the required equation. We already know, from the third chapter, how to proceed further, in order to find this equation itself. It would be better, however, instead of the equation for p to find that for ζ; for if we have ζ, we have also p.

VII.—A GENERAL METHOD BY WHICH, FROM THE KNOWN VALUE OF A GIVEN FUNCTION OF THE ROOTS OF AN EQUATION, TO FIND THE VALUE OF EVERY OTHER FUNCTION OF THESE ROOTS.

SECTION CXLI.

ALL the methods which we have hitherto applied to the solution of equations, are founded either on analysis or transformation. The first, from its very nature, cannot be general, because every equation will not admit of being analyzed into others of lower degrees. Consequently, in the general solution of equations, we have no other mode left us but to transform the given equations into others, which in themselves are either solvible by the methods already known, or may be made so by analysis.

Now, let it be assumed that we have transformed in any way, no matter which, the given equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

into another

$$t^n + A't^{n-1} + B't^{n-2} + C't^{n-3} + \&c. = 0$$

then the roots of the last equation must stand in some one relation to the roots of the first, or, in other words, t must admit of being expressed by some function of the roots x', x'', x''', &c. Now I affirm, that it is always

allowable to assume t to be a rational function of these roots. For, let $F:\ (x')\ (x'')\ (x''')\ldots(x^{(n)})$ be any irrational function of these roots, and let $t=F:\ (x')\ (x'')\ (x''')\ldots(x^{(n)})$; then this equation, as has been already shown in the fifth chapter, can always be made rational by removing the irrational magnitudes. We shall thus get an equation

$$t^{\mu} + A''t^{\mu-1} + B''t^{\mu-2} + C''t^{\mu-3} + \&c. = 0$$

in which the coefficients A'', B'', C'', &c. are all rational functions of x', x'', x''', &c. Now if we eliminate from this equation and the equation $t^{m} + A't^{m-1} + B't^{m-2} +$ &c. $=0$ all the powers of t, as far as the first, we then obtain for t a rational function only.

In the first place ∴ it is only necessary, in the transformation of the equations, to find such rational functions of x', x'', x''', &c., for which the transformed equation is either immediately solvible, or at least may be made to depend on solvible equations. But this is not all; it is not sufficient to know the values of the assumed function; we must also be able, from these values, to find the roots x', x'', x''', &c. I shall first handle the second subject, and, according to Mr. Lagrange, in the third volume of the New Berlin Memoirs, show the method by which, from the known values of a given function, the value of every other function may be found, consequently also the roots themselves. Here two cases must be taken into consideration, viz. first, the case in which the given and the required function are homogeneous; secondly, the case in which they are not so.

For the sake of greater perspicuity, when I treat of the

values of a function, I shall sometimes distinguish the values of forms from numerical values; the first are the different forms themselves, which arise from the transposition of the roots x', x'', x''', &c.; the latter, the actual values of these forms expressed by given magnitudes.

SECTION CXLII.

PROB. Let it be assumed that the given equation

$$\text{I. } x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

by the introduction of a new magnitude $t = f: (x')(x'')(x''')\ldots(x^{(n)})$, according to the method in the third chapter, is transformed into an equation

$$\text{II. } t^\pi + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + \ldots + U = 0$$

which is completely solvible, consequently all of whose roots may be found: from these known numerical values of the function t, it is required to find the numerical values of any other function $y = \phi : (x')(x'')(x''')\ldots(x^{(n)})$ respecting which it is assumed that it is homogeneous to the former.

Solution 1. Since the functions t, y, according to the hypothesis, are homogeneous, then, by the transposition of the roots x', x'', x''', &c. the former must contain exactly as many unequal values as the latter. The function t, however, has π values, because the equation II, by which it is represented, has been assumed to be of the πth degree, consequently the second function has also π values. I shall denote the values of forms of t by t', t'', t''', $\ldots t^{(\pi)}$, and the values of forms of y by y', y'', y''', $\ldots y^{(\pi)}$,

and assume besides, that those which have the same number of dashes, arise from the same transpositions.

2. Since t and y are homogeneous functions, consequently any expression whatever, which is compounded of these functions, can have no more unequal values of forms than these functions themselves. Consequently, also, such an expression as $t^\lambda y$ can have no more than π different values, and these are $t'^\lambda y'$, $t''^\lambda y''$, $t'''^\lambda y''' + \ldots\ldots +$ $(t^{(\pi)\lambda} y^{(\pi)})$. If we take the sum of all these values, we obtain the function

$$(\psi)\ldots\ldots t'^\lambda y' + t''^\lambda y'' + t'''^\lambda y''' + \ldots + (t^{(\pi)})^\lambda y^{(\pi)}$$

and this function has the property of remaining the same, however we transpose the roots x', x'', x''', &c.; it is ∴ symmetrical with respect to the former roots, and consequently, let λ be any number whatever, may always be expressed rationally by the coefficients A, B, C, &c. of the given equation.

3. If we denote the numerical values, which the function (ψ) contains, when we substitute 0, 1, 2, 3,...$\pi-1$ sucessively for λ, by z_0, z_1, z_2, z_3,......$z_{\pi-1}$, we obtain the following π equations:

$$\begin{array}{l}
y' + y'' + y''' + \ldots\ldots + y^{(\pi)} = z_0 \\
t'y' + t''y'' + t'''y''' + \ldots\ldots + t^{(\pi)}y^{(\pi)} = z_1 \\
t'^2y' + t''^2y'' + t'''^2y''' + \ldots\ldots + (t^{(\pi)})^2y^{(\pi)} = z_2 \\
t'^3y' + t''^3y'' + t'''^3y''' + \ldots\ldots + (t^{(\pi)})^3y^{(\pi)} = z_3 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
t'^{\pi-1}y' + t''^{\pi-1}y'' + t'''^{\pi-1}y''' + \ldots + (t^{(\pi)})^{\pi-1}y^{(\pi)} = z_{\pi-1}
\end{array}$$

in which z_0, z_1, z_2, z_3,......$z_{\pi-1}$, are all known magnitudes expressed by the coefficients of the equation I.

4. Now, let t', t'', t''', ... $t^{(\pi)}$, instead of the values of forms of the function, denote its numerical values, then these are no other than the roots of the equation II, consequently, by the hypothesis, are all known. Therefore in the foregoing π equations there are no other unknown magnitudes but y', y'', y''',...$y^{(\pi)}$; and since their number is π, consequently we have exactly the same number as of equations; they may $\therefore$, with a few exceptions (which will be inquired into hereafter), always be calculated and expressed rationally by the magnitudes t', t'', t''' ...... $t^{(\pi)}$ and z_0, z_1, z_2, z_3, ... $z_{\pi-1}$, consequently also by the magnitudes t', t'', t''' ...... $t^{(\pi)}$, and the coefficients A, B, C, &c. of the given equation.

EXAMPLE. When $\pi = 1$, we only have

$$y' = z_0$$

which must also be the case, because then t and y are symmetrical functions of x', x'', x''', &c. and $\therefore$ y no longer depends on t, but only on the coefficients A, B, C, &c.

When $\pi = 2$, we have the two equations

$$y' + y'' = z_0$$
$$t'y' + t''y'' = z_1$$

and hence

$$y' = \frac{z_1 - t''z_0}{t' - t''}, \quad y'' = \frac{z_1 - t'z_0}{t'' - t'}$$

When $\pi = 3$, we have the three equations

$$y' + y'' + y''' = z_0$$
$$t'y' + t''y'' + t'''y''' = z_1$$
$$t'^2y' + t''^2y'' + t'''^2y''' = z_2$$

and hence we obtain

$$y' = \frac{z_2 - (t'' + t''')z_1 + t''t'''z_0}{(t' - t'')(t' - t''')}$$

$$y'' = \frac{z_2 - (t' + t''')z_1 + t't'''z_0}{(t'' - t')(t'' - t''')}$$

$$y''' = \frac{z_2 - (t' + t'')z_1 + t't''z_0}{(t''' - t')(t''' - t'')}$$

In the same way, when $\pi = 4$, we find the following values for $y', y'', y''', y^{\mathrm{v}}$.

$$\frac{z_3 - (t'' + t''' + t^{\mathrm{v}})z_2 + (t''t''' + t''t^{\mathrm{v}} + t'''t^{\mathrm{v}})z_1 - t''t'''t^{\mathrm{v}}z_0}{(t' - t'')(t' - t''')(t' - t^{\mathrm{v}})}$$

$$\frac{z_3 - (t' + t''' + t^{\mathrm{v}})z_2 + (t't''' + t't^{\mathrm{v}} + t'''t^{\mathrm{v}})z_1 - t't'''t^{\mathrm{v}}z_0}{(t'' - t')(t'' - t''')(t'' - t^{\mathrm{v}})}$$

$$\frac{z_3 - (t' + t'' + t^{\mathrm{v}})z_2 + (t't'' + t't^{\mathrm{v}} + t''t^{\mathrm{v}})z_1 - t't''t^{\mathrm{v}}z_0}{(t''' - t')(t''' - t'')(t''' - t^{\mathrm{v}})}$$

$$\frac{z_3 - (t' + t'' + t''')z_2 + (t't'' + t't''' + t''t''')z_1 - t't''t'''z_0}{(t^{\mathrm{v}} - t')(t^{\mathrm{v}} - t'')(t^{\mathrm{v}} - t''')}$$

from which the law of the progression may be very easily seen.

SECTION CXLIII.

Prob. All that has been said in the problem in the foregoing § holds, with this single difference, that all the roots of the equation II, as was there assumed, are not known, but merely one of them: required now to find the numerical value of the function y corresponding to this numerical value of the function t.

Solution 1. Let t' be the known root of the equation II. If we divide this equation by $t - t'$, we obtain another equation

III. $t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \ldots\ldots\ldots + U' = o$
in which

$$\begin{aligned} P' &= t' + P \\ Q' &= t'^2 + Pt' + Q \\ R' &= t'^3 + Pt'^2 + Qt' + R \\ S' &= t'^4 + Pt'^3 + Qt'^2 + Rt' + S \\ &\&c. \end{aligned}$$

and the roots of this equation are t'', t''', $t^{\prime v}$, ... $t^{(\pi)}$.

2. But since in this case the single root t' was assumed to be known, we must merely endeavour to express y' by t'; and this object is most easily attained in the following way by means of the method of elimination given in § LVIII. Multiply the equations in 3 of the foregoing §, beginning with the last but one, and proceeding upwards, by P', Q', R', &c. viz. the last but one by P', the one preceding it by Q', and so on to the first, which is multiplied by U', and then add the results thus obtained to the last equation; by these means we obtain

$$\begin{aligned} & z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0 \\ = & y'\ (t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \ldots\ldots + U') \\ + & y''\ (t''^{\pi-1} + P't''^{\pi-2} + Q't''^{\pi-3} + \ldots\ldots + U') \\ + & y'''\ (t'''^{\pi-1} + P't'''^{\pi-2} + Q't'''^{\pi-3} + \ldots\ldots + U') \\ & \&c. \end{aligned}$$

3. Since t'', t''', $t^{\prime v}$, ... $t^{(\pi)}$, are the roots of the equation III, then all that which has been multiplied by y'', y''', $y^{\prime v}$, ... $y^{(\pi)}$ in the second part of the equation just found, $= o$. We only $\therefore$ retain

$$\begin{aligned} & z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0 \\ = & y'(t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \ldots\ldots + U') \end{aligned}$$

and hence it follows

$$y' = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0}{t'^{\pi-1} + P't'^{\pi-2} + Q't'^{\pi-3} + \ldots\ldots + U'}$$

4. If in this we substitute for t' every other root of the equation II, we then obtain the numerical values of $y'', y''', y'_{v}, \ldots\ldots y^{(\pi)}$. If $\therefore$ t and y which are undetermined, denote two corresponding values of the functions just given, we then have generally

$$y = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots + U'z_0}{t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \ldots\ldots + U'}$$

and it is then

$$\begin{aligned} P' &= t + P \\ Q' &= t^2 + Pt + Q \\ R' &= t^3 + Pt^2 + Qt + R \\ &\text{\&c.} \end{aligned}$$

Example. In § XXXIX we find, that when $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ is the given equation, the function $t = x'x'' + x'''x^{\prime v}$ depends on the following equation of the third degree:

$$t^3 - Bt^2 + (AC - 4D)\,t - (C^2 - 4BD + A^2D) = 0.$$

I shall now assume, that we have so far solved this equation, that we have found one of its roots, and that we now wished to determine from it the value of another function $y = (x'x'' - x'''x^{\prime v})^2$, which is homogeneous to the former.

Since here $\pi = 3$, we then have

$$y = \frac{z_2 + P'z_1 + Q'z_0}{t^2 + P't + Q'}$$

Further, since $P = -B$, $Q = AC - 4D$, we have

$$P' = t + P = t - B$$
$$Q' = t^2 + Pt + Q = t^2 - Bt + AC - 4D$$

It only remains now to determine the values of z_0, z_1, z_2. But

$$t' = x'x'' + x'''x^{\mathrm{IV}},\quad y' = (x'x'' - x'''x^{\mathrm{IV}})^2$$
$$t'' = x'x''' + x''x^{\mathrm{IV}},\quad y'' = (x'x''' - x''x^{\mathrm{IV}})^2$$
$$t''' = x'x^{\mathrm{IV}} + x''x''',\quad y''' = (x'x^{\mathrm{IV}} - x''x''')^2$$

which, when we take the numerical expressions from the annexed Tables, give the following values:

$$z_0 = y' + y'' + y''' = [2^2] - 6[1^4]$$
$$= B^2 - 2AC - 4D$$
$$z_1 = t'y' + t''y'' + t'''y''' = [3^2] - [1^2 2^2]$$
$$= B^3 - 3ABC + 3C^2 + 3A^2D - 4BD$$
$$z_2 = t'^2y' + t''^2y'' + t'''^2y''' = [4^2] - 6[2^4]$$
$$= B^4 - 4AB^2C + 2A^2C^2 + 4BC^2 + 4A^2BD$$
$$- 4B^2D - 8ACD$$

If we substitute the values of P', Q', z_0, z_1, z_2 here found, we obtain

$$y = \frac{\begin{pmatrix}(B^2 - 2AC - 4D)t^2 - (ABC - 3C^2 - 3A^2D)t \\ + 16D^2 - 4B^2D + BC^2 + A^2BD - 4ACD\end{pmatrix}}{3t^2 - 2Bt + AC - 4D}$$

and by means of this expression we are now enabled, for each numerical value of the function t, to find a numerical value of the function y.

Remark. By means of the differential calculus, we can give the denominator of the general expression for y in 4 a more simple form. Thus, since

$$(t - t')(t^{\pi-1} + P't^{\pi-2} + Q't^{\pi-3} + \ldots\ldots + U')$$
$$= (t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + \ldots\ldots + U)$$

we then have, when we differentiate both sides, in reference to t, and divide by the differential dt,

$$(t-t')\,[\pi-1)t^{\pi-2}+(\pi-2)\,P't^{\pi-3}+(\pi-3)\,Q't^{\pi-4}+\&c.]$$
$$+t^{\pi-1}+P't^{\pi-2}+Q't^{\pi-3}+R't^{\pi-4}+\&c.$$
$$=$$
$$\pi t^{\pi-1}+(\pi-1)Pt^{\pi-2}+(\pi-2)Qt^{\pi-3}+(\pi-3)\,Rt^{\pi-4}+\&c.$$

If in these we substitute t' for t, we obtain the equation

$$t'^{\pi-1}+P't'^{\pi-2}+Q't'^{\pi-3}+R't'^{\pi-4}+\&c.$$
$$=\pi t'^{\pi-1}+(\pi-1)\,Pt'^{\pi-2}+(\pi-2)\,Qt'^{\pi-3}+\&c.$$

and since this equation must be correct, whatever root we assume for t', we then have generally

$$t^{\pi-1}+P't^{\pi-2}+Q't^{\pi-3}+\&c.=$$
$$\pi t^{\pi-1}+(\pi-1)\,Pt^{\pi-2}+(\pi-2)\,Qt^{\pi-3}+(\pi-3)Rt^{\pi-4}+\&c.$$

Consequently the value of y may also be expressed in the following way

$$y=\frac{z_{\pi-1}+P'z_{\pi-2}+Q'z_{\pi-3}+\ldots\ldots\ldots\ldots\ldots\ldots+U'z_0}{\pi t^{\pi-1}+(\pi-1)\,Pt^{\pi-2}+(\pi-2)\,Qt^{\pi-3}+\ldots+T}$$

SECTION CXLIV.

If the formula of the foregoing § be generally applicable, we are enabled, from the given value of any function $f:(x')\,(x'')\,(x''')\ldots\ldots(x^{(n)})$ to find the value of every other function $\phi:\,(x')\,(x'')\,(x''')\ldots\ldots(x^{(n)})$, homogeneous to it, and that immediately merely by a rational expression. But it is also actually applicable in all imaginable cases, with the single exception of the one in which the value of t is such, that the denominator of the expression for $y=0$; a case which was mentioned in § LX. In order to see how the case is here, I shall consider the denominator $t'^{\pi-1}+P't'^{\pi-2}+Q't'^{\pi-3}+\&c.$

in the expression for y in 3 of the foregoing §. It is, from its origin, no other than the product of the factors $t' - t''$, $t' - t'''$, $t' - t^{v}$, ... $t' - t^{(\pi)}$. If ∴ it vanishes, then one or other of these factors = o, and t' ∴ must be equal to one, or even more of the roots t'', t''', t^{v}, ... $t^{(\pi)}$. Hence it follows, that the case in which the denominator in the expression for y vanishes, can only obtain, when the equation II has equal roots. But now it may likewise be seen, why this expression cannot give the value of y'. For so long as a number of roots t', t'', t''', ... $t^{(\nu)}$ are different from one another, t' gives the value of y', t'' the value of y'', &c. But if they are equal to one another, then the single root t' must at once give the ν values y', y'', y''', ... $y^{(\nu)}$; but since the expression found for y is rational, this is impossible. Hence it may be further concluded, that the ν values y', y'', y''', ... $y^{(\nu)}$ must be given by a single irrational expression, which contains exactly ν values, or, which is the same, that they must depend on an equation of the νth degree, whose coefficients are all rational. How this equation may be found, will be seen immediately.

SECTION CXLV.

Auxiliary Rule.

Prob. Let Π denote any function of x, and let the equation

$$y = (x - a)^m \Pi$$

be given: required to find the value of the differential proportional $\frac{d^m y}{dx^m}$ for the case, where $x = a$.

Solution 1. Let $m = 1$; $\therefore y = (x-a)\,\Pi$. If we differentiate this equation, we find

$$dy = (x-a)\,d\Pi + \Pi dx$$

If in this equation we put $x = a$, then the first term of the second part vanishes, and we consequently have, when Π' denotes what Π becomes when we put $x = a$,

$dy = \Pi' dx$, and $\dfrac{dy}{dx} = \Pi'$.

2. Let $m = 2$; $\therefore y = (x-a)^2\Pi$. If we differentiate this equation twice successively, we find

$$dy = (x-a)^2\,d\Pi + 2\,(x-a)\,\Pi dx$$
$$d^2y = (x-a)^2\,d^2\Pi + 4\,(x-a)\,d\Pi dx + 1\,.\,2\,\Pi dx^2$$

If we put $x = a$ in the second equation, the two first terms of the second part vanish, and we then have

$d^2y = 1\,.\,2\,\Pi' dx^2$; consequently $\dfrac{d^2y}{dx^2} = 1\,.\,2\,\Pi'$.

3. Let $m = 3$; $\therefore y = (x-a)^3\,\Pi$. If we differentiate this equation three times in succession, we then obtain successively,

$$dy = (x-a)^3 d\,\Pi + 3\,(x-a)^2\Pi dx$$
$$d^2y = (x-a)^3 d^2\Pi + 6\,(x-a)^2 d\Pi dx + 2\,.\,3\,(x-a)\Pi dx^2$$
$$d^3y = (x-a)^3 d^3\Pi + 9\,(x-a)^2 d^2\Pi dx + 18\,(x-a) d\Pi dx^2$$
$$+1\,.\,2\,.\,3\,\Pi dx^3$$

and when we put $x = a$, $d^3y = 1\,.\,2\,.\,3\,\Pi' dx^3$, $\therefore$ $\dfrac{d^3y}{dx^3} = 1\,.\,2\,.\,3\Pi'$.

4. Generally, as is easily seen from the continuation of the calculation, we find for $d^m y$, after differentiating the

equation $y=(x-a)^m\Pi$ m times, a differential expression, whose last term is $1 . 2 . 3 \ldots m\Pi dx^m$, and in which all the remaining terms contain the factor $x-a$. If $\therefore$ we put $x=a$, we then obtain $d^my = 1 . 2 . 3 \ldots\ldots m\Pi' dx^m$, and consequently

$$\frac{d^my}{dx^m} = 1 . 2 . 3 \ldots\ldots\ldots m\Pi'$$

SECTION CXLVI.

PROB. When t and y denote two homogeneous functions of the roots x', x'', x''', &c. of the given equation

$$\text{I. } x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \&c. = 0$$

from the known value of the function t it is required to find the value of the function y in the case where the equation

$$\text{II. } t^\pi + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + \&c. = 0$$

on which the first depends, contains equal roots, amongst which is the known value of t.

Solution 1. In the remark in § CXLIII, we find the following expression for y:

$$y = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots\ldots + U'z_0}{\pi t^{\pi-1} + (\pi-1)\,Pt^{\pi-2} + (\pi-2)\,Qt^{\pi-3} + \&c.}$$

in which $P' = t + P$, $Q' = t^2 + Pt + Q$, &c.; and from this general expression we obtain the particular values of y', y'', y''', &c., when we substitute t', t'', t''', &c. for t. Now we wish, in the first place, to give this expression a form which will be more convenient for our purpose.

2. Since t', t'', t''', &c. are the roots of the equation II, then

$$t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + \&c. =$$
$$(t-t')\,(t-t'')\,(t-t''')\,(t-t^{\mathrm{IV}})\ldots\ldots(t-t^{(\pi)})$$

If we differentiate this equation in reference to t, we obtain, after dividing by dt,

$$\pi t^{\pi-1} + (\pi-1)\,Pt^{\pi-2} + (\pi-2)\,Qt^{\pi-3} + \&c. =$$
$$(t-t'')\;\;(t-t''')\;\;(t-t^{\mathrm{V}})\;\ldots\ldots\ldots\ldots\;(t-t^{(\pi)})$$
$$+\;(t-t')\;\;(t-t''')\;(t-t^{\mathrm{V}})\;\ldots\ldots\ldots\ldots\;(t-t^{(\pi)})$$
$$+\;(t-t')\;\;(t-t'')\;\;(t-t^{\mathrm{V}})\;\ldots\ldots\ldots\ldots(t-t)^{(\pi)})$$
&c.

If in this we substitute t', t'', t''', &c. successively for t, we obtain

$$\pi t'^{\pi-1} + (\pi-1)\,Pt'^{\pi-2} + (\pi-2)\,Qt'^{\pi-3} + \&c. =$$
$$(t'-t'')\;\;(t'-t''')\;\;(t'-t^{\mathrm{V}})\;\ldots\ldots\ldots\;(t'-t^{(\pi)})$$
$$\pi t''^{\pi-1} + (\pi-1)\,Pt''^{\pi-2} + (\pi-2)\,Qt''^{\pi-3} + \&c. =$$
$$(t''-t')\;(t''-t''')\;(t''-t^{\mathrm{V}})\;\ldots\ldots\ldots\;(t''-t^{(\pi)})$$
$$\pi t'''^{\pi-1} + (\pi-1)\,Pt'''^{\pi-2} + (\pi-2)\,Qt'''^{\pi-3} + \&c. =$$
$$(t'''-t')\;(t'''-t'')\;(t'''-t^{\mathrm{V}})\;\ldots\ldots\;(t'''-t^{(\pi)})$$
&c.

3. Now, if we denote that which the numerator of the expression for t becomes by the substitution of t', t'', t''', &c. for t, by Ω', Ω'', Ω''', &c., we then obtain, by means of the results in 2,

$$y' = \frac{\Omega'}{(t'-t'')\;\;(t'-t''')\;\;(t'-t^{\mathrm{V}})\ldots\ldots(t'-t^{(\pi)})}$$
$$y'' = \frac{\Omega''}{(t''-t')\;(t''-t''')\;(t''-t^{\mathrm{V}})\ldots\ldots(t''-t^{(\pi)})}$$
$$y''' = \frac{\Omega'''}{(t'''-t')\,(t'''-t'')\,(t'''-t^{\mathrm{V}})\ldots\ldots(t'''-t^{(\pi)})}$$
&c.

From the form of these values it is evident, that when $t'=t''$, the denominators in the values of y' and y'', are both $= 0$; whence, by § CXLIV, it may be concluded, that these values cannot be determined singly by a rational expression, but depend on an equation of the second degree. In like manner, when we put $t' = t'' = t'''$, the denominators in the values of y', y'', y''', all vanish, and consequently in this case these values must depend on a single equation of the third degree; and in a similar way it holds, when more values of t are equal to one another.

4. In the first place we assume, that the equation II has no more than two equal roots t', t''. First let them be unequal, and let them differ by an infinitely small magnitude h, so that $t'' = t' + h$. Further, let, for shortness' sake,

$$(t' - t''') (t' - t^{v}) \ldots\ldots\ldots (t' - t^{(n)}) = \Pi'$$
$$(t'' - t''') (t'' - t^{v}) \ldots\ldots\ldots (t'' - t^{(n)}) = \Pi''$$

we then have

$$y' = \frac{\Omega'}{(t' - t'')\,\Pi'} = \frac{\Omega'}{-h\Pi'}$$
$$y'' = \frac{\Omega''}{(t'' - t')\,\Pi''} = \frac{\Omega''}{h\Pi''}$$

and $\therefore$

$$y' + y'' = \frac{1}{h}\left[\frac{\Omega''}{\Pi''} - \frac{\Omega'}{\Pi'}\right].$$

5. If we omit the infinitely small magnitude in this last equation, we then can put $\Pi'' = \Pi'$, and we consequently have

$$y' + y'' = \frac{\Omega'' - \Omega'}{h} \cdot \frac{1}{\Pi'}$$

But according to Taylor's Theorem

$$\Omega'' - \Omega' = \frac{d\Omega'}{dt'} \cdot \frac{h}{1} + \frac{d^2\Omega'}{dt'^2} \cdot \frac{h^2}{1.2} + \&c.$$

If ∴ we divide this expression by h, and then put $h = o$, we obtain

$$y' + y'' = \frac{d\Omega'}{dt'} \cdot \frac{1}{\Pi'}$$

6. We now assume, that the equation II has three equal roots, which are t', t'', t'''. As before, consider again these roots at first as differing, by an infinitely small magnitude, and put $t'' = t' + h$, $t''' = t' + k$; further, put

$$(t' - t^{\text{iv}})(t' - t^{\text{v}})(t' - t^{\text{vi}}) \ldots \ldots (t' - t^{(n)}) = \Pi'_1$$
$$(t'' - t^{\text{iv}})(t'' - t^{\text{v}})(t'' - t^{\text{vi}}) \ldots \ldots (t'' - t^{(n)}) = \Pi''_1$$
$$(t''' - t^{\text{iv}})(t''' - t^{\text{v}})(t''' - t^{\text{vi}}) \ldots \ldots (t''' - t^{(n)}) = \Pi'''_1$$

Then we have (3)

$$y' = \frac{\Omega'}{(t' - t'')(t' - t''')\Pi'_1} = \frac{1}{hk} \cdot \frac{\Omega'}{\Pi'_1}$$

$$y'' = \frac{\Omega''}{(t'' - t')(t'' - t''')\Pi''_1} = \frac{1}{h(h-k)} \cdot \frac{\Omega''}{\Pi''_1}$$

$$y''' = \frac{\Omega'''}{(t''' - t')(t''' - t'')\Pi'''_1} = \frac{1}{k(k-h)} \cdot \frac{\Omega'''}{\Pi'''_1}$$

If we add these three results, we obtain

$$y' + y'' + y''' = \frac{1}{hk} \cdot \frac{\Omega'}{\Pi'_1} + \frac{1}{h(h-k)} \cdot \frac{\Omega''}{\Pi''_1} + \frac{1}{k(k-h)} \cdot \frac{\Omega'''}{\Pi'''_1}$$

or, when we omit the infinitely small magnitude in this

last equation, and put $\Pi_1''' = \Pi_1'' = \Pi_1'$,

$$y' + y'' + y''' = \frac{1}{\Pi_1'}\left(\frac{\Omega'}{hk} + \frac{\Omega''}{h(h-k)} + \frac{\Omega'''}{k(k-h)}\right).$$

7. But by Taylor's Theorem

$$\Omega'' = \Omega' + \frac{d'\Omega}{dt'}\cdot\frac{h}{1} + \frac{d^2\Omega'}{dt'^2}\cdot\frac{h^2}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{h^3}{1.2.3} + \&c.$$

$$\Omega''' = \Omega' + \frac{d'\Omega}{dt'}\cdot\frac{k}{1} + \frac{d^2\Omega'}{dt'^2}\cdot\frac{k^2}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{k^3}{1.2.3} + \&c.$$

If we substitute this sum in the expression for $y' + y'' + y'''$, and omit what ought to be left out, we then obtain

$$y' + y'' + y''' = \left(\frac{d^2\Omega'}{dt'^2}\cdot\frac{1}{1.2} + \frac{d^3\Omega'}{dt'^3}\cdot\frac{h+k}{1.2.3} + \&c.\right)\frac{1}{\Pi_1'}$$

Now, if we put h and $k = o$, we get

$$y' + y'' + y''' = \frac{d^2\Omega'}{dt'^3}\cdot\frac{1}{1.2\Pi_1'}$$

8. In like manner, if four roots t', t'', t''', t^{v}, of the equation II are equal to one another, when in the beginning we assume these roots as differing by an infinitely small magnitude, and $t'' = t' + h$, $t''' = t + k$, $t^{\text{v}} = t' + l$, but after completing the calculation, we put h, k and $l = o$, we then find the following result:

$$y' + y'' + y''' + y^{\text{v}} = \frac{d^3\Omega'}{dt'^3}\cdot\frac{1}{1.2.3\Pi_2'}$$

when we put

$$(t' - t^{\text{v}})\,(t' - t^{\text{v}\prime})\,(t' - t^{\text{v}\prime\prime})\ldots\ldots(t' - t^{(n)}) = \Pi_2'$$

9. Hence we may perceive the law. Thus, if ν roots

$t', t'', t''', t^{v}, \ldots\ldots t^{(\nu)}$ are equal to one another, we have

$$y' + y'' + y''' \ldots\ldots y^{(\nu)} = \frac{d^{\nu-1}\Omega'}{dt'^{\nu-1}} \cdot \frac{1}{1 \,.\, 2 \,.\, 3 \ldots \nu - 1 \,.\, \Pi'}$$

when we put

$$(t' - t^{(\nu+1)})\,(t' - t^{(\nu+2)}) \ldots\ldots (t' - t^{(\pi)}) = \Pi'.$$

10. The expression Π' contains the roots $t^{\nu+1}$, $t^{\nu+2}$, $\ldots\ldots t^{(\pi)}$. Now, since it may happen that we know no other root of the equation II, except t', it remains to be shown, how we can determine this expression directly from the above equation.

11. By the assumed nature of the equation II, when we put

$$(t - t^{(\nu+1)})\,(t - t^{(\nu+2)})\,(t - t^{(\nu+3)}) \ldots\ldots (t - t^{(\pi)}) = \Pi$$

we have

$$(t - t')^{\nu}\Pi = t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + \&c.$$

If we differentiate this equation ν times successively with reference to t, and then substitute t' for t, we then obtain (foregoing §)

$$1 \,.\, 2 \,.\, 3 \ldots\ldots \nu \,.\, \Pi' = \frac{d^{\nu}\,(\pi t'^{\pi} + Pt'^{\pi-1} + Qt'^{\pi-2} + \&c.)}{dt'^{\nu}}$$

or, when we actually differentiate it once

$$1 \,.\, 2 \,.\, 3 \ldots \nu \,.\, \Pi' = \frac{d^{\nu-1}\,(\pi t'^{\pi-1} + (\pi - 1)\,Pt'^{\pi-2} + (\pi - 2)\,Qt'^{\pi-3} + \&c.}{dt'^{\nu-1}}$$

12. If we substitute the value of Π', which we derive from hence, in 9 we then obtain

$$y' + y'' + \dots + y^{\nu} = \frac{\nu d^{\nu-1}\Omega'}{d^{\nu-1}(\pi t'^{\pi-1} + (\pi-1)Pt'^{\pi-2} + \&c.)}$$

the differentials taken with reference to t'.

13. We have ∴ found the sum of the values corresponding to the equal values of y. But in like manner also, we may find the sum of their squares, cubes, and so on. To effect this, we only require in the equations in 3, § CXLII, for the function y to substitute its square y^2, its cube y^3, &c. Since by these means the magnitudes z_0, z_1, $z_2, \dots\dots z_{\pi-1}$, only undergo any change, nothing remains to be done, but to change the expression $\Omega = z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \dots\dots + U'z_0$ accordingly, and moreover to retain the formula just found for $y' + y'' + y''' + \dots\dots + y^{(\nu)}$. Having obtained these sums, we may likewise always find the equation, which has the values y', y'', $y''', \dots\dots y^{(\nu)}$ as roots, and this equation must necessarily be solved, if we wish to find the above values.

Remark. From what has been here said, we see the reason why it was said in 10, § CXXXVI, that it would be sufficient to solve the equation for the coefficient p, in order to find the other coefficients q, r, &c. immediately, and without the solution of any other equation. For since p, q, r, &c. are all homogeneous functions of x', x'', x''', &c. from the known numerical value of one of them, we may represent the numerical values of all the others by mere rational expressions; because the cases in which the denominators of these expressions vanish, belong to the exceptions, and can only occur in particular

equations, and not in general ones, of which we treated in the above-mentioned §.

SECTION CXLVII.

For the sake of the use which we might, perhaps, make of this, I shall now arrange the results found in the foregoing § together, and for the greater generality, instead of the function y itself, I shall assume any power of it y^κ.

If we denote the symmetrical functions expressed by the coefficients of the given equation

$$\begin{array}{l}
y'^\kappa + y''^\kappa + y'''^\kappa + \ldots\ldots + (y^{(\pi)})^\kappa \\
t'y'^\kappa + t''y''^\kappa + t'''y'''^\kappa + \ldots\ldots + (t^{(\pi)})(y^{(\pi)})^\kappa \\
t'^2y'^\kappa + t''^2y''^\kappa + t'''^2y'''^\kappa + \ldots\ldots + (t^{(\pi)})^2(y^{(\pi)})^\kappa \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
t'^{\pi-1}y'^\kappa + t''^{\pi-1}y''^\kappa + t'''^{\pi-1}y'''^\kappa + \ldots + (t^{(\pi)})^{\pi-1}(y^{(\pi)})^\kappa
\end{array}$$

in the order in which they succeed each other, by z_0, z_1, z_2, ...$z_{\pi-1}$, and put, for shortness' sake,

$$\pi t'^{\pi-1} + (\pi-1)Pt'^{\pi-2} + (\pi-2)Qt'^{\pi-3} + \&c. = \Phi'$$

$$z_{\pi-1} + P'z_{\pi-2} + Q'z_{\pi-3} + \ldots\ldots\ldots\ldots + U'z_0 = \Omega'$$

(in which $P' = t' + P$, $Q' = t'^2 + Pt' + Q$, $R' = t'^3 + Pt'^2 + Qt' + R$, &c.); we have for a simple root of the transformed equation for t,

$$y'^\kappa = \frac{\Omega'}{\Phi'};$$

for a double root

$$y'^\kappa + y''^\kappa = \frac{2d\Omega'}{d\Phi'};$$

for a three-fold root

$$y'^\kappa + y''^\kappa + y'''^\kappa = \frac{3d^2\Omega'}{d^2\Phi'};$$

for a four-fold root

$$y'^{\kappa} + y''^{\kappa} + y'''^{\kappa} + y'^{\text{v}\kappa} = \frac{4d^3\Omega'}{d^3\Phi'};$$

and, in general, for an ν-fold root

$$y'^{\kappa} + y''^{\kappa} + y'''^{\kappa} + \ldots\ldots + (y^{(\nu)})^{\kappa} = \frac{\nu d^{\nu-1}\Omega'}{d^{\nu-1}\Phi'};$$

all the differentials taken in reference to t'.

By means of these formulæ, we may find the sums of powers of all those values of y, which belong to the complex root t'. Having now found these sums of powers, we may also, by § IX, find the equation on which they depend. I shall now elucidate what has been just advanced by an example.

Example. I shall assume, that in order to solve the equation

$$\text{I.}\quad x^4 - 3x^3 - 3x^2 + 11x - 6 = 0$$

we have transformed it into another

$$\text{II.}\quad t^6 - 9t^5 + 21t^4 + 9t^3 - 54t^2 + 32 = 0$$

when we put $t = x' + x''$. I shall further assume, that we are able to find a root of this last equation, and that we now wished from it to determine the value of the function $y = x'x''$.

Here we have the following corresponding values of t and y

$$x'+x'',\ x'+x''',\ x'+x^{\text{iv}},\ x''+x''',\ x''+x^{\text{iv}}+x'''+x^{\text{iv}}$$

$$x'x'',\quad x'x''',\quad x'x^{\text{iv}},\quad x''x''',\quad x''x^{\text{iv}},\quad x'''x^{\text{iv}}$$

and these give, when $\kappa = 1$,

$$z_0 = x'x'' + x'x''' + x'x^{\text{iv}} + x''x''' + x''x^{\text{iv}} + x'''x^{\text{iv}}$$

$$= [1^2]$$

$$z_1 = (x' + x'')\,x'x'' + (x' + x''')\,x'x''' + (x + x^{\prime\mathrm{v}})\,x'x^{\prime\mathrm{v}} + \&c.$$
$$= [12]$$
$$z_2 = (x' + x'')^2 x'x'' + (x' + x''')^2 x'x''' + (x'x^{\prime\mathrm{v}})^2 x'x^{\prime\mathrm{iv}} + \&c.$$
$$= [13] + 2[2^2]$$
$$z_3 = (x' + x'')^3 x'x'' + (x' + x''')^3 x'x''' + (x' + x^{\prime\mathrm{iv}})^3 x'x^{\prime\mathrm{iv}} + \&c.$$
$$= [14] + 3\,[23]$$
$$z_4 = (x' + x'')^4 x'x'' + (x'x''')^4 x'x''' + (x' + x^{\prime\mathrm{iv}})^4 x'x^{\prime\mathrm{iv}} + \&c.$$
$$= [15] + 4\,[24] + 6\,[3^2]$$
$$z_5 = (x' + x'')^5 x'x'' + (x' + x''')^5 x'x''' + (x' + x^{\prime\mathrm{v}})^5 x'x^{\prime\mathrm{v}} + \&c.$$
$$= [16] + 5\,[25] + 10\,[34]$$

If we take the numerical expressions from the annexed Tables, and then put for A, B, C, D, their values $3, -3, -11, -6$, we then find $z_0 = -3$, $z_1 = 24$, $z_2 = 90$, $z_3 = 390$, $z_4 = 1542$, $z_5 = 6174$. If we substitute these values in the expression for Ω', we then obtain, since here $P' = t' - 9$, $Q' = t'^2 - 9t' + 21$, $R' = t'^3 - 9t'^2 + 21t' + 9$, $S' = t'^4 - 9t'^3 + 21t'^2 + 9t' - 54$, $T' = t'^5 - 9t'^4 + 21t'^3 + 9t'^2 - 54t'$, after the usual reduction:

$$\Omega' = z_5 + P'z_4 + Q'z_3 + R'z_2 + S'z_1 + T'z_0$$
$$= -3t'^5 + 51\,t'^4 - 189\,t'^3 + 57\,t'^2 + 300\,t'$$

Also

$$\phi' = 6\,t'^5 - 45\,t'^4 + 84\,t'^3 + 27\,t'^2 - 108\,t'$$

consequently

$$y' = \frac{-3\,t'^5 + 51\,t'^4 - 189\,t'^3 + 57\,t'^2 + 300\,t'}{6\,t'^5 - 45\,t'^4 + 84\,t'^3 + 27\,t'^2 - 108\,t'}$$

One root of the equation II, is $t = 1$. If we substitute this root for t' in the value of y' here found, we get $y' = x'x'' = -6$. Of the accuracy of this result we can convince ourselves by solving the two equations $x' + x'' = 1$, $x'x'' = -6$; for by these means we obtain

3 and -2 for x' and x'', and these are actually two roots of the equation I.

Another root of the equation II, is $t=2$; and this root substituted for t' in the value of y', gives $y'=1$. But from $x'+x''=2$, and $x'x''=1$, we find $x'=x''=1$; whence it follows, that $x=1$ is also a root of the equation II, and that a double one.

But $t=4$ is also a root of the equation II. If we substitute this root for t' in the value of x', we find $y=\frac{o'}{o}$, which denotes that y' can be determined from t' in no other way than by an equation of the second degree. If, however, we differentiate Ω' and ϕ', we then find

$$d\Omega'=(-15t'^4+204t'^3-567t'^2+114t'+300)\,dt'$$
$$d\Phi'=(30t'^4-180t'^3+252t'^2+54t'-108)\,dt'$$

$\therefore$

$$y'+y''=\frac{2d\Omega'}{d\Phi'}=$$
$$2\,.\,\frac{-15t'^4+204t'^3+567t'^2+114t'+300}{30t'^4-180t'^3+252t'^2+54t'-108}.$$

If in this we put $t'=4$, we then get

$$y'+y''=6.$$

In order to determine y' and y'' singly, we must now find the value of $y'^2+y''^2$.

With this view, we put $\kappa=2$; we then have

$z_0=x'^2x''^2+\&c.=[2^2]$

$z_1=(x'+x'')\,x'^2x''^2+\&c.=[23]$

$z_2=(x'+x'')^2x'^2x''^2+\&c.=[24]+2\,[3^2]$

$z_3=(x'+x'')^3x'^2x''^2+\&c.=[25]+3\,[34]$

$z_4=(x'+x'')^4x'^2x''^2+\&c.=[26]+4\,[35]+\ 6\,[4^2]$

$z_5=(x'+x'')^5x'^2x''^2+\&c.=[27]+5\,[36]+10\,[45].$

If we take the numerical expressions from the annexed Tables, and then put for A, B, C, D, their values 3, -3, -11, -6, we then find $z_0 = 63$, $z_1 = 102$, $z_2 = 336$, $z_3 = 1188$, $z_4 = 4668$, $z_5 = 18492$. If we substitute these values in the expression for Ω', and at the same time for P', Q', R', S', T', put the above expressions, after the requisite reduction, we find

$$\Omega' = z_5 + P'z_4 + Q'z_3 + R'z_2 + S'z_1 + T'z_0$$
$$= 63t'^5 - 465t'^4 + 741t'^3 + 873t'^2 - 1452t' - 1056$$

$\therefore$

$$d\Omega' = (315t'^4 - 1860t'^3 + 2223t'^2 + 1746t' - 1452)\,dt'$$

the values of Φ' and $d\Phi'$ remain the same as before. We obtain $\therefore$

$$y'^2 + y''^2 = 2 \cdot \frac{315t'^4 - 1860t'^3 + 2223t'^2 + 1746t' - 1452}{30t'^4 - 180t'^3 + 252t'^2 + 54t' - 108}$$

If in this we put $t' = 4$, we obtain

$$y'^2 + y''^2 = 18.$$

We have now $\therefore$ the two equations

$$y' + y'' = 6,\ y'^2 + y''^2 = 18,$$

whence it follows, that the two values y', y'', depend on the quadratic equation

$$y^2 - 6y + 9 = 0$$

which contains the double root 3; and $\therefore$ $y' = y'' = 3$. That the result is correct, appears immediately, when we solve the two equations $x' + x'' = 4$, $x'x'' = 3$; for these give 1 and 3 for the values of x' and x'', which are actually two roots of the equation I. Besides, because here y' depends on an equation of the second degree, we may infer from hence, that $t = 4$ must be a double root of the equation II; which is also correct.

If we put $t = -1$, which is also a double root of the equation II, we find, when in the above expression for y', we put -1 for t', $y' = \frac{0}{0}$, as required. But if in the two expressions found for $y' + y''$ and $y'^2 + y''^2$, we put -1 for t', we then obtain

$$y' + y'' = -4,\ y'^2 + y''^2 = 8$$

and consequently the values of y', y'', depend on the equation

$$y^2 + 4y + 4 = 0$$

which has the double root $y = -2$. We have $\therefore$ $y' = y'' = -2$. But when we solve the two equations $x' + x'' = -1$, $x'x'' = -2$, we then obtain for x' and x'' the values 1 and -2, which are actually two roots of the equation I.

Besides that for y, as well as for $t = 4$, and for $t = -1$, we found such quadratic equations as have double roots, is merely accidental, and this will only be the case, when the equal values of y also correspond to the equal values of t.

SECTION CXLVIII.

PROB. Let t and y be any two functions of the roots of a given equation: required to find a general method by which, from the known value of one, to find the value of the other, however the functions are constituted.

Solution 1. In order to solve the problem in its most general form, we shall assume, that both functions contain all the roots of the given equation. This supposition is

always allowable; for if one function does not contain all the roots at the same time, we then can, as was already observed in § XLIX, add those that are wanting with the coefficient o. Thus, if we had the function $x'x''x'''$, and the given equation were of the fifth degree, it would only be necessary, instead of these, to put $x'x''x''' + o \,.\, x^{IV} + o \,.\, x^{V}$.

2. The method in § CXLII for determining the numerical values of y from the numerical values of t assumed as known, in the case in which both these functions are symmetrical, may also be applied, when they are not so, by merely making the alterations which are requisite on this account. It was said in the above place, that, when $t', t'', t''', \ldots\ldots t^{(\pi)}$ denote the unequal values of forms of t, and $y', y'', y''', \ldots\ldots t^{(\pi)}$ the unequal values of forms of y, the function $t'^{\lambda}y' + t''^{\lambda}y'' + t'''^{\lambda}y''' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ is symmetrical, because the function $t^{\lambda}y$ can have no more unequal values of forms than those of which the former function is composed. This is correct, when the functions t, y, are no longer symmetrical, because they do not in this case change, or remain unchanged, at the same time.

3. But the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots. + (t^{(\pi)})^{\lambda}y^{(\pi)}$, in every imaginable state of the functions t, y, are assuredly always symmetrical, when $t', t'', t''', \ldots\ldots t^{(\pi)}$ and $y', y'', y''', \ldots\ldots y^{(\pi)}$ denote not only the unequal values of forms, but generally all the possible values, which arise from the transposition of the roots x', x'', x''', &c. whether equal or unequal. That in certain cases, and in

certain forms of the functions t, y, we often get a much less number of these values, is nothing to the purpose; because here we only are treating of the general method applicable to every case.

4. The method in § CLXIII for determining the numerical value of a function y from a single known numerical value of t, may in like manner be extended to functions which are not homogeneous, provided by t', t'', t''', ... $t^{(\pi)}$ and y', y'', y''', ... $y^{(\pi)}$, we merely denote all the possible values of forms of t and y, which arise from the transposition of the roots x', x'', x''', &c., and the transformed equation II be composed of all the values of forms of t, and not, as has always been the case hitherto, only of the unequal ones. This equation, however, will be found by the following method. I shall assume, that amongst all the π values of forms there are μ unequal ones, and that the equation for these last $t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + rt^{\mu-3} + \text{\&c.} = 0$ is already found. Further, if we put $\pi = \mu\nu$, then ν is necessarily a whole number, because by § LV, Corollary, μ is always a submultiple of π, and all the π values of forms, taken ν and ν together, will then be equal. The equation II, which, as is now required, is composed of all the values of forms t', t'', t''', ... $t^{(\pi)}$, is consequently no other than

$$(t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + rt^{\mu-3} + \text{\&c.})^{\nu} = 0$$

and it may . · . be obtained by solving this equation. If the values of forms t', t'', t''', ... $t^{(\pi)}$ be all different

from one another, then $\nu = 1$, and the equation II is the equation $t^\mu + pt^{\mu-1} + qt^{\mu-2} + \&c. = o$ itself.

5. With respect to the equation on which the numerical value of the function y depends, two cases must be distinguished; viz. 1st, the case in which the given equation is the most general one of its degree, and consequently whose coefficients are in no way combined; 2ndly, the case in which the coefficients are determinate numbers, or else have some relation to one another.

6. In the first case, the equation II can only contain roots which are all unequal, when the values of forms t', t'', t''' ...... $t^{(\pi)}$ are all different from one another; and if this be the case, as we have seen in the foregoing §, the numerical value of y may be expressed rationally by the numerical value of t. But if the above values of forms of t, consequently also the roots of the equation II, are equal, taken ν and ν together, then each of these roots is ν-fold, and consequently the numerical value of y (when all the particular relations between the functions t and y are first laid aside), necessarily depends on an equation of the νth degree, which may always be found (§ CXLVI); and this equation gives the ν values of y, which at the same time correspond to this root.

7. In the second case, on the other hand, it may happen, that this or that root t' of the equation II, besides the $\nu - 1$ equal values, which arise from the

identity of the values of forms, has also other equal ones, which have their bases in the particular property of the given equation itself, and consequently in a case of this kind the numerical value of y, which corresponds to the root t', must necessarily be given by an equation of a higher degree than the νth.

8. Hitherto we have not noticed, in the general inquiries respecting the dependence of the numerical values of the functions t and y, the particular nature of these functions; it is now time to consider this. We have already seen, in the preceding §, that, when the above functions are symmetrical, the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ becomes symmetrical, when for t', t'', t''', ... $t^{(\pi)}$ we merely take the unequal values of forms; by which means not only the calculation is essentially shortened, but likewise in the case, in which the transformed equation for t has equal roots, the numerical values of y, which correspond to these equal roots, are expressed by lower equations than we should have obtained if we had introduced all the values of forms of t. But a similar abbreviation may generally be used, when the functions t and y are such, that when the nature of one of them is expressed by the equation

$$A' = A'' = A''' = \ldots = A^{(k)} = A^{(k+1)} = \ldots\ldots = A^{(p)}$$

between the p types A', A'', A''', ...$A^{(k)}$, $A^{(k+1)}$,...... $A^{(p)}$, the nature of the other is determined by the equation

$$A' = A'' = A''' = \ldots = A^{(k)}$$

merely between the k types A', A'', A''', ... $A^{(k)}$. For

if we try to find all the unequal types, which a function contains, whose nature is expressed by the type-equation $A' = A'' = A''' = \ldots\ldots = A^{(\iota)}$, and then find all the values of forms of the function t and y corresponding to these unequal types; consequently, when t', t'', t''', $\ldots\ldots t^{(\pi)}$ and y', y'', y''', $\ldots\ldots y^{(\pi)}$ denote these values, the function $t'^{\lambda}y' + t''^{\lambda}y'' + \ldots\ldots + (t^{(\pi)})^{\lambda}y^{(\pi)}$ will necessarily be symmetrical, because there is no value of forms of $t^{\lambda}y$, which is not included amongst those of which this aggregate is composed.

9. As for the formation of the transformed equation for t in the assumed property of the functions t, y, we must distinguish the two cases, where t or y is that function, whose nature is determined by the equation $A' = A'' = A''' = \ldots\ldots = A^{(\iota)}$. If the first be supposed, then the values of forms t', t'', t''', $\ldots\ldots t^{(\pi)}$, are all different from one another, and the equation II, which is composed of these values of forms, is actually, as in the case of homogeneous functions, only the result of the unequal values of forms. But if the second supposition be taken, then amongst the values of forms t', t'', t''', $\ldots t^{(\pi)}$ there are several equal ones; and when we put the numbers of the unequal ones amongst these, consequently the number of the unequal values of forms which that function has whose nature is determined by the type-equation $A' = A'' = A''' = \ldots = A^{(\nu)}$, equal to μ, the number π is a multiple of the number μ. If $\therefore$ we put $\pi = \mu\nu$, and assume that $t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + \&c. = 0$ is the equation, which is merely composed of the unequal values of forms of t, then the equation II, which is composed of

the values of forms t', t'', t''',......$t^{(\pi)}$, is no other than the developement of the equation

$$(t^{\mu} + pt^{\mu-1} + qt^{\mu-2} + \&c.)^{\nu} = 0.$$

10. Further, since to each root t of the equation II there are ν corresponding values of y, consequently the numerical value of y depends necessarily on an equation of the νth degree. If the functions t and y be homogeneous, then $\nu = 1$, and consequently this value depends only on an equation of the first degree, as required. But all this only obtains so long as the given equations are general ones; for in particular equations it might certainly happen, as was already observed, that the equation for y were of a higher degree.

11. Besides the relations given in 8, between the functions t, y, there are numberless others, in which the calculation may, in like manner, be simplified. Such a simplification as this is always practicable, when in all the values of forms of t, which arise from all the possible transpositions of the roots x', x'', x''', &c., such as t', t'', t''',......$t^{(\mu)}$ may be omitted, which are all either different, or the periods of the different values occur more than once, and at the same time are such, that the function $t'^{\lambda}y' + t''^{\lambda}y'' + t'''^{\lambda}y''' + \ldots\ldots + (t^{(\mu)})^{\lambda}y^{(\mu)}$ is symmetrical.

12. Although there are cases where the calculation may be simplified, when, instead of all the values of forms of the function t, we only use those which possess

the properties just mentioned, yet for the determination of the value of y from the value of t, there is no further disadvantage arising from it (with the exception of the calculations being extremely prolix). It may indeed be objected, that in this case the equation for y rises to a higher degree than is necessary, and that it may happen, that we cannot solve an equation of this kind, notwithstanding, perhaps, that in the calculation properly arranged, we arrive at a solvible equation. But since in this case amongst the roots of the former equation, there must be more than one which are equal, and in the sequel it will be shown, that an equation of this kind may always be reduced to another, which only contains the unequal roots, consequently in the present case the lowest rational equation for y, $\therefore$ this objection is removed of itself.

SECTION CXLIX.

PROB. Let t and y be two functions of the roots of a general equation of any degree: required to give the degree of the equation, by which the numerical value of y is determined from the known numerical value of t.

Solution. In the function t perform all the transpositions of the roots x', x'', x''', &c. for which its value of form remains unchanged; in the function y perform the same transpositions as in t. Let ν be the number of the unequal values of forms of y, which we obtain by these means; then the equation between y and t, with reference to y, is of the νth degree. For since the equal

values of forms of t have only a single numerical value, the ν unequal values of forms, on the other hand, ν different numerical values, consequently ν numerical values of y belong to a single numerical value of t; $\therefore$ the former can be determined from the last in no other way than by an equation of the νth degree.

Example I. With respect to the general equation of the fourth degree, let $t = f:\ (x')\ (x'')\ (x''')\ ({}^{iv})$, $y = \phi:\ (x')\ (x'')\ (x''')\ (x^{iv})$, and let the nature of these functions be expressed by the type-equations

$$f:\ (x')\ (x'')\ (x''')\ (x^{iv}) = f:\ (x'')\ (x')\ (x''')\ (x^{iv}) =$$
$$f:\ (x')\ (x'')\ (x^{iv})\ (x''') = f:\ (x''')\ (x^{iv})\ (x')\ (x'')$$

$$\phi:\ (x')\ (x'')\ (x''')\ (x^{iv}) = \phi:\ (x'')\ (x')\ (x''')\ (x^{iv})$$
$$= \phi:\ (x')\ (x'')\ (x^{iv})\ (x''')$$

Now, if we try to find the equal values of forms of t (§ LV), and then perform the same transpositions in y, we then obtain the following corresponding values of t and y:

$f:\ (x')\ (x'')\ (x''')\ (x^{iv})$	$\phi:\ (x')\ (x'')\ (x''')\ (x^{iv})$
$f:\ (x')\ (x'')\ (x^{iv})\ (x''')$	$\phi:\ (x')\ (x'')\ (x^{iv})\ (x''')$
$f:\ (x'')\ (x')\ (x''')\ (x^{iv})$	$\phi:\ (x'')\ (x')\ (x''')\ (x^{iv})$
$f:\ (x'')\ (x')\ (x^{iv})\ (x''')$	$\phi:\ (x'')\ (x')\ (x^{iv})\ (x''')$
$f:\ (x''')\ (x^{iv})\ (x')\ (x'')$	$\phi:\ (x''')\ (x^{iv})\ (x')\ (x'')$
$f:\ (x^{iv})\ (x''')\ (x')\ (x'')$	$\phi:\ (x^{iv})\ (x''')\ (x')\ (x'')$
$f:\ (x''')\ (x^{iv})\ (x'')\ (x')$	$\phi:\ (x''')\ (x^{iv})\ (x'')\ (x')$
$f:\ (x^{iv})\ (x''')\ (x'')\ (x')$	$\phi:\ (x^{iv})\ (x''')\ (x'')\ (x')$

Of the eight values of y, which we have here found, the four first, as well as the four last, on account of the sup-

posed nature of this function, are equal to one another; consequently two values of y belong to a single value of t. Therefore the equation, which gives y in terms of t, is of the second degree.

To the numberless functions of the assumed nature, the following ones belong, viz.: $t = x'x'' + x'''x^{IV}$, $y = x'x'' = x'x'' + 0\,x''' + 0\,x^{IV}$, or $y = x' + x'' = x' + x'' + 0\,x''' + 0\,x^{IV}$. Consequently, if the numerical value of $x'x'' + x'''x^{IV}$ be known, we may find from it both the numerical value of $x'x''$, and that of $x' + x''$, by the solution of an equation of the second degree, which agrees with § XLI, where we merely had to solve an equation of the second degree.

Example II. For any general equation, let $t = x'x''x''' + x^{IV}$, $y = x' - x'' = x' - x'' + 0(x''' + x^{IV})$. Now, in order to find the degree of the lowest rational equation, by which y may be determined from t, proceed as follows:

Equal values of forms of t	Corresponding values of forms of y
$x'x''x''' + x^{IV}$	$x' - x'' + 0\,(x''' + x^{IV})$
$x'x'''x'' + x^{IV}$	$x' - x''' + 0\,(x'' + x^{IV})$
$x''x'x''' + x^{IV}$	$x'' - x' + 0\,(x''' + x^{IV})$
$x''x'''x' + x^{IV}$	$x'' - x''' + 0\,(x' + x^{IV})$
$x'''x'x'' + x^{IV}$	$x''' - x' + 0\,(x'' + x^{IV})$
$x'''x''x' + x^{IV}$	$x''' - x'' + 0\,(x' + x^{IV})$

Consequently six different values of y belong to a single value of t, viz.: $x' - x''$, $x' - x'''$, $x'' - x'$, $x'' - x'''$,

$x''' - x', x''' - x''$; and $\therefore$ y can only be determined from t by an equation of the sixth degree, when its coefficients are rational functions of t. Besides, since the values of forms of y, taken two and two, are equal, this equation $\therefore$ only contains even powers of y.

SECTION CL.

PROB. Let t, y, be any two functions of the roots $x'x''x'''$ &c. of a general equation: required an operation to find the lowest equation, by which the numerical value of y may be determined from the numerical value of t, under the condition that the unequal values of forms of t only are made use of.

Solution. Find, as in the foregoing §, the equal values of forms of t, and the corresponding values of forms of y, and from these last take away the unequal ones; let them be $y', y'', y''' \ldots\ldots\ldots y^{(\nu)}$: then the required equation will be of the νth degree, and it has the above values for roots. Let

$$y^{\nu} + py^{\nu-1} + qy^{\nu-2} + ry^{\nu-3} + \&c. = 0$$

be this equation; then $p = y' + y'' + y''' + \&c.$ $q = y'y'' + y'y''' + y''y''' + \&c.$ $r = y'y''y''' + \&c.$ &c. Consequently, since the functions p, q, r, &c. with reference to $y', y'', y''', \ldots\ldots y^{(\nu)}$, are symmetrical, then in those transpositions of the roots x', x'', x''', &c. for which the function t continues unchanged, they in like manner undergo no change. If $\therefore$ we denote the unequal values of forms of t, by $t', t'', t''', \ldots\ldots\ldots t^{(\pi)}$, then

$t'^{\lambda}p'$, $t''^{\lambda}p''$, $t'''^{\lambda}p'''$, ... $(t^{(\pi)})^{\lambda}p^{(\pi)}$, are all the possible unequal values of forms of $t^{\lambda}p$, and in the same way $t'^{\lambda}q'$, $t''^{\lambda}q''$, $t'''^{\lambda}q'''$, ... $(t^{(\pi)})^{\lambda}q^{(\pi)}$ are all the possible values of forms of $t^{\lambda}q$, and so on. But if this be symmetrical with reference to x', x'', x''', &c. .... then also are the functions $t'^{\lambda}p' + t''^{\lambda}p'' + t'''^{\lambda}p''' + \ldots$ $\ldots + (t^{(\pi)})^{\lambda}p^{(\pi)}$, $t'^{\lambda}q' + t''^{\lambda}q'' + t'''^{\lambda}q''' + \ldots\ldots\ldots +$ $(t^{(\pi)})^{\lambda}q^{(\pi)}$ necessarily symmetrical with reference to x', x'', x''', &c., and consequently the unequal values of forms t', t'', t''', ... $t^{(\pi)}$ are sufficient for the determination of p, q, r, &c. Therefore the operation given in § CXLVI may be applied immediately, and, without any alteration to the coefficients p, q, r, &c. Thus, if we wish to determine p, we find, in the first place, the transformed equation for the function t according to the third chapter; it is

$$t^{\pi} + Pt^{\pi-1} + Qt^{\pi-2} + Rt^{\pi-3} + \&c. = 0.$$

Having found this, we immediately get

$$p = \frac{z_{\pi-1} + P'z_{\pi-2} + Q'_{\pi-3} + \ldots\ldots + U'z_0}{\pi t^{\pi-1} + \pi-1 \,.\, Pt^{\pi-2} + \pi-2 \,.\, Qt^{\pi-3} + \&c.}$$

in which $P' = t + P$, $Q' = t^2 + Pt + Q$, &c., and the symbol of the form z_k denotes the numerical value of the symmetrical function $t'^{k}p' + t''^{k}p'' + t'''^{k}p''' + \ldots$ $\ldots\ldots + (t^{(\pi)})^{k}p^{(\pi)}$. For the coefficients q, r, &c. the same equation and the same expression obtain, with this exception, that by z_k we must understand the numerical values of the functions $t'^{k}q' + t''^{k}q'' + t'''^{k}q''' + \ldots\ldots$ $\ldots + (t^{(\pi)})^{k}q^{(\pi)}$, $t'^{k}r' + t''^{k}r'' + t'''^{k}r''' + \ldots\ldots\ldots +$ $(t^{(\pi)})^{k}r^{(\pi)}$, &c.

SECTION CLI.

So long as x', x'', x''', &c. may be considered as the roots of a general equation $x^{\pi} + Ax^{\pi-1} + Bx^{\pi-2} + \&c. = o$, consequently of one whose coefficients A, B, C, &c. are undetermined, we shall always find rational functions of t for the coefficients p, q, r, &c. But if these roots relate to a particular equation, then it may happen, according to the nature of the function t, that the common denominator $\pi t^{\pi-1} + \pi-1 \, . \, Pt^{\pi-2} + \pi-2 \, . \, Qt^{\pi-3} + \&c.$ in the expressions for p, q, r, &c. is equal to o, and that it even continues equal to o, when it is differentiated more than once. We now assume, that we must differentiate it $\mu-1$ times before the denominator ceases to vanish, consequently it follows, from § CLXVI, that the coefficients p, q, r, &c. depend upon the same number of equations of the μth degree

$$p^{\mu} + a'p^{\mu-1} + b'p^{\mu-2} + c'p^{\mu-3} + \&c. = o$$
$$q^{\mu} + a''q^{\mu-1} + b''q^{\mu-2} + c''q^{\mu-3} + \&c. = o$$
$$r^{\mu} + a'''r^{\mu-1} + b'''r^{\mu-2} + c'''r^{\mu-3} + \&c. = o$$
$$\&c.$$

which may always be found by the method there given, and in which the coefficients a', b', c', &c. a'', b'', c'', &c. a''', b''', c''', &c. are all rational functions of t.

All that has been said in this chapter respecting the function y, may also be applied to the function x. Thus, if we wish to determine a root, say x', from the known value of a function $t = f : (x') \, (x'') \, (x''') \ldots (x^{\pi})$, nothing further is necessary, than to put $y = x'$, and to proceed besides in the way already pointed out.

We now perceive the reason why we are not able, from

the known value of a symmetrical function of the roots of an equation, whatever the nature of this function may be, to determine these roots. For since a function of this kind, in all the transpositions of the roots, always retains the same value, .·. it must necessarily give all the roots at once; and, however we begin it, we shall consequently always again get an equation, which is not different from the given one.

VIII.—A GENERAL METHOD FOR THE SOLUTION OF EQUATIONS OF ALL DEGREES.

SECTION CLII.

IN § CXLI we have seen that the requisites for the general solution of equations may be reduced to two; viz. first, to find such functions of the roots, by means of which the equation, into which we have transformed the given one, is adapted to the solution; and secondly, to determine the roots from the known value of the assumed functions. The second requisite we have handled in the foregoing chapter; the first, together with its application to the general solution of equations, will form the subject of this chapter.

In order to render the notation more easy, and the inspection more convenient, I shall henceforth omit in the types the letter x, together with the superfluous brackets, and for the dashes substitute numbers; thus: f: (12345 ......n) instead of f: $(x') (x'') (x''') (x^{\prime v})...(x^{(n)})$, and f: (342651) instead of f: $(x''') (x^{\prime v}) (x'') (x^{v\prime}) (x^{v}) (x')$.

SECTION CLIII.

RULE. When from the period of n types A_1, A_2, A_3, A_4......A_μ....A_ν......A_n, which may be derived from the equation

$$f:\ (123456 \ldots\ldots \overline{n-1n}) = f:\ (234567 \ldots\ldots n1)$$

we take away any two A_μ, A_ν and find all the possible types, which may be derived from the transformation-rule $A_\mu = A_\nu$; then we shall get a period, which, in the case where $\nu - \mu$ and ν are prime numbers to each other, consists of all the n types $A_1, A_2, A_3, \ldots\ldots A_n$; on the other hand, when $\nu - \mu$ and n have a common measure m, this period only consists of $\frac{n}{m}$ of these types. The types, which we successively obtain by deduction, succeed each other in the following order:

$$A_\mu,\ A_\nu,\ A_{2\nu-\mu},\ A_{3\nu-2\mu},\ A_{4\nu-3\mu},\ \&c.$$

so that the dashes μ, ν, $2\nu - \mu$, $3\nu - 2\mu$, $4\nu - 3\mu$, &c. form an arithmetical progression, with the difference $\nu - \mu$, when from all the terms of this progression, which exceed the number n, we omit this number as often as possible.

Thus the equation

$$f:\ (12345678) = f:\ (23456781)$$

gives the period

$$\begin{array}{l}
A_1 \ldots\ldots\ldots\ldots f:\ (12345678) \\
A_2 \ldots\ldots\ldots\ldots f:\ (23456781) \\
A_3 \ldots\ldots\ldots\ldots f:\ (34567812) \\
A_4 \ldots\ldots\ldots\ldots f:\ (45678123) \\
A_5 \ldots\ldots\ldots\ldots f:\ (56781234) \\
A_6 \ldots\ldots\ldots\ldots f:\ (67812345) \\
A_7 \ldots\ldots\ldots\ldots f:\ (78123456) \\
A_8 \ldots\ldots\ldots\ldots f:\ (81234567)
\end{array}$$

If we equate every two of these types, we obtain

For the equation	The period
$A_1 = A_3$	$A_1,\ A_3,\ A_5,\ A_7,$
$A_1 = A_4$	$A_1,\ A_4,\ A_7,\ A_2,\ A_5,\ A_8,\ A_3,\ A_6$
$A_1 = A_5$	$A_1,\ A_5$
$A_1 = A_6$	$A_1,\ A_6,\ A_3,\ A_8,\ A_5,\ A_2,\ A_7,\ A_4$
$A_1 = A_7$	$A_1,\ A_7,\ A_5,\ A_3$
$A_1 = A_8$	$A_1,\ A_8,\ A_7,\ A_6,\ A_5,\ A_4,\ A_3,\ A_2$
$A_2 = A_3$	$A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8,\ A_1$
$A_2 = A_4$	$A_2,\ A_4,\ A_6,\ A_8$
$A_2 = A_5$	$A_2,\ A_5,\ A_8,\ A_3,\ A_6,\ A_1,\ A_4,\ A_7$
&c.	&c.

The reason of this is easily found, and depends on the properties of numbers.

Corollary I. The rule is also correct, when, instead of the type-equation $A_\mu = A_\nu$ we take the type-equation $A_\nu = A_\mu$, if in the progression which is taken away, viz. ν, μ, $2\mu - \nu$, $3\mu - 2\nu$, $4\mu - 3\nu$, $5\mu - 4\nu$, &c., when we come to a negative term or o, we add the number n so many times, till it becomes positive. Thus we have

For the equation	The period
$A_2 = A_1$	$A_2,\ A_1,\ A_8,\ A_7,\ A_6,\ A_5,\ A_4,\ A_3$
$A_3 = A_1$	$A_3,\ A_1,\ A_7,\ A_5$
$A_4 = A_1$	$A_4,\ A_1,\ A_6,\ A_3,\ A_8,\ A_5,\ A_2,\ A_7$
$A_5 = A_1$	$A_5,\ A_1$
$A_6 = A_1$	$A_6,\ A_1,\ A_4,\ A_7,\ A_2,\ A_5,\ A_8,\ A_3$
$A_7 = A_1$	$A_7,\ A_1,\ A_3,\ A_5$
$A_8 = A_1$	$A_8,\ A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7.$
&c.	&c.

Corollary II. If ∴ n be a prime number, we then always obtain the same period again, whichever two of the types A_1, A_2, A_3, ... A_n we put equal to one-another.

SECTION CLIV.

Transpositions of the kind, which the equation $A_u = A_\nu$ or $A_\nu = A_\mu$, gives in the preceding §, are called recurring transpositions, and the periods which are obtained from them, recurring periods.

The characteristic feature of transpositions of this kind consists in this, that each root is removed one place forwards or backwards, and in the first case the last takes the place of the first; but in the second case, the first takes the place of the last, so that there is a kind of circular motion; as if, for instance, a number of persons stand in a circle, having their backs to each other, and all walking at the same time either backwards or forwards.

The transpositions are called recurring ones, when only some of the roots move in the manner just mentioned, but the remaining ones retain their places. Thus the equation $f:$ (12345678) $= f:$ (34512678) only gives recurring transpositions of the first five roots. The law of the preceding § is also true for this, when we take for n merely the number of the roots to be transposed, and the remaining ones are considered, as though they did not exist.

SECTION CLV.

Rule. If the equation $x^n + Ax^{n-1} + Bx^{n-2} +$ &c. $=0$, by the introduction of a function $t = f:$ (12345...n),

be transformed into an equation of two terms $t^n - K = 0$: then the roots of this last equation $t', t'', t''', \ldots\ldots t^{(n)}$ are always the numerical values of such values of forms of the function t, as, taken together, form a period.

Proof. The roots of the equation $t^n - K = 0$ may always, as may be seen from the fifth chapter, be expressed by $t', at', a^2t', a^3t', \ldots\ldots a^{n-1}t'$, when a denotes a primitive root of the equation $t^n - 1 = 0$. Therefore $t'' = at'$, $t''' = at''$, $t^{v} = at'''$, ... $t^{(n)} = at^{(n-1)}$, $t' = at^{(n)}$. Now, let $A_1, A_2, A_3, A_4, \ldots\ldots A_n$, denote the values of forms, which correspond to the roots t', t'', t''', ... $t^{(n)}$, then also must $A_2 = aA_1$, $A_3 = aA_2$, $A_4 = aA_3$, ... $A_n = aA_{n-1}$, $A_1 = aA_n$; and since every such equation $A_\nu = aA_{\nu-1}$, independently of the particular values which we may assign to the roots x', x'', x''', &c. must be true, it will also remain true, when in the two parts of this equation we transpose the above roots in any way, provided it be the same, because this is precisely the same as when the values of these roots are changed. If $\therefore$ we assume, that in $A_{\nu-1}$, we have so transposed the roots x', x'', x''', &c. that it becomes A_ν, and that by the same transposition A_ν is transformed into any other value of form A_π; then we have also $A_\pi = aA_\nu$. But if also $A_{\nu+1} = aA_\nu$, consequently $A_{\nu+1} = A_\pi$. Hence it follows that $A_{\nu+1}$ is generated by the same transposition from A_ν, as A_ν is from $A_{\nu-1}$; $\therefore$ A_2 is produced from A_1, as A_3 is from A_2, as A_4 from A_3; and so on; lastly, as A_n is from A_{n-1}, and A_1 from A_n. Since $\therefore$ all the values of forms $A_1, A_2, A_3, A_4, \ldots\ldots\ldots A_n$ are deduced from one another by the same transposition-rule, and

from the last A_n, the first A_1 is again obtained; $\therefore$ it follows, that these n values of forms constitute a period.

SECTION CLVI.

PROB. Amongst all the possible functions of the roots x', x'', of the general equation of the second degree $x^2 - Ax + B = 0$, find that one which is fit for its solution; under the supposition that we know not how to solve any other equations, than those of the first degree, and those of the form $t^2 - K = 0$.

Solution 1. Let $t = f: (12)$ be that function, which is fit for the solution of the given equation. Since it has two values, viz. $f: (12)$, $f: (21)$, then the equation for t, taken generally, is of the second degree. If we only wished it to be of the first degree, then must $f: (12) = f: (21)$; but then $f: (12)$ would be symmetrical; and the roots x', x'', could be determined from the known value of t only by the solution of the given equation itself (§ CXLIX). There now remains nothing further than to assume that the two values of forms $f: (12)$, $f: (21)$, are the roots of an equation of the form $t^2 - K = 0$, because it was assumed that we know not how to solve an equation of the second degree of any other form but this. That they may be so, appears from hence, that they form a period (preceding §).

2. Since $K = -t't''$, when t', t'' denote the two roots of the equation $t^2 - K = 0$; then also $K = -f: (12) \times f: (21)$; and since this product remains

the same when we substitute x' for x'', $\therefore K$ is a symmetrical function of these roots. Consequently this magnitude may be expressed rationally by the coefficients of the given equation.

3. Since $f:(12)$, $f:(21)$ are the roots of the equation $t^2 - K = 0$, therefore $f:(12) = -f:(21)$ and this is the only condition which we have to fulfil. Having once found the numerical value of $t = f:(12)$, then also the roots x', x'', may be determined without the solution of any other equation, because the values of forms $f:(12)$, $f:(21)$ are different (§ CXLIV).

4. This condition, however, is evidently sufficient, when we put $f:(12) = \phi:(12) - \phi:(21)$, where it is allowable, for $\phi:(12)$ to assume every arbitrary function of x', x'' which is not symmetrical. For from $f:(12) = \phi:(12) - \phi:(21)$ we obtain by the substitution of x' for x'', $f:(21) = \phi:(21) - \phi:(12)$: consequently $f:(12) = -f:(21)$, as was required.

5. Hence it follows, that all functions of the form $\phi:(12) - \phi:(21)$ are fit for the solution of the given equation.

SECTION CLVII.

Prob. Solve actually the general equation of the second degree $x^2 - Ax + B = 0$.

Solution 1. We have seen in the foregoing §, that all functions of the form $t = \phi:(12) - \phi:(21)$ are fit for

the solution. Amongst the infinite number of functions which we can assume for $\phi:(12)$, the root x is the most simple. Put $\therefore$ $\phi:(12)=x'$, then $t=\phi:(12)-\phi:(21)$ $=x'-x''$. But the equation $t^2-K=0$ gives $K=t^2=$ $(x'-x'')^2=x'^2+x''^2-2x'x''=[2]-2[1^2]=A^2-4B$; the transformed equation consequently is

$$t^2-(A^2-4B)=0$$

and this gives $t=\pm\sqrt{(A^2-4B)}$. We $\therefore$ have the two equations

$$x'+x''=A$$
$$x'-x''=\pm\sqrt{(A^2-4B)}$$

and hence $x'=\dfrac{A\pm\sqrt{(A^2-4B)}}{2}$, $x''=\dfrac{A\mp\sqrt{(A^2-4B)}}{2}$, as was required.

SECTION CLVIII.

Prob. Find the functions which are fit for the solution of the general equation of the third degree

$$x^3-Ax^2+Bx-C=0$$

under the supposition, that we know not how to solve any other equations than those of the first and second degrees, and those of the form $t^3-K=0$.

Solution 1. Let $t=f:(123)$ represent all those functions which are fit for the solution of the given equation. Since the roots x', x'', x''', admit of six transpositions, consequently the function t contains six values; and these are

$$f:(123),\quad f:(231),\quad f:(312)$$
$$f:(213),\quad f:(132),\quad f:(321)$$

Consequently, taken generally, the equation for t is of the sixth degree.

2. The six values of forms in 1 are arranged in recurring periods. Thus, in the first horizontal row, we have the recurring period $f\colon (123)$, and in the second $f\colon (213)$. If we assume, that the three values of forms of the first period are the roots of the equation of two terms, viz. $t^3 - K = o$, then K is the product of these three roots, and $\therefore = f\colon (123) \times f\colon (231) \times f\colon (312)$. But this product, as may be easily seen, is such, that in all the recurring transpositions of the three roots x', x'', x''', it undergoes no change: for if we perform these transpositions, we then obtain the period

$$\begin{array}{l} f\colon (123) \times f\colon (231) \times f\colon (312) \\ f\colon (231) \times f\colon (312) \times f\colon (123) \\ f\colon (312) \times f\colon (123) \times f\colon (231) \end{array}$$

and these three values of forms of K are evidently not different from one another. The six values of forms of K, which arise from all the transpositions of the roots x', x'', x''', are $\therefore$ equal, taken three and three, and consequently this function has no more than two different values; and these are

$$\begin{array}{l} f\colon (123) \times f\colon (231) \times f\colon (312) \\ f\colon (213) \times f\colon (132) \times f\colon (321) \end{array}$$

of which one can be derived from the other merely by putting the roots x', x'' for each other.

3. Since $\therefore$ K has no more than two different values, consequently this function depends on an equation of no higher degree than the second, and the roots of this equa-

tion are these two values. But these values admit of a more simple form; for since $K = t^3$, and $t = f\colon (123)$, then also $K = (f\colon (123))^3$; and since the second value, as we have already seen in 2, is obtained from the first, merely by putting the roots x', x'' for each other, consequently $(f\colon (213))^3$ is the second value of K.

4. Let

$$K^2 - pK + q = 0$$

be the equation, on which the function K depends, $\therefore$ p the sum, and q the product, of the two values of K. Consequently

$$p = (f\colon (123))^3 + (f\colon (213))^3$$
$$q = (f\colon (123))^3 \times (f\colon (213))^3$$

and these functions p, q are such, that in all the transpositions of the roots x', x'', x''', they suffer no change. Since p and q are symmetrical functions of the roots x', x'', x''', they may always $\therefore$ be expressed rationally by the coefficients of the given equation.

5. We have consequently reduced the transformed equation for t, which originally was of the sixth degree, to two equations

$$t^3 - K = 0$$
$$K^2 - pK + q = 0$$

and we are always able to determine the coefficients p, q, from A, B, C, when the function $f\colon (123)$ is known. Having once determined the coefficients p, q, we then obtain, by the solution of the second equation, the two values of K, and if these be successively substituted in the first equation, by its solution we obtain the six values of t.

6. Since all the values of t are different from one another, we may always determine (which is known from the foregoing chapter) the values of the roots x', x'', x''', immediately from the values of the function t already found, and that without the solution of any other equation, however constituted this function may be.

7. It now only remains to determine the function $t = f: (123)$ in such a way, that the three values of forms $f: (123)$, $f: (231)$, $f: (312)$, may be the roots of an equation of the form $t^3 - K = o$. If this be the case, then these three values must have such a relation to each other, that

$$f: (123) = \alpha f: (231) = \alpha^2 f: (312).$$

In order to perform this, we assume any other arbitrary function $\phi: (123)$, and put

$$f: (123) = A\phi: (123) + B\phi: (231) + C\phi: (312)$$

in which A, B, C, denote coefficients hitherto unknown. From this equation, by a proportional transposition of the roots, we obtain

$$f: (231) = A\phi: (231) + B\phi: (312) + C\phi: (123)$$
$$f: (312) = A\phi: (312) + B\phi: (123) + C\phi: (231)$$

and when we substitute these values in the foregoing proportional equation, we get

$$A\phi: (123) + B\phi: (231) + C\phi: (312)$$
$$= \alpha\ \big(A\phi: (231) + B\phi: (312) + C\phi: (123)\big)$$
$$= \alpha^2 \big(A\phi: (312) + B\phi: (123) + C\phi: (231)\big)$$

If we equate the coefficients of these values of forms, we obtain, for the determination of A, B, C, the following equations:

$$A = \alpha C = \alpha^2 B$$
$$B = \alpha A = \alpha^2 C$$
$$C = \alpha B = \alpha^2 A.$$

Since $\alpha^3 = 1$, the first gives $B = \alpha A$, $C = \alpha^2 A$, and these values verify also the second and third. The coefficient A remains undetermined, and we may ∴ put it equal to 1. Consequently

$$f : (123) = \phi : (123) + \alpha\phi : (231) + \alpha^2\phi : (312).$$

8. We can, as was observed already, for $\phi : (123)$ assume every arbitrary function; yet, for another reason, those which undergo no change in the recurring transpositions of all the three roots, cannot be used. For in the case where $\phi : (123)$ is a function of this kind, we have $\phi : (123) = \phi : (231) = \phi : (312)$, and consequently $f : (123) = (1 + \alpha + \alpha^2)\ \phi : (123) = 0$, because $1 + \alpha + \alpha^2 = 0$. This restriction ∴ might with good reason be omitted, since it is a necessary consequence.

SECTION CLIX.

PROB. Required to solve actually the general equation of the third degree $x^3 - Ax^2 + Bx - C = 0$.

Solution 1. In the foregoing § we saw, that all functions of the form $\phi : (123) + \alpha\phi : (231) + \alpha^2\phi : (312)$ are fit for the solution of equations of the third degree. Consequently there are numberless ways in which these equations may be solved. The most simple supposition

is $\phi:\ (123) = x'$; then $\phi:\ (231) = x''$, $\phi:\ (312) = x'''$; $\therefore$

$$f:\ (123) = x' + \alpha x'' + \alpha^2 x'''.$$

2. Hence we obtain

$$(f:\ (123))^3 = [3] + 6[1^3] +$$
$$3\alpha(x''x'^2 + x'x'''^2 + x'''x''^2) + 3\alpha^2(x'''x'^2 + x'x''^2 + x''x'''^2)$$

and when in this we substitute x' for x'', we get

$$(f:\ (213))^3 = [3] + 6[1^3] +$$
$$3\alpha(x'x''^2 + x''x'''^2 + x'''x'^2) + 3\alpha^2(x'''x''^2 + x''x'^2 + x'x'''^2)$$

or when, for shortness' sake, we put $[3] + 6[1^3] = P$, $x''x'^2 + x'x'''^2 + x'''x''^2 = Q$, $x'''x'^2 + x'x''^2 + x''x'''^2) = R$,

$$(f:\ (123)^3) = P + 3\alpha Q + 3\alpha^2 R$$
$$(f:\ (213))^3 = P + 3\alpha R + 3\alpha^2 Q.$$

3. Hence, by 4 of the foregoing §, we further obtain

$$p = (f:\ (123))^3 + (f:\ (213))^3$$
$$= 2P + 3(\alpha + \alpha^2)(Q + R)$$

or, since $\alpha + \alpha^2 = -1$, and $Q + R = [12]$,

$$p = 2[3] + 12[1^3] - 3[12]$$

and when for the numerical expressions we put their values taken from the annexed Tables,

$$p = 2A^3 - 9AB + 27C.$$

4. Further, by the foregoing §

$$q = (f:\ (123))^3 \times (f:\ (213))^3 =$$
$$(P + 3\alpha Q + 3\alpha^2 R)(P + 3\alpha R + 3\alpha^2 Q)$$
$$= P\big(P + 3(\alpha+\alpha^2)(Q+R)\big) + 9(\alpha+\alpha^2)QR + 9(Q^2 + R^2)$$

or, since $\alpha + \alpha^2 = -1$, $Q+R = [12]$, $QR = [3^2] + 3[2^3] + [1^2 4]$, $Q^2 + R^2 = [24] + 2[123]$

$$q = \left([3] + 6[1^3]\left([3] + 6[1^3] - 3[12]\right)\right.$$
$$\left. + 9\left([24] + 2[123] - [3^2] - 3[2^3] - [1^24]\right)\right)$$

or when for the numerical expressions we put their values,

$$q = A^6 - 9A^4B + 27A^2B^2 - 27B^3$$
$$= (A^2 - 3B)^3.$$

5. Consequently, the two equations in 5 of the foregoing § are

$$t^3 - K = 0$$
$$K^2 - (2A^3 - 9AB + 27C)K + (A^2 - 3B)^3 = 0.$$

6. Let K', K'' be the two values of K, and t', t'', the corresponding values of t, then

$$t' = f:\ (123) = x' + \alpha x'' + \alpha^2 x''' = \sqrt[3]{K'}$$
$$t'' = f:\ (213) = x'' + \alpha x' + \alpha^2 x''' = \sqrt[3]{K''}.$$

We have ∴ for the determination of x', x'', x''', the three equations

$$x' + x'' + x''' = A$$
$$x' + \alpha x'' + \alpha^2 x''' = \sqrt[3]{K'}$$
$$x'' + \alpha x' + \alpha^2 x''' = \sqrt[3]{K''}.$$

If we multiply the third equation by α^2, and then add it to the two first, after dividing by 3, we then get, since $1 + \alpha + \alpha^2 = 0$,

$$x' = \frac{A + \sqrt[3]{K'} + \alpha^2\sqrt[3]{K''}}{3},$$

If we multiply the second by α^2, and then add it to the other two, we then get

$$x'' = \frac{A + \alpha^2\sqrt[3]{K'} + \sqrt[3]{K''}}{3}.$$

Lastly, if we multiply the two last equations by α, and then add them to the first, we obtain

$$x''' = \frac{A + \alpha(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}.$$

7. But since each of the two irrational magnitudes $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, has three values, for instance, the first the values $\alpha\sqrt[3]{K'}$, $\alpha^2\sqrt[3]{K'}$, $\alpha^3\sqrt[3]{K'}$, and the second the values $\alpha\sqrt[3]{K''}$, $\alpha^2\sqrt[3]{K''}$, $\alpha^3\sqrt[3]{K''}$, we must first determine which are to be taken. I assert, in the first place, that the two roots $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, must always be combined with one and the same power of α. For let

$$x' + \alpha x'' + \alpha^2 x''' = \alpha^\nu \sqrt[3]{K'}$$

in which the exponent ν may either be one, two, or three. If in this equation we put the roots x' and x'' for one another, we then obtain

$$x'' + \alpha x' + \alpha^2 x''' = \alpha^\nu \sqrt[3]{K''}$$

because in this case K' is transformed into K'', $\therefore$ also $\sqrt[3]{K'}$ into $\sqrt[3]{K''}$. Hence it follows, that in the two last of the three equations, consequently also in the results derived from them, the roots $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, must always be combined with the same power of α. It only remains now to determine the exponent ν.

8. With this view, if we put in the values of x', x'', x''', found in 6, $\alpha^{\nu}\sqrt[3]{K'}$, $\alpha^{\nu}\sqrt[3]{K''}$ for $\sqrt[3]{K'}$, $\sqrt[3]{K''}$, respectively, then

$$x' = \frac{A + \alpha^{\nu}\sqrt[3]{K'} + \alpha^{\nu+2}\sqrt[3]{K''}}{3}$$

$$x'' = \frac{A + \alpha^{\nu+2}\sqrt[3]{K'} + \alpha^{\nu}\sqrt[3]{K''}}{3}$$

$$x''' = \frac{A + \alpha^{\nu+1}(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}$$

Now, since these three roots must also be found, when for α the other primitive root α^2 is substituted, then also

$$x''' = \frac{A + \alpha^{2\nu+2}(\sqrt[3]{K'} + \sqrt[3]{K''})}{3}.$$

must be a root. But since this one is not to be found amongst the three here given, consequently no *other* assumption is allowable, except this one, that $\alpha^{\nu+1} = \alpha^3 = 1$; $\therefore \nu = 2$. Consequently $\alpha^2\sqrt[3]{K'}$, $\alpha^2\sqrt[3]{K''}$, must be substituted for $\sqrt[3]{K'}$, $\sqrt[3]{K''}$. If we actually do this, we find the three following equations:

$$(A + \sqrt[3]{K'} + \sqrt[3]{K''}) : 3$$

$$(A + \alpha\sqrt[3]{K'} + \alpha^2\sqrt[3]{K''}) : 3$$

$$(A + \alpha^2\sqrt[3]{K'} + \alpha\sqrt[3]{K''}) : 3$$

in which it is only necessary for K' and K'' to substitute the two roots of the second equation in 5.

9. If, for the sake of greater simplicity, we put $A = 0$, then this equation is transformed into

$$K^2 - 27CK - 27B^3 = 0$$

and the two values of K are

$$27[\tfrac{1}{2}C \pm \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}].$$

If we substitute these values in the three roots in 3, we obtain

$$\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{4}C^2+\tfrac{1}{27}B^3)}]} + \sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2+\tfrac{1}{27}B^3)}]}$$

$$\alpha\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{4}C^2+\tfrac{1}{27}B^3)}]} + \alpha^2\sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2+\tfrac{1}{27}B^3)}]}$$

$$\alpha^2\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{8}C^2+\tfrac{1}{27}B^3)}]} + \alpha\sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2+\tfrac{1}{27}B^3)}]}$$

which agrees with Cardan's formula.

SECTION CLX.

PROB. Find the functions which are fit for the solution of the general equation of the fourth degree, viz.

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

under the supposition that we only know how to solve equations of lower degrees, and those of the form $t^4 - K = 0$.

Solution 1. Arrange the twenty-four values of forms of f: (1234) in recurring transpositions, under and opposite each other; (the symbolical function and the brackets are omitted for shortness' sake)

1 2 3 4	2 3 4 1	3 4 1 2	4 1 2 3
2 3 1 4	3 1 4 2	1 4 2 3	4 2 3 1
3 1 2 4	1 2 4 3	2 4 3 1	4 3 1 2
2 1 3 4	1 3 4 2	3 4 2 1	4 2 1 3
1 3 2 4	3 2 4 1	2 4 1 3	4 1 3 2
3 2 1 4	2 1 4 3	1 4 3 2	4 3 2 1.

Thus, in the first vertical column, we first put f: (1234) with its recurring transpositions of the three first roots,

this gives the recurring period $f:(1234)$, $f:(2314)$, $f:(3124)$. Then we in like manner put $f:(2134)$ with its recurring transpositions of the three first roots, and we obtain the period $f:(2134)$, $f:(1324)$, $f:(3214)$. From the value of form $f:(1234)$ by a recurring transposition of all the four roots, we further derive the values of forms $f:(2341)$, $f:(3412)$, $f:(4123)$, and place them near $f:(1234)$ in a horizontal row; we do the same with the remaining five values of forms in the first vertical column, so that in each horizontal row there is a recurring period.

2. Since the four values of forms in the first horizontal row form a period, they may ∴ be the roots of an equation of the form

$$t^4 - K = o$$

(§ 155). Now, since $-K$ is the product of all the four roots, we have

$$-K = f:(1234) \times f:(2341) \times f:(3412) \times f:(4123)$$

and this product is such, that in all the transpositions of the roots we can obtain no more than the following six different values:

$$f:(1234) \times f:(2341) \times f:(3412) \times f:(4123)$$
$$f:(2314) \times f:(3142) \times f:(1423) \times f:(4231)$$
$$f:(3124) \times f:(1243) \times f:(2431) \times f:(4312)$$
$$f:(2134) \times f:(1342) \times f:(3421) \times f:(4213)$$
$$f:(1324) \times f:(3241) \times f:(2413) \times f:(4132)$$
$$f:(3214) \times f:(2143) \times f:(1432) \times f:(4321)$$

which arise merely from the transposition of the three roots x', x'', x'''.

3. From the equation $t^4 - K = o$, we obtain $K = t^4 = (f: (1234))^4$. Consequently also $(f: (1234))^4$ must be such a function, that in the recurring transpositions of all the four roots it remains the same, and consequently has no more different values than those which arise from the transposition of the roots x', x'', x'''. Therefore the six values of K can also be expressed thus:

$$(f: (1234))^4,\ (f: (2314))^4,\ (f: (3124))^4$$
$$(f: (2134))^4,\ (f: (1324))^4,\ (f: (3214))^4.$$

4. Since the function K has still six different values, consequently it necessarily depends on an equation of the sixth degree. If this equation be solvible, then it must admit of being reduced to such equations, whose solution is assumed to be known. I shall .·. assume, that the three functions $(f: (1234))^4$, $(f: (2314))$,^{4} $(f: (3124))^4$, which arise from the recurring transposition of the three roots x', x'', x''', are the roots of an equation of the third degree

$$\text{I. } K^3 - pK^2 + qK - r = o$$

consequently the coefficients p, q, r, are no longer rational, because otherwise K can have no more than three values. They must .·. depend on certain equations, which we shall now seek.

5. Since $(f: (1234))^4$, $(f: (2314))^4$, $(f: (3124))^4$, are the roots of the equation I, then

$$p = (f: (1234))^4 + (f: (2314))^4 + (f: (3124))^4$$

$$q = (f:\ (1234))^4 \times (f:\ (2314))^4$$
$$+ (f:\ (1234))^4 \times (f:\ (3124))^4$$
$$+ (f:\ (2314))^4 \times (f:\ (3124))^4$$
$$r = (f:\ (1234))^4 \times (f:\ (2314))^4 \times (f:\ (3124))^4.$$

The functions p, q, r, are evidently such, that in the recurring transpositions of the three roots x', x'', x''' they undergo no change. But in the recurring transpositions of all the four roots $x', x'', x''', x^{\mathrm{v}}$, they in like manner suffer no change, because the functions $(f:\ (1234))^4$, $(f:\ (2314))^4$, $(f:\ (3124))^4$, remain the same after this operation (3).

6. Consequently the functions p, q, r, can have no more than two different values, viz. those which arise merely from the transposition of the roots x', x''. If ∴ we put, for shortness' sake, $p = f' :\ (1234)$, then p has no more than the two values $f' :\ (1234)$, $f' :\ (2134)$. Let these two values be the roots of the following equation of the second degree:

$$p^2 - p'p + q' = 0$$

then

$$p' = f' :\ (1234) \times f' :\ (2134)$$
$$q' = f' :\ (1234) \times f' :\ (2134).$$

The functions p', q', are ∴ such, that when x' is substituted for x'', they remain unchanged. But since in the recurring transpositions of the three roots x', x'', x''', as also in the recurring transpositions of all the four roots, they also suffer no change; they ∴, are necessarily symmetrical, and consequently admit of being expressed rationally by the coefficients of the given equation.

What has been here said of p, may, in like manner, be said of q and r. Consequently these coefficients also depend on equations of the second degree with rational coefficients.

7. The function $t = f: (1234)$.·. depends on the equation of two terms of the fourth degree, viz.

$$t^4 - K = 0$$

and the coefficient K depends again on the equation of the third degree

$$K^3 - pK^2 + qK - r = 0$$

whose coefficients p, q, r, are represented by three equations of the second degree

$$p^2 - p' p + q' = 0$$
$$q^2 - p'_1 q + q'_1 = 0$$
$$r^2 - p'_2 r + q'_2 = 0$$

whose coefficients p', q', p'_1, q'_1, p'_2, q'_2, are all rational functions of the coefficients A, B, C, D.

8. It only remains now to determine the function $f: (1234)$ in such a way, that the values of forms $f: (1234)$, $f: (2341)$, $f: (3412)$, $f: (4123)$ may be the roots of an equation of the form $t^4 - K = 0$. If this be the case, then they must have such a relation to one another, that

$f: (1234) = af: (2341) = a^2 f: (3412) = a^3 f: (4123)$.
Now, in order to perform this, we put in a way similar to that in 7, § CLVIII,

$$f: (1234) =$$
$$A\phi: (1234) + B\phi: (2341) + C\phi: (3412) + D\phi: (4123)$$

and derive from hence the values of $f: (2341)$, $f: (3412)$,

f: (4123). If we substitute now these values in the foregoing proportional equation, we then get

$$\begin{aligned}
&A\varphi: (1234) + B\varphi: (2341) + C\varphi: (3412) + D\varphi: (4123) \\
&= \alpha\left(A\varphi: (2341) + B\varphi: (3412) + C\varphi: (4123) + D\varphi: (1234)\right) \\
&= \alpha^2\left(A\varphi: (3412) + B\varphi: (4123) + C\varphi: (1234) + D\varphi: (2341)\right) \\
&= \alpha^3\left(A\varphi: (4123) + B\varphi: (1234) + C\varphi: (2341) + D\varphi: (3412)\right)
\end{aligned}$$

and when we put the coefficients of these values of forms equal to one another

$$\begin{aligned}
A &= \alpha D = \alpha^2 C = \alpha^3 B \\
B &= \alpha A = \alpha^2 D = \alpha^3 C \\
C &= \alpha B = \alpha^2 A = \alpha^3 D \\
D &= \alpha C = \alpha^2 B = \alpha^3 A.
\end{aligned}$$

The first equation gives $B = \alpha A$, $C = \alpha^2 A$, $D = \alpha^3 A$; and these values verify also the second, third, and fourth equations. We have consequently

$$f: (1234) =$$
$$\phi: (1234) + \alpha\phi: (2341) + \alpha^2\phi: (3412) + \alpha^3\phi: (4123).$$

SECTION CLXI.

Prob. Solve actually the general equation of the fourth degree

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

under the same conditions as those of the foregoing problem.

Solution 1. We have seen in the foregoing §, that all functions of the form $\phi: (1234) + \alpha\phi: (2341) + \alpha^2\phi: (3412) + \alpha^3\phi: (4123)$ are adapted to the solution. If, for the sake of greater perspicuity, we put

$\phi: (1234) = x'$, then

$$f: (1234) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x'^{v}$$

or, when we briefly substitute for α one of the primitive roots of the equation $x^4 - 1 = 0$, say $+\sqrt{-1}$,

$$f: (1234) = x' - x''' + (x'' - x'^{v}) \sqrt{-1}.$$

Hence we obtain

$$f: (2314) = x'' - x' + (x''' - x'^{v}) \sqrt{-1}$$
$$f: (3124) = x''' - x'' + (x' - x'^{v}) \sqrt{-1}.$$

2. By 5 of the foregoing §, we ∴ have

$$p = \left(x'\ x''' + (x''-x'^{v})\sqrt{-1}\right)^4 + \left(x''-x' + (x'''-x'^{v})\sqrt{-1}\right)^4$$
$$+ \left(x'''-x'' + (x'-x'^{v})\sqrt{-1}\right)^4$$

$$q = \left(x'-x''' + (x''-x'^{v})\sqrt{-1}\right)^4 \times \left(x''-x' + (x'''-x'^{v})\sqrt{-1}\right)^4$$
$$+ \left(x'-x''' + (x''-x'^{v})\sqrt{-1}\right)^4 \times \left(x'''-x'' + (x'-x'^{v})\sqrt{-1}\right)^4$$
$$+ \left(x''-x' + (x'''-x'^{v})\sqrt{-1}\right)^4 \times \left(x'''-x'' + (x'-x'^{v})\sqrt{-1}\right)^4$$

$$r = \left(x'-x''' + (x''-x'^{v})\sqrt{-1}\right)^4 \times \left(x''-x' + (x'''-x'^{v})\sqrt{-1}\right)^4$$
$$\times \left(x'''-x'' + (x'-x'^{v})\sqrt{-1}\right)^4$$

The functions p, q, r, are evidently such, that each of them can only have a single value which is different, viz. that which arises from the one here given, when we substitute x'' for x', and, vice versâ, x' for x''. Consequently each of these functions depends on an equation of the second degree, which may be found by the method already well known from the foregoing §. As the subject contains no difficulty, and the calculation is rather diffuse, I shall dwell no longer upon it.

3. In the preceding chapter we have seen, that in two homogeneous functions the value of one may always be determined from the value of the other by a rational

expression, so long as we have to do with general equations. Consequently also the values of q, r may be found immediately from the known value of p. Now, since p has two values, consequently the magnitudes p, q, r may be determined in two different ways. Every such determination gives an equation

$$K^3 - pK^2 + qK - r = 0$$

and we $\therefore$ obtain generally six values of K. If we put $K = f' : (1234)$, then the six values of forms, which correspond to these numerical values, are those which arise from the transposition of the first three roots (§ CLX, 3); $\therefore f' : (1234)$, $f' : (2314)$, $f' : (3124)$, $f' : (2134)$, $f' : (1324)$, $f' : (3214)$, of which the three first correspond to the three roots of one equation for K, and the three last to the three roots of the other.

4. Let K', K'', K''', be the three roots, which correspond to the values of forms $f' : (1234)$, $f' : (2314)$, $f' : (3124)$. If we substitute these values of K in the equation $t^4 - K = 0$, we then obtain for t three values $\sqrt[4]{K'}$, $\sqrt[4]{K''}$, $\sqrt[4]{K'''}$, and to these $\therefore$ the values of forms $f : (1234)$, $f : (2314)$, $f : (3124)$ correspond. Now, since $f : (1234) = x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\text{IV}}$, we then have, including the equation $x' + x'' + x''' + x^{\text{IV}} = A$, the four following equations:

$$\begin{aligned}
x' + x'' + x''' + x^{\text{IV}} &= A \\
x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{\text{IV}} &= \sqrt[4]{K'} \\
x'' + \alpha x''' + \alpha^2 x' + \alpha^3 x^{\text{IV}} &= \sqrt[4]{K''} \\
x''' + \alpha x' + \alpha^2 x'' + \alpha^3 x^{\text{IV}} &= \sqrt[4]{K'''}.
\end{aligned}$$

and the two values of K are

$$27[\tfrac{1}{2}C \pm \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}].$$

If we substitute these values in the three roots in 3, we obtain

$$\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]} + \sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]}$$

$$\alpha\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]} + \alpha^2\sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]}$$

$$\alpha^2\sqrt[3]{[\tfrac{1}{2}C + \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]} + \alpha\sqrt[3]{[\tfrac{1}{2}C - \sqrt{(\tfrac{1}{4}C^2 + \tfrac{1}{27}B^3)}]}$$

which agrees with Cardan's formula.

SECTION CLX.

Prob. Find the functions which are fit for the solution of the general equation of the fourth degree, viz.

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

under the supposition that we only know how to solve equations of lower degrees, and those of the form $t^4 - K = 0$.

Solution 1. Arrange the twenty-four values of forms of f: (1234) in recurring transpositions, under and opposite each other; (the symbolical function and the brackets are omitted for shortness' sake)

1 2 3 4	2 3 4 1	3 4 1 2	4 1 2 3
2 3 1 4	3 1 4 2	1 4 2 3	4 2 3 1
3 1 2 4	1 2 4 3	2 4 3 1	4 3 1 2
2 1 3 4	1 3 4 2	3 4 2 1	4 2 1 3
1 3 2 4	3 2 4 1	2 4 1 3	4 1 3 2
3 2 1 4	2 1 4 3	1 4 3 2	4 3 2 1.

Thus, in the first vertical column, we first put f: (1234) with its recurring transpositions of the three first roots,

found the other roots, if we had given ourselves the trouble to solve the four equations in 4.

SECTION CLXII.

Prob. Find functions, which are fit for the solution of the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

under the supposition, that we know not how to solve any other equation but those of lower degrees, and those of the form $t^5 - K = 0$.

Solution 1. Arrange the 120 values of forms of the function $t = f:\ (12345)$ in recurring periods, as follows: (symbolical functions and brackets are omitted)

1 2 3 4 5	2 3 4 5 1	3 4 5 1 2	4 5 1 2 3	5 1 2 3 4
2 3 1 4 5	3 1 4 5 2	1 4 5 2 3	4 5 2 3 1	5 2 3 1 4
3 1 2 4 5	1 2 4 5 3	2 4 5 3 1	4 5 3 1 2	5 3 1 2 4
2 1 3 4 5	1 3 4 5 2	3 4 5 2 1	4 5 2 1 3	5 2 1 3 4
1 3 2 4 5	3 2 4 5 1	2 4 5 1 3	4 5 1 3 2	5 1 3 2 4
3 2 1 4 5	2 1 4 5 3	1 4 5 3 2	4 5 3 2 1	5 3 2 1 4
2 3 4 1 5	3 4 1 5 2	4 1 5 2 3	1 5 2 3 4	5 2 3 4 1
3 1 4 2 5	1 4 2 5 3	4 2 5 3 1	2 5 3 1 4	5 3 1 4 2
1 2 4 3 5	2 4 3 5 1	4 3 5 1 2	3 5 1 2 4	5 1 2 4 3
....	....	....	....	....
....	....	....	....	....
....	....	....	....	....
4 3 2 1 5	3 2 1 5 4	2 1 5 4 3	1 5 4 3 2	5 4 3 2 1

Thus in the first vertical column we find the 24 transpositions of the four first roots arranged under one another,

in the same way as they were in 1, § CLX. The four following columns contain the recurring transpositions of all the five roots, and in such a way, that in each horizontal row there is a period.

2. According to this arrangement, the 120 values of forms of the function $f:$ (12345) may .·. be generated in the following way. From the two values of forms, $f:$ (12345), $f:$ (21345), which form a period of recurring transpositions of the two first roots, we derive, by recurring transpositions of the three first roots, the six values of forms, $f:$ (12345), $f:$ (23145), $f:$ (31245), $f:$ (21345), $f:$ (13245), $f:$ (32145). From these, by recurring transpositions of the first four roots, we get the 24 values of forms which are contained in the first vertical column in 1; and lastly, from these again, by recurring transpositions of all the five roots, we derive all the 120 values of forms.

3. Let the five values of forms $f:$ (12345), $f:$ (23451), $f:$ (34512), $f:$ (45123), $f:$ (51234), be the roots of the equation of two terms

$$t^5 - K = 0$$

then K is their product, consequently

$$K = f:\ (12345) \times f:\ (23451) \times f:\ (34512)$$
$$\times f:\ (45123) \times f:\ (51234).$$

If, for the sake of brevity, we put $K = f':$ (12345), then $f:$ (12345) is a function such, that in all the recurring transpositions of the five roots, it remains the same, because in each such transposition, one of the five factors, of

which it is composed, merely changes place with another. Consequently all the values of K include 24 times five equal values, and $\therefore$ this function can contain no more than 24 unequal values, and they are those which correspond to the 24 transpositions in the first vertical column in 1, and consequently those which arise exclusively from the transposition of the four roots, x', x'', x''', $x^{\prime V}$.

4. The equation for t, which, taken generally, is of the 120th degree, is consequently already reduced, by the introduction of the function K, to an equation of the 24th degree. Each root of this last equation gives five values of t, viz. $\sqrt[5]{K}$, $\alpha\sqrt[5]{K}$, $\alpha^2\sqrt[5]{K}$, $\alpha^3\sqrt[5]{K}$, $\alpha^4\sqrt[5]{K}$, and $\therefore$ all the 24 roots together give all the 120 values of t.

5. Since $K = t^5$ and $t = f: (12345)$, then also $K = f^5: (12345) = f: (12345)^5$. The 24 roots of the equation for K are $\therefore$ no other than the results of the transpositions of the four first roots in $(f: (12345))^5$. We must now endeavour to reduce this equation.

6. With this view I shall assume, that the four values of forms $f^5: (12345)$, $f^5: (23415)$, $f^5: (34125)$, $f^5: (41235)$, which together constitute a period of recurring transpositions of the four first roots, are the roots of an equation of the fourth degree

$$K - pK^3 + qK^2 - rK + s = 0;$$

then the coefficients p, q, r, s, are symmetrical functions of these four values of forms, and, consequently, in each recurring transposition of the roots x', x'', x''' $x^{\prime V}$, they remain

the same, because in each such transposition, one of these values merely changes place with another. Therefore they can contain no more unequal values than those which arise from the transposition of the roots x', x'', x'''; consequently six values. Therefore the coefficients p, q, r, s, depend on equations of the sixth degree only.

7. Since p, q, r, s, are homogeneous functions, because they all change only when the roots x', x'', x''', are transposed, we are $\therefore$ always enabled, from the known value of one of these coefficients, say p, to find directly the corresponding values of the remaining ones q, r, s, (§ CXLIII). It is consequently quite sufficient to solve the equation for p. Moreover, the six corresponding values of p, q, r, s, give six such equations as those in 6; and since each of these equations gives four values of K, consequently all the six equations together give the 24 values of K.

8. If we put $p = f'' : (12345)$, then $f'' : (12345)$, $f'' : (23145)$, $f'' : (31245)$, $f'' : (21345)$, $f'' : (13245)$, $f'' : (32145)$, are the six unequal values of forms of p, which form two periods of recurring transpositions of the three roots x', x'', x'''. I shall now assume that the three values of forms of the first period $f'' : (12345)$, $f'' : (23145)$, $f'' : (31245)$, are the roots of an equation of the third degree

$$p^3 - p'p^2 + q'p - r' = 0;$$

$\therefore p'$, q', r', are symmetrical functions of these three values, and consequently in each recurring transposition of the three roots x', x'', x''', they remain the same. They

$\therefore$ can have no more than two different values, viz. those which arise merely from the substitution of x' for x''. Besides, if p' be found, then also q', r' may be found immediately, because these three functions are homogeneous.

9. Let $p' = f''' : (12345)$, then $f''' : (12345)$, $f''' : (21345)$ are the only two unequal values of forms of this function. If $\therefore$ we assume, that they are the roots of the equation

$$p'^2 - p'' p' + q'' = 0$$

then p'', q'' are symmetrical functions of the roots x', x'', x''', x^{IV}, x^{V}, and may consequently be expressed rationally by the coefficients A, B, C, D, E, of the given equation.

10. We have now $\therefore$ reduced the equation for t of the 120th degree to the following equations:

$$
\begin{aligned}
&\text{I.} \quad t^3 - K = 0 \\
&\text{II.} \quad K^4 - pK^3 + qK^2 - rK + s = 0 \\
&\text{III.} \quad p^3 - p'p^2 + q'p - r' = 0 \\
&\text{IV.} \quad p'^2 - p''p' + q'' = 0.
\end{aligned}
$$

Having found the equation IV, we obtain from it two values of p'. If we substitute one of these values of p' in the equation III, and for q', r', the corresponding values, we then obtain, by the solution of this equation, three values of p. Lastly, if we substitute one of these values in the equation II, and for q, r, s, their corresponding values, we then obtain four values of K, and from one of these values that of t.

11. We now wish to inquire, how the function t must be constituted, in order that the five values of forms $f:(12345)$, $f:(23451)$, $f:(34512)$, $f:(45123)$, $f:(51234)$, may be the roots of an equation of the form $t^5 - K = o$; for this was the supposition with which we set out. If this condition be fulfilled, we then must have:

$$f:(12345) = \alpha f:(23451) = \alpha^2 f:(34512)$$
$$= \alpha^3 f:(45123) = \alpha^4 f:(51234).$$

Of this kind, however, are all the functions of the form

$$\phi:(12345) + \alpha\phi:(23451) + \alpha^2\phi:(34512)$$
$$+ \alpha^3\phi:(45123) + \alpha^4\phi:(51234).$$

Consequently all functions of this form are fit for the solution of an equation of the fifth degree, under the supposition that we are able, from the known value of this function, to determine the roots x', x'', x''', $x^{\prime v}$, x^v.

12. But I affirm, that this last supposition is always correct, whatever function we may assume for $\phi:(12345)$. For since t, in every recurring transposition of all the roots, changes its value, ∴ it can have at most only twenty-four equal values, viz. those which are in the first vertical column in t. Now, since amongst these values there is not a single one which has x^v in the first place, consequently, these can only at the same time give the roots x', x'', x''', $x^{\prime v}$, but the root x^v would always admit of being determined rationally from t. Therefore, equations of the fifth degree may be solved in an infinite number of ways, and we shall see in the sequel, that this is generally the case with all equations.

13. Since $f' : (12345) = (f : (12345))^5$, and $p = f' : (12345) + f' : (23415) + f' : (34125) + f' : (41235)$, then also, since we have put $p = f'' : (12345)$,

$$f'' : (12345) = (f : (12345))^5 + (f : (23415))^5 + (f : (34125))^5 + (f : (41235))^5.$$

Further, since $p' = f'' : (12345) + f'' : (23145) + f'' : (31245) = f''' : (12345)$, then also, when the requisite transpositions of the roots are made,

$$\begin{aligned} f''' : (12345) = {} & (f : (12345))^5 + (f : (23415))^5 + \\ & (f : (34125))^5 + (f : (41235))^5 + \\ & (f : (23145))^5 + (f : (31425))^5 + \\ & (f : (14235))^5 + (f : (42315))^5 + \\ & (f : (31245))^5 + (f : (12435))^5 + \\ & (f : (24315))^5 + (f : (43125))^5. \end{aligned}$$

If in this we substitute x' for x'', we obtain

$$\begin{aligned} f''' : (21345) = {} & (f : (21345))^5 + (f : (13425))^5 + \\ & (f : (34215))^5 + (f : (42135))^5 + \\ & (f : (13245))^5 + (f : (32415))^5 + \\ & (f : (24135))^5 + (f : (41325))^5 + \\ & (f : (32145))^5 + (f : (21435))^5 + \\ & (f : (14325))^5 + (f : (13425))^5. \end{aligned}$$

Hence we see, that the two functions $f''' : (12345)$ and $f'' : (21345)$, taken together, give all the possible values of $(f : (12345))^5$, which arise from the transposition of the four first roots, consequently all the unequal values of this function. Therefore, since $p'' = f''' : (12345) + f''' : (21345)$, p'' is a symmetrical function of x', x'', x''', x^{iv}, x^{v}, which is obtained immediately from the function $(f : (12345))^5$, when we take the sum of all

the values of this function, which arise from the transposition of the four roots x', x'', x'''; x^{IV}.

14. Further, because $q''=f''':(12345)\times f''':(21345)$, we then immediately obtain the function q'', by taking the product of the above two values for $f''':$ (12345) and $f''':$ (21345). Besides, it is evident, that both in p'' and q'' the root a must vanish, because otherwise t would have more than 120 values.

SECTION CLXIII.

Prob. Solve actually the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

Solution 1. Since for $\phi:$ (12345) in the foregoing §, we can assume any arbitrary function, in order ∴ to simplify the calculation, I shall assume for it the root x, and put $\phi:(12345) = x'$. Then $\phi:(23451) = x''$, $\phi:(34512)=x'''$, $\phi:(45123)=x^{IV}$, $\phi:(51234)=x^{V}$; hence

$$t = f:(12345) = x' + ax'' + a^2x''' + a^3x^{IV} + a^4x^{V}$$

and ∴

$$(f:(12345))^5 = (x' + ax'' + a^2x''' + a^3x^{IV} + a^4x^{V})^5$$

to which expression we can also give, as in § CXL, the form

$$(ax' + a^2x'' + a^3x''' + a^4x^{IV} + a^5x^{V})^5.$$

2. If we solve this expression according to the powers

of α, it has, for the reasons given in 12, § CXXXIX, the following form:

$$\xi' + \xi''\alpha + \xi'''\alpha^2 + \xi^{\prime v}\alpha^3 + \xi^{v}\alpha^4$$

and then

$$\xi' = [5] + 120\,[1^5] +$$

$$+20\left\{\begin{array}{c} x'^3x''x^{v} + x'^3x'''x'^{v} + x'x''^3x''' + x''^3x'^{v}x^{v} + \\ x'x'''^3x^{v} + x''x'''^3x'^{v} + x'x''x'^{v3} + x''x'^{v3}x^{v} \\ + x'x'^{v}x^{v3} + x''x'''x^{v3} \end{array}\right\}$$

$$+30\left\{\begin{array}{c} x'x''^2x^{v2} + x''x'''^2x'^{v2} + x'^2x''x'''^2 + x''x'^{v2}x^{v2} + \\ x'^2x'''x^{v2} + x''^2x'''x'^{v2} + x'^2x''^2x'^{v} + x'''^2x'^{v}x^{v2} \\ + x'^2x'^{v2}x^{v} + x''^2x'''^2x^{v} \end{array}\right\}$$

$$\xi'' =$$

$$5\,(x'^4x'' + x''^4x''' + x'''^4x'^{v} + x'x^{v4} + x'^{v4}x^{v})$$
$$+10\,(x'^3x'^{v2} + x'^2x''^3 + x''^3x^{v2} + x''^2x'^{v3} + x'''^2x^{v3})$$
$$+20\left(\begin{array}{c} x'^3x'''x^{v} + x'x''^3x'^{v} + x'x'''x'^{v3} + x''x'''^3x^{v} \\ + x''x'^{v}x^{v3} \end{array}\right)$$
$$+30\left(\begin{array}{c} x'^2x''^2x^{v} + x'x''^2x'''^2 + x'^2x'^{v}x^{v2} + x''x'''^2x'^{v2} \\ + x'''x'^{v2}x^{v2} \end{array}\right)$$
$$+60\left(\begin{array}{c} x'^2x''x'''x'^{v} + x'x''x'''x^{v2} + x'x''x'^{v2}x^{v} \\ + x'x'''^2x'^{v}x^{v} + x''^2x''x'^{v}x^{v} \end{array}\right)$$

$$\xi''' =$$

$$5\,(x'^4x''' + x''^4x'^{v} + x'x'^{v4} + x'''^4x^{v} + x''x^{v4})$$
$$+10\,(x'^3x''^2 + x''^3x'''^2 + x'^2x^{v3} + x'''^3x'^{v2} + x'^{v3}x^{v2})$$
$$+20\left(\begin{array}{c} x'^3x'^{v}x^{v} + x'x''^3x^{v} + x'x''x'''^3 + x''x'''x'^{v3} \\ + x'''x'^{v}x^{v3} \end{array}\right)$$
$$+30\left(\begin{array}{c} x'^2x''x'^{v2} + x'^2x'''^2x'^{v} + x'x'''^2x^{v2} + x''^2x'''x^{v2} \\ + x''^2x'^{v2}x^{v} \end{array}\right)$$
$$+60\left(\begin{array}{c} x'^2x''x'''x^{v} + x'x''^2x'''x'^{v} + x'x''x'^{v}x^{v2} \\ + x'x'''x'^{v2}x^{v} + x''x'''^2x'^{v}x^{v} \end{array}\right)$$

$$\xi^{\prime v} =$$

$$5\,(x^{\prime 4}x^{\prime v} + x^{\prime}x^{\prime\prime\prime 4} + x^{\prime\prime 4}x^{v} + x^{\prime\prime}x^{\prime v 4} + x^{\prime\prime\prime}x^{v 4})$$
$$+10\,(x^{\prime 2}x^{\prime\prime 3} + x^{\prime 3}x^{v 2} + x^{\prime 2}x^{\prime\prime\prime 3} + x^{\prime\prime 2}x^{\prime v 3} + x^{\prime v 2}x^{v 3})$$
$$+20\left(\begin{array}{c} x^{\prime 3}x^{\prime\prime}x^{\prime\prime\prime} + x^{\prime\prime 3}x^{\prime\prime\prime}x^{\prime v} + x^{\prime}x^{\prime\prime}x^{v 3} + x^{\prime}x^{\prime v 3}x^{v} \\ + x^{\prime\prime\prime 3}x^{\prime v}x^{v} \end{array}\right)$$
$$+30\,(x^{\prime 2}x^{\prime\prime 2}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime v 2} + x^{\prime 2}x^{\prime v}x^{v 2})$$
$$+60\left(\begin{array}{c} x^{\prime 2}x^{\prime\prime}x^{\prime v}x^{v} + x^{\prime 2}x^{\prime\prime\prime}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime\prime\prime}x^{v 2} \\ + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime 2}x^{\prime v} + x^{\prime}x^{\prime\prime\prime}x^{\prime v}x^{v 2} \end{array}\right)$$

$$\xi^{v} =$$

$$5\,(x^{\prime 4}x^{v} + x^{\prime}x^{\prime\prime 4} + x^{\prime\prime}x^{\prime\prime\prime 4} + x^{\prime\prime\prime}x^{\prime v 4} + x^{\prime v}x^{v 4})$$
$$+10\,(x^{\prime 3}x^{\prime\prime 2} + x^{\prime 2} + x^{\prime v 3} + x^{\prime\prime 3}x^{\prime v 2} + x^{\prime\prime 2}x^{v 3} + x^{\prime\prime\prime 3}x^{v 2})$$
$$+20\left(\begin{array}{c} x^{\prime 3}x^{\prime\prime}x^{\prime v} + x^{\prime}x^{\prime\prime\prime 3}x^{\prime v} + x^{\prime\prime 3}x^{\prime\prime\prime}x^{v} + x^{\prime}x^{\prime\prime\prime}x^{v 3} \\ + x^{\prime\prime}x^{\prime v 3}x^{v} \end{array}\right)$$
$$+30\left(\begin{array}{c} x^{\prime 2}x^{\prime\prime}x^{v\ 2} + x^{\prime}x^{\prime\prime 2}x^{v 2} + x^{\prime\prime 2}x^{\prime\prime\prime 2}x^{\prime v} \\ + x^{\prime}x^{\prime v 2}x^{v 2} + x^{\prime\prime\prime 2}x^{\prime v 2}x^{v} \end{array}\right)$$
$$+60\left(\begin{array}{c} x^{\prime 2}x^{\prime\prime\prime}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime 2}x^{\prime v}x^{v} + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime 2}x^{v} \\ + x^{\prime}x^{\prime\prime}x^{\prime\prime\prime}x^{\prime v 2} + x^{\prime\prime}x^{\prime\prime\prime}x^{\prime v}x^{v 2} \end{array}\right)$$

3. Now, if in the expression $\xi^{\prime} + \xi^{\prime\prime}\alpha + \xi^{\prime\prime\prime}\alpha^2 + \xi^{\prime v}\alpha^3 + \xi^{v}\alpha^4$, we perform all the possible transpositions of the roots $x^{\prime}$, $x^{\prime\prime}$, $x^{\prime\prime\prime}$, $x^{\prime v}$, and denote the sum of all the results thus obtained by

$$\zeta^{\prime} + \zeta^{\prime\prime}\alpha + \zeta^{\prime\prime\prime}\alpha^2 + \zeta^{\prime v}\alpha^3 + \zeta^{v}\alpha^4$$

we then find

$$\zeta^{\prime} = 24\,[5] + 8\,.\,20\,[1^2 3] + 8\,.\,30\,[12^2] + 24\,.\,120\,[1^5]$$
$$\zeta^{\prime\prime} = 6\,.\,5\,[14] + 6\,.\,10\,[23] + 4\,.\,20\,[1^2 3] + 4\,.\,30\,[12^2] + 6\,.\,60\,[1^3 2]$$

and for $\zeta^{\prime\prime\prime}$, $\zeta^{\prime v}$, ζ^{v}, the same values as for $\zeta^{\prime\prime}$. Therefore

$$\zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta^{iv}\alpha^3 + \zeta^{v}\alpha^4 =$$
$$\zeta' + \zeta''(\alpha + \alpha^2 + \alpha^3 + \alpha^4) = \zeta' - \zeta'';$$

because $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = [1] - 1 = -1.$

4. But by 13 of the preceding § the coefficient p'' is the sum of all the values of forms of $(f: (12345))^5$, which arise from the transposition of the four first roots; we have ∴ also

$$p'' = \zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta^{iv}\alpha^3 + \zeta^{v}\alpha^4 = \zeta' - \zeta''.$$

Now, if we substitute for ζ', ζ'', their values already found, we then get

$$p'' = 24\,[5] - 30\,[14] - 60\,[23] + 80\,[1^2 3] \\ + 120\,[12^2] - 360\,[1^3 2] + 2880\,[1^5]$$

and when for the numerical expressions we substitute their values taken from the annexed Tables,

$$p'' = 24\,A^5 - 150\,A^3B + 150\,AB^2 + 250\,A^2C \\ - 250\,BC - 1250\,AD + 6250\,E.$$

5. By a method not much different from this, we may also find the coefficient q''. Having, however, found p'' and q'', then the solution of the equation IV in 10 of the foregoing §, gives the value of p'. Having obtained p', then we may also find the coefficients q', r' of the equation III; and the solution of this equation gives the value of p. From the known value of p we may now again find the coefficients q, r, s, of the equation I. But the calculations by this method would be extremely troublesome, and almost impracticable. I shall, in the third part of this collection, show how it may be shortened essentially, and at the same time give the complete

solution of equations of the fifth, sixth, and seventh degrees.

6. Let K', K'', K''', K^{iv}, be the four roots of the equation II, consequently

$$f': (12345) = K' \,, f': (23415) = K''$$
$$f': (34125) = K''', f': (41235) = K^{iv}.$$

If we substitute K', K'', K''', K^{iv}, for K in the equation I, we then get for t four values $\sqrt[5]{K'}$, $\sqrt[6]{K''}$, $\sqrt[5]{K'''}$, $\sqrt[5]{K^{iv}}$, and the values of forms f: (12345), f: (23415), f: (34125), f: (41235) correspond to these values; we ∴ have

$$f: (12345) = \sqrt[5]{K'} \,, f: (23415) = \sqrt[5]{K''}$$
$$f: (34125) = \sqrt[5]{K'''}, f: (41235) = \sqrt[5]{K^{iv}}.$$

If we substitute here for f: (12345), its value $x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{iv} + \alpha^4 x^{v}$, we then get, including the equation $x' + x'' + x''' + x^{iv} + x^{v} = A$, the five following equations:

$$x' + x'' + x''' + x^{iv} + x^{v} = A$$
$$x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{iv} + \alpha^4 x^{v} = \sqrt[5]{K'}$$
$$x'' + \alpha x''' + \alpha^2 x^{iv} + \alpha^3 x' + \alpha^4 x^{v} = \sqrt[5]{K''}$$
$$x''' + \alpha x^{iv} + \alpha^2 x' + \alpha^3 x'' + \alpha^4 x^{v} = \sqrt[5]{K'''}$$
$$x^{iv} + \alpha x' + \alpha^2 x'' + \alpha^5 x''' + \alpha^4 x^{v} = \sqrt[5]{K^{iv}}.$$

7. If we multiply the four last equations by α, and then add them to the first, since $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = [1] = 0$, we get immediately

$$x = \frac{A + \alpha(\sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{\text{IV}}})}{5}$$

and the remaining roots x', x'', x''', x^{IV}, are all of the form $aA + b\sqrt[5]{K'} + c\sqrt[5]{K''} + d\sqrt[5]{K'''} + e\sqrt[5]{K^{\text{IV}}}$ in which a, b, c, d, e, denote certain functions of α. Hence now we may conclude, in a similar way as in equations of the third and fourth degrees, that

$$x = \frac{A + \sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{\text{IV}}}}{5}$$

is a root of the given equation.

8. Having ∴ solved the equation II, we immediately have a root of the given equation, and the remaining roots may be determined from the five equations in 6, by elimination, if, after having performed the calculation, we merely substitute $\alpha^4\sqrt[5]{K'}$, $\alpha^4\sqrt[5]{K''}$, $\alpha^4\sqrt[5]{K'''}$, $\alpha^4\sqrt[5]{K^{\text{IV}}}$ for $\sqrt[5]{K'}$, $\sqrt[5]{K''}$, $\sqrt[5]{K'''}$, $\sqrt[5]{K^{\text{IV}}}$.

SECTION CLXIV.

Prob. Find functions, which are fit for the solution of the general equation of the sixth degree

$$x^6 - Ax^5 + Bx^4 - Cx^3 + Dx^2 - Ex + F = 0$$

under the supposition, that we know not how to solve any other equations than those of lower degrees, and those of the form $t^6 - K = 0$.

Solution 1. With the view of arranging the 1 . 2 . 3 . 4 . 5 . 6 = 720 values of forms of f: (123456) in a re-

curring order, if we put the 120 values of forms in I, § CLXII, in a vertical column under each other, and add to each the root x^{VI} in the last place, we then have 120 values of forms, all of which end with x^{VI}. From each of these, if we derive, by a recurring transposition of all the six roots, five others, we then get 120 periods, each consisting of six values of forms, consequently the 720 values of forms of $f:$ (123456).

2. I shall now assume, that the six values of forms of the first period $f:$ (123456), $f:$ (234561), $f:$ (345612), $f:$ (456123), $f:$ (561234), $f:$ (612345), are the roots of the equation

$$t^6 - K = 0;$$

then $-K$ is the product of all these roots, and hence

$$-K = f:(123456) \times f:(234561) \times f:(345612) \times f:(456123) \times f:(561234) \times f:(612345).$$

This product, however, evidently undergoes no change in each recurring transposition of all the six roots x', x'', x''', x^{IV}, x^{V}, x^{VI}, because in each such transposition, one factor merely changes place with another; consequently the 720 values of forms of K, taken six and six together, are equal. Therefore K can have no more than 120 different values, and these 120 values are no other than those which have x^{VI} in the last place, and which consequently arise merely from the transposition of the five remaining roots.

3. Since $K = t^6$, and $t = f:$ (123456), then also $K = \left(f: (123456)\right)^6$. Consequently the function $\left(f: (123456)\right)^6$ can have no more different values than

those which exclusively arise from the transposition of the roots x', x'', x''', x^{IV}, x^{V}. For shortness' sake, I shall denote them by $f' : (123456)$, and assume, that the five values of forms $f' : (123456)$, $f' : (234516)$, $f' : (345126)$, $f' : (451236)$, $f' : (512346)$, which arise from the recurring transpositions of the five first roots x', x'', x''', x^{IV}, x^{V}, are the roots of the following equation of the fifth degree:

$$K^5 - pK^4 + qK^3 - rK^2 + sK - u = 0,$$

then p, q, r, s, u, are symmetrical functions of the above values of forms, and consequently undergo no change in each recurring transposition of the first five roots. But since they also remain the same in the recurring transpositions of all the six roots, they consequently can contain no more different values, than those which arise exclusively from the transpositions of the roots x', x'', x''', x^{IV}. Therefore each of these functions depends only on an equation of the 24th degree. Since they are homogeneous, it will be sufficient to have determined one of these functions.

4. If for the sake of brevity, we put $p = f'' : (123456)$, then $f'' : (123456)$ can only undergo a change when the four first roots are transposed. I shall now assume, that the four values of forms $f'' : (123456)$, $f'' : (234156)$, $f'' : (341256)$, $f'' : (412356)$, are the roots of the following equation of the fourth degree:

$$p^4 - p'p^3 + q'p^2 - r'p + s' = 0;$$

then the coefficients p', q', r', s', are symmetrical functions of these values of forms, and consequently in each

recurring transposition of the four first roots they remain unchanged; and since they also remain the same in the recurring transpositions of the first five and of all the six roots, consequently, amongst the 720 values of forms, there are no more than six which are different, viz. those which arise exclusively from the transposition of the three first roots x', x'', x'''.

5. I put $p' = f''' : (123456)$, and assume that the three values of forms $f''' : (123456)$, $f''' : (231456)$, $f : (312456)$, are the roots of the following equation of the third degree:

$$p'^3 - p''p'^2 + q''p' - r'' = 0;$$

then p'', q'', r'', are symmetrical functions of these values of forms, and consequently in the recurring transpositions of the three first roots suffer no change; and since they also remain unchanged in the recurring transpositions of the four and five first, likewise of all the six roots, ∴ each of these functions can have no more than two different values, viz. those which arise from the transposition of the two first roots.

6. If ∴ we put $p'' = f^{iv} : (123456)$, and assume that the two values of forms $f^{iv} : (123456)$, $f^{iv} : (213456)$, are the roots of the equation

$$p''^2 - p'''p'' + q''' = 0,$$

then the functions p''', q''' undergo no change in the transposition of the two first, three first, four first, five first, and all the six roots, consequently they are symmetrical, and ∴ may be expressed rationally by means of the coefficients A, B, C, D, E, F, of the given equation.

7. The equation for t, which originally was of the 720th degree, has consequently been reduced by these successive operations to the following equations:

$$\begin{array}{ll} \text{I.} & t^6 - K = 0 \\ \text{II.} & K^5 - pK^4 + qK^3 - rK^2 + sK - u = 0 \\ \text{III.} & p^4 - p'p^3 + q'p^2 - r'p + s' = 0 \\ \text{IV.} & p'^3 - p''p'^2 + q''p' - r'' = 0 \\ \text{V.} & p''^2 - p'''p'' + q''' = 0 \end{array}$$

which are so constituted, that the coefficients of each of them depend on the solution of all the following equations. The equation V gives two values of p'', $\therefore$ the equation IV six values of p', consequently the equation III 24 values of p, and $\therefore$ the equation *II* 120 values of K, consequently the equation I 720 values of t.

8. Therefore, if the function t be such, that the values of forms $f\colon (123456)$, $f\colon (234561)$, $f\colon (345612)$, $f\colon (456123)$, $f\colon (561234)$, $f\colon (612345)$, are the roots of the equation $t^6 - K = 0$, consequently it is always fit for the solution of an equation of the 6th degree. But nothing more will be required to effect this, than that

$$\begin{aligned} &f\colon (123456) = af\colon (234561) = a^2f\colon (345612) \\ &= a^3f\colon (456123) = a^4f\colon (561234) = a^5f\colon (612345) \end{aligned}$$

when a denotes a primitive root of the equation $t^6 - 1 = 0$. But all functions of the form

$$\begin{aligned} &\phi\colon (123456) + a\phi\colon (234561) + a^2\phi\colon (345612) \\ &+ a^3\phi\colon (456123) + a^4\phi\colon (561234) + a^5\phi\colon (612345) \end{aligned}$$

are of this nature. Consequently all functions of this

kind are fit for the solution of equations of the 6th degree. Besides, we need not be apprehensive of not being able to determine the roots of the given equation from these functions; for, since the equal values of forms of $f:$ (123456), if indeed it should have any, can only be found amongst those which have x^{vi} in the last place, consequently x^{vi} cannot be amongst the roots, which correspond to the equal values of $t = f:$ (123456), and $\therefore$, by the foregoing chapter, this root at least must be determined from the known value of t by a rational expression.

SECTION CLXV.

PROB. To solve actually the general equation of the 6th degree in the foregoing §.

Solution 1. If, to facilitate the operation, we put $\phi:$ (123456) $= x'$, then
$f:$ (123456) $= x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{iv} + \alpha^4 x^{v} + \alpha^5 x^{vi}$.
We have $\therefore$

$$f^6: (123456) = (f: (123456))^6$$
$$= (x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{iv} + \alpha^4 x^{v} + \alpha^5 x^{vi})^6 =$$
$$\alpha^{-6} (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{iv} + \alpha^5 x^{v} + \alpha^6 x^{vi})^6$$
$$= (\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{iv} + \alpha^5 x^{v} + \alpha^6 x^{vi})^6.$$

This transformation was performed merely in order to make the dashes over x agree with the powers of α, by which means the solution of the polynomial is rendered more easy.

2. By 2 of the foregoing §, $f^{6}:$ (123456) is the sum

of all the results which are obtained from the recurring transpositions of the five first roots in $f' : (123456)$, consequently also the sum of all the results, which arise from the transposition of the five first roots in $(f : (123456))^6$. Further, by 3 of the foregoing §, $f''' : (123456)$ is the sum of all the recurring transpositions of the four first roots in $f'' : (123456)$, consequently also the sum of all the recurring transpositions of the four and five first roots in $(f : (123456))^6$. By 4, $f^{\text{v}} : (123456)$ is the sum of all the recurring transpositions of the three first roots in $f''' : (123456)$, consequently also the sum of all the recurring transpositions of the three first, four first, and five first roots in $(f : (123456))^6$. Now, since by 5, $p''' = f^{\text{v}} : (123456) + f^{\text{v}} : (213456)$, ∴ also p''' is the sum of all the recurring transpositions of the two first, three first, four first, and five first roots in $(f : (123456))^6$, consequently the sum of all the values, which are obtained from this function by the transposition of the five first roots.

3. In order ∴ to find the coefficient p''', we must first solve the power

$$(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{I\text{v}} + \alpha^5 x^{\text{v}} + \alpha^6 x^{\text{v}I})^6;$$

this solution, since $\alpha^6 = 1$, $\alpha^7 = \alpha$, $\alpha^8 = \alpha^2$, &c. assumes the following form

$$\zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta^{I\text{v}}\alpha^3 + \zeta^{\text{v}}\alpha^4 + \zeta^{\text{v}I}\alpha^5$$

in which ζ', ζ'', ζ''', $\zeta^{I\text{v}}$, ζ^{v}, $\zeta^{\text{v}I}$, are functions of x', x'', x''', $x^{I\text{v}}$, x^{v}, $x^{\text{v}I}$, without α. If we then perform the 120 transpositions of the roots x', x'', x''', $x^{I\text{v}}$, x^{v}, then the sum of all the results thus obtained, gives the coefficient p'''.

4. Further, if we multiply the two functions $f''':$ (123456), $f''':$ (213456) together, we then obtain also the coefficient q'''. The equation V of the foregoing §, will then give the coefficient p''; and having found this, we may directly find q'', r'' by the foregoing chapter. Now, the equation IV gives the coefficient p', ∴ also q', r', s', and lastly the equation III gives the coefficient p, and at the same time also the remaining coefficients of the equation II.

5. Let K', K'', K''', $K^{\prime v}$, K^{v}, denote the five roots of the equation II; then $\sqrt[6]{K'}$, $\sqrt[6]{K''}$, $\sqrt[6]{K'''}$, $\sqrt[6]{K^{\prime v}}$, $\sqrt[6]{K^{v}}$, are the five values of t, and the values of forms $f:$ (123456), $f:$ (234516), $f:$ (345126), $f:$ (451236), $f:$ (512346) correspond to these values; we ∴ have the six equations

$$\begin{aligned}
x' + x'' + x''' + x^{\prime v} + x^{v} + x^{v\prime} &= A \\
x' + ax'' + a^2x''' + a^3x^{\prime v} + a^4x^{v} + a^5x^{v\prime} &= \sqrt[6]{K'} \\
x'' + ax''' + a^2x^{\prime v} + a^3x^{v} + a^4x' + a^5x^{v\prime} &= \sqrt[6]{K''} \\
x''' + ax^{\prime v} + a^2x^{v} + a^3x' + a^4x'' + a^5x^{v\prime} &= \sqrt[6]{K'''} \\
x^{\prime v} + ax^{v} + a^2x' + a^3x'' + a^4x''' + a^5x^{v\prime} &= \sqrt[6]{K^{\prime v}} \\
x^{v} + ax' + a^2x'' + a^3x''' + a^4x^{\prime v} + a^5x^{v\prime} &= \sqrt[6]{K^{v}}.
\end{aligned}$$

6. Hence now, when we multiply the five last equations by a, and add them to the first, we immediately obtain

$$x^{v\prime} = \frac{A + a(\sqrt[6]{K'} + \sqrt[6]{K''} + \sqrt[6]{K'''} + \sqrt[6]{K^{\prime v}} + \sqrt[6]{K^{v}})}{6}$$

and when for the same reasons as in the case of lower equations, we put $\alpha^5\sqrt[6]{K'}, \alpha^5\sqrt[6]{K''}, \alpha^5\sqrt[6]{K'''}, \alpha^5\sqrt[6]{K^{\prime v}}$, $\alpha^5\sqrt[6]{K^v}$, for $\sqrt[6]{K'}$, $\sqrt[6]{K''}$, $\sqrt[6]{K'''}$, $\sqrt[6]{K^{\prime v}}$, $\sqrt[6]{K^v}$, so that α vanish, we get a root of the given equation

$$x = \frac{A+\sqrt[6]{K'}+\sqrt[6]{K''}+\sqrt[6]{K'''}+\sqrt[6]{K^{\prime v}}+\sqrt[6]{K}}{6}$$

and we obtain the remaining ones from the above six equations by elimination.

SECTION CLXVI.

Prob. Solve the general equation of the nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0.$$

Solution 1. Let $t = f: (12345 \ldots\ldots n)$ be a function of such a nature, that the n values of forms, which arise from the recurring transposition of all the n roots, are the roots of the equation

$$t^n - K = 0.$$

Since then K is the product of all these values of forms, it must consequently remain the same in all these transpositions. Therefore its values of forms, $1 . 2 . 3 . 4 \ldots\ldots n$, taken n and n together, are equal. Hence it follows, that K has no more than $1 . 2 . 3 . 4 \ldots\ldots n-1$ different values, and that these values are those which arise exclusively from the transposition of the $n - 1$ first roots.

2. Since $\therefore$ K still depends on an equation of the $1 . 2 . 3 . 4 \ldots\ldots n - 1$th degree, we must consequently

endeavour to reduce this equation. To effect this, I put $K = f' : (12345 \ldots\ldots n)$, and assume that the $n - 1$ values of forms of this function, which arise from a recurring transposition of the $n - 1$ first roots, are the roots of the following equation:

$$K^{n-1} = pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0.$$

Then the coefficients p, q, r, &c. are symmetrical functions of these values of forms, and consequently in the recurring transpositions of the $n - 1$ first roots remain the same. But since they also undergo no change in the recurring transpositions of all the n roots, because by that means $f : (1234 \ldots\ldots n)$ suffers no change, consequently only those of their values are different, which arise from the transposition of the $n - 2$ first roots. Therefore each of these coefficients has only $1 . 2 . 3 . 4 \ldots\ldots n - 2$ different values, and consequently each of them depends on an equation of the $1 . 2 . 3 . 4 \ldots\ldots n-2$th degree only.

3. Since it is quite sufficient to have found p, because p, q, r, &c. are homogeneous functions, I shall put $p = f'' : (12345 \ldots\ldots n)$, and assume, that the $n - 2$ values of forms of this function, which arise from a recurring transposition of the $n - 2$ first roots, are the roots of the equation

$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} + \&c. = 0;$$

then p', q', r', &c. are symmetrical functions of these values, and they $\therefore$ remain the same in the recurring transpositions of the $n - 2$, $n - 1$ first roots, and also of all the n roots. Therefore these coefficients have only $1 . 2 . 3 . 4 \ldots\ldots n-3$ different values, viz. those which

arise from the transposition of the $n-3$ first roots, and they consequently depend only on equations of the $1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n-3$th degree.

4. In a similar way we successively form the equations

$$p'^{\,n-3} - p'' p'^{\,n-4} + q'' p'^{\,n-5} - r'' p'^{\,n-6} + \&c. = 0$$
$$p''^{\,n-4} - p''' p''^{\,n-5} + q''' p''^{\,n-6} - r''' p''^{\,n-7} + \&c. = 0$$
$$p'''^{\,n-5} - p'^{\mathrm{v}} p'''^{\,n-6} + q'^{\mathrm{v}} p'''^{\,n-7} - r'^{\mathrm{v}} p'''^{\,n-8} + \&c. = 0$$
$$\&c.$$

viz. the first from the recurring period of the $n-3$ first roots of the function $p' = f''' : (1234 \ldots\ldots n)$; the second from the recurring period of the $n-4$ first roots of the function $p'' = f'^{\mathrm{v}} : (1234 \ldots\ldots n)$; the third from the recurring period of the $n-5$ first roots of the function $p''' = f^{\mathrm{v}} : (1234 \ldots\ldots n)$; and so on. We continue this operation till we arrive at an equation of the second degree

$$(p^{(n-4)})^2 - p^{(n-3)} p^{(n-4)} + q^{(n-3)} = 0;$$

then $p^{(n-3)}$, $q^{(n-)}$, are such functions of x', x'', x''', ... $x^{(n)}$, as remain the same in the recurring transpositions of all the n roots, also in the recurring transpositions of the $n-1$, $n-2$, $n-3$ first roots, and so on, and likewise of the two first roots. They $\therefore$ undergo no change in all the transpositions of the roots, and consequently they are symmetrical. Therefore they may be expressed rationally by the coefficients of the given equation.

5. We consequently have a series of equations

$$t^n - K = 0$$
$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$
$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} + \&c. = 0$$
$$p'^{n-3} - p''p'^{n-4} + q''p'^{n-5} - r''p'^{n-6} + \&c. = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(p^{(n-4)})^2 - p^{(n-3)}p^{(n-4)} + q^{(n-3)} = 0$$

which are so constituted, that the first coefficient of each depends on all the following ones. Now if the first coefficients p, p', p'', p''', &c. are found, then also the remaining ones q, r, &c. q', r', &c. q'', r'', &c. &c., may be found by the foregoing chapter.

6. It only remains now to assume for $t = f: (12345\ldots\ldots n)$ a function, such, that its values of forms, which arise from the recurring transposition of all the n roots x', x'', x''', &c. may be the roots of an equation of the form $t^n - K = 0$. With this view, we assume any other function $z = \phi: (12345 \ldots\ldots x)$ at pleasure. Let $z', z'', z''', z^{IV}, \ldots\ldots z^{(n)}$ denote the values of forms of z, which arise from the recurring transposition of all the n roots, and α a primitive root of the equation $x^n - 1 = 0$: assert, that then

$$t = z' + \alpha z'' + \alpha^2 z''' + \alpha^3 z^{IV} + \ldots\ldots + \alpha^{n-1} z^{(n)}$$

is always a function of the required property. For since in this function, in each recurring transposition of the roots x', x'', x''', &c., the values of forms z', z'', z''', $z^{IV}, \ldots\ldots z^{(n)}$ are in like manner transposed in a recurring order (for by these means z' changes place with z'', z'' with z''', z''' with z^{IV}, and so on, lastly $z^{(n)}$ with z'), there-

fore the function t, in the recurring transpositions of the roots x', x'', x''', &c. has the following values:

$$t' = z' + az'' + a^2z''' + a^3z^{IV} + \dots\dots\dots + a^{(n-1)}z^{(n)}$$
$$t'' = z'' + az''' + a^2z^{IV} + a^3z^{V} + \dots\dots\dots + a^{(n-1)}z'$$
$$t''' = z''' + az^{IV} + a^2z^{V} + a^3z^{VI} + \dots\dots\dots + a^{(n-1)}z'$$
$$t^{IV} = z^{IV} + az^{V} + a^2z^{VI} + a^3z^{VII} + \dots\dots\dots + a^{(n-1)}z'''$$
&c.

and we immediately see, that $t'' = a^{n-1}t'$, $t''' = a^{n-2}t'$, $t^{IV} = a^{n-3}t'$, and so on. Therefore the functions t', t'', t''', &c. have exactly those relations which they ought to have, in order that they may be the roots of an equation of the form $t^n - K = 0$.

7. That the value of the roots x', x'', x''', &c., may always be determined from the known value of the function $z' + az'' + a^2z''' + a^3z^{IV} + \dots\dots\dots + a^{n-1}z^{(n)}$ let the function z be what it may, appears from this, that the root $x^{(n)}$ never can correspond to the equal values of this function, if it should have any, because these equal values must necessarily be amongst those which arise from the transposition of the $n-1$ roots x', x'', x''', x^{IV}, $\dots\dots\dots x^{(n-1)}$. Consequently this one root at least may always be determined from the known value of t, without solving any equation. But then the remaining ones may also be found, when the solution of equations below the nth degree is pre-supposed.

8. Therefore all equations may be found in numberless ways. If, in order to make the calculation more simple, we put $z = x$, we then have

$$t = x' + ax'' + a^2x''' + a^3x^{IV} + \dots\dots\dots + a^{n-1}x^{(n)}.$$

Hence we immediately obtain

$$K = t^n = (x' + \alpha x'' + \alpha^2 x''' + \ldots\ldots + \alpha^{n-1} x^{(n)})^n$$

9. Now, in order to solve actually the given equation, we must first of all endeavour to determine the coefficients $p^{(n-3)}$, $q^{(n-3)}$ of the last reduced equation in 4. Since p is the sum of the $n-1$ values of forms of K, which arise from the recurring transposition of the $n-1$ first roots, and p' again the sum of all the values of forms of p, which arise from the recurring transposition of the $n-2$ first roots; consequently, also, p' is the sum of the $n-1 . n-2$ values of forms of K, which arise from the recurring transpositions of the $n-1$ and of the $n-2$ first roots. Further, since p'' is the sum of the $n-3$ values of forms of p', which arise from the recurring transposition of the $n-3$ first roots, ∴ also p'' is the sum of the $n-1 . n-2 . n-3$ values of forms of K, which arise from the recurring transpositions of the $n-1$, $n-2$, and $n-3$ first roots. If we proceed further in this way, we then find that the coefficient $p^{(n-3)}$ is the sum of all the $n-1 . n-2 . n-3 \ldots . 3 . 2$ values of forms of K, which arise from the recurring transpositions of the $n-1$, $n-2$, $n-3$ roots, and so on, lastly, of the two first roots, or, which is the same, that $p^{(n-3)}$ is the sum of all the $1 . 2 . 3 . 4 \ldots n-1$ values of forms of K, which arise from all the transpositions of the $n-1$ first roots.

10. In order, therefore, to find the coefficient $p^{(n-3)}$, we must in the first place solve the expression for K in 8

$$(x' + \alpha x'' + \alpha^2 x''' + \alpha^3 x^{IV} + \ldots\ldots + \alpha^{n-2} x^{(n)})$$

according to the powers of α. This solution, since $\alpha^n = 1$, $\alpha^{n+1} = \alpha$, $\alpha^{n+2} = \alpha^2$, &c. will then have the following form:

$$\xi + \xi''\alpha + \xi'''\alpha^2 + \xi^{IV}\alpha^3 + \ldots\ldots + \xi^{(n)}\alpha^{n-1};$$

in which ξ', ξ'', ξ''',......$\xi^{(n)}$ are certain functions of x', x'', x''',......$x^{(n)}$. Now, if in these we transpose the roots x', x'', x''', ... $x^{(n-1)}$ in all possible ways, while $x^{(n)}$ retains its place, and then add the results together, we obtain an expression of the form

$$\zeta' + \zeta''\alpha + \zeta'''\alpha^2 + \zeta' \alpha^3 + \ldots + \zeta^{(n)}\alpha^{(n-1)}$$

which, since it is the value of $p^{(n-3)}$, is necessarily symmetrical with reference to the roots x', x'', x'''......$x^{(n)}$, and $\therefore$ may be expressed rationally by the coefficients A, B, C &c. of the given equation.

11. In order to find the coefficient $q^{(n-3)}$ of the last equation in 5, in the expression $\xi' + \xi''\alpha + \xi'''\alpha^2 + \ldots + \xi^{(n)}\alpha^{(n-1)}$ complete the $n.n-1.n-2$......3 recurring transpositions of the $n-1$, $n-2$, $n-3$, and so on, first roots, exclusive of the two first; then substitute x' for x'', and again make the same transpositions. Now, if we multiply the sum of the $n.n-1.n-2$.........3 first results by the sum of the $n.n-1.n-2$......3 last, we then obtain a symmetrical function of x', x'', x''',...$x^{(n)}$, which will give the value of $q^{(n-3)}$, expressed by the coefficients A, B, C, &c.

12. If we write the reduced equations in 5 backwards,

we have the following series:

$$(p^{(n-4)})^2 - p^{(n-3)}(p^{(n-4)}) \;+ q^{(n-3)} = 0$$
$$(p^{(n-5)})^3 - p^{(n-4)}(p^{(n-5)})^2 + q^{(n-4)}(p^{(n-5)}) - r^{(n-4)} = 0$$
$$(p^{(n-6)})^4 - p^{(n-5)}(p^{(n-6)})^3 + q^{(n-5)}(p^{(n-6)})^2 - r^{(n-5)}(p^{(n-6)})$$
$$+ s^{(n-5)} = 0$$

. . . .

$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$
$$t^n - K = 0.$$

Having found the coefficients $p^{(n-3)}$, $q^{(n-3)}$ by 10 and 11, then the solution of the first equation gives the coefficient $p^{(n-4)}$ of the second equation; and by the foregoing chapter, we may hence immediately determine the values of $q^{(n-4)}$ and $r^{(n-4)}$ by rational expressions, because $p^{(n-4)}$, $q^{(n-4)}$ and $r^{(n-4)}$, are homogeneous functions. The coefficients of the second equation are consequently fully determined, and its solution gives the coefficient $p^{(n-5)}$ of the third equation; $\therefore$ also, as before, the coefficients $q^{(n-5)}$, $r^{(n-5)}$, $s^{(n-5)}$; and the solution of this last equation again gives the coefficient $p^{(n-6)}$ of the following equation. If we proceed further in this way, we at last find the coefficients p, q, r, &c. of the equation for K.

13. Let K', K'', K''', K^{IV}......$K^{(n-1)}$, be the $n-1$ roots of this equation; corresponding to these, as we have assumed in 2, are those values of forms of $K = f' : (1234\ldots n)$, which arise from the recurring transposition of the $n-1$ first roots. Each of these values of K substituted in the equation $t^n - K = 0$, gives n values of t, consequently all together they give $n.n-1$ values of t; and these are:

$$\sqrt[n]{K'},\quad \alpha\sqrt[n]{K'},\quad \alpha^2\sqrt[n]{K'},\ldots\ldots\ldots\alpha^{n-1}\sqrt[n]{K'}$$
$$\sqrt[n]{K''},\quad \alpha\sqrt[n]{K''},\quad \alpha^2\sqrt[n]{K''},\ldots\ldots\ldots\alpha^{n-1}\sqrt[n]{K''}$$
$$\sqrt[n]{K'''},\quad \alpha\sqrt[n]{K'''},\quad \alpha^2\sqrt[n]{K'''},\ldots\ldots\ldots\alpha^{n-1}\sqrt[n]{K'''}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sqrt[n]{K^{(n-1)}},\quad \alpha\sqrt[n]{K^{(n-1)}},\quad \alpha^2\sqrt[n]{K^{(n-1)}}\ldots\alpha^{n-1}\sqrt[n]{K^{(n-1)}}$$

and corresponding to these are the $n.n-1$ values of forms of $t=f:\ (1234\ldots\ldots n)$, which arise from the recurring transposition of all the n and $n-1$ first roots. These numerical values of the function t have such a relation to the values of forms of $f:\ (1234\ldots\ldots n)$, that if we put $f:\ (1234\ldots\ldots n)=\alpha^{\nu}\sqrt[n]{K'}$, the remaining $n-2$ values of forms, which arise from $f:\ (1234\ldots\ldots n)$ by the recurring transpositions of the $n-1$ first roots, correspond to the numerical values $\alpha^{\nu}\sqrt[n]{K''}$, $\alpha^{\nu}\sqrt[n]{K'''}$, $\alpha^{\nu}\sqrt[n]{K^{\prime v}},\ldots\ldots\alpha^{\nu}\sqrt[n]{K^{(n-1)}}$. Now since we have assumed that $t=x'+\alpha x''+\alpha^2x'''+\alpha^3x^{\prime v}+\ldots\ldots+\alpha^{n-1}x^{(n)}$, we have $\therefore$ the following n equations:

$$\begin{aligned}
&x' + x'' + x''' + \ldots + x^{(n-1)} + x^{(n)} = A\\
&x' + \alpha x'' + \alpha^2x''' + \ldots + \alpha^{n-2}x^{(n-1)} + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K'}\\
&x'' + \alpha x''' + \alpha^2x^{\prime v} + \ldots + \alpha^{n-2}x' + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K''}\\
&x''' + \alpha x^{\prime v} + \alpha^2x + \ldots + \alpha^{n-2}x'' + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K'''}\\
&\cdots\cdots\cdots\cdots\cdots\\
&x^{(n-1)} + \alpha x' + \alpha^2x'' + \ldots + \alpha^{n-2}x^{(n-2)} + \alpha^{n-1}x^{(n)} = \alpha^{\nu}\sqrt[n]{K^{(n-1)}}.
\end{aligned}$$

14. If we multiply the $n-1$ last equations by α, and

then add them to the first, we obtain, after dividing by n,

$$x^{(n)} = \frac{A}{n} + \frac{\alpha^{\nu+1}}{n}(\sqrt[n]{K'} + \sqrt[n]{K''} + \sqrt[n]{K'''} + \ldots + \sqrt[n]{K^{(n-1)}})$$

and the remaining roots are all of the form

$$aA + b\sqrt[n]{K'} + c\sqrt[n]{K''} + d\sqrt[n]{K'''} + \ldots + l\sqrt[n]{K^{(n-1)}}$$

in which a, b, c, d, ... l denote certain functions of α. But since these roots must always remain the same, whatever primitive root we substitute for α, $\therefore$ amongst the remaining roots x', x'', x''', ... $x^{(n-1)}$ there must at least be another, which has the form of the root $x^{(n)}$; and since this is not possible, α must quite vanish from the value of $x^{(n)}$ already found, and $\therefore$ $\alpha^{\nu+1} = \alpha n = 1$, consequently $\nu = n - 1$. Therefore

$$x = \frac{A}{n} + \frac{1}{n}(\sqrt[n]{K'} + \sqrt[n]{K''} + \sqrt[n]{K'''} + \ldots + \sqrt[n]{K^{(n-1)}})$$

is a root of the given equation, and the remaining ones may be determined from the equations in 13, when we substitute in them $n - 1$ for ν.

Remark. The solution which I have here given, has only this one fault, that we do not by its means obtain all the roots at once, but only one, and that the remaining ones must afterwards be sought by a very troublesome elimination. I shall $\therefore$ give another solution, which has not this fault, and in other respects also is perhaps preferable to the former one. For the sake of perspicuity I shall begin with an equation of the fifth degree.

SECTION CLXVII.

PROB. Solve the general equation of the fifth degree

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

so as to obtain all the roots at once.

Solution 1. As in the preceding §, let $t^5 - K = 0$ be the equation for the recurring period of all the roots of the function

$$t = x' + ax'' + a^2x''' + a^3x^{IV} + a^4x^{V}$$

then

$$K = t^5 = (x' + ax'' + a^2x''' + a^3x^{IV} + a^4x^{V})^5$$

or also

$$K = (ax' + a^2x'' + a^3x''' + a^4x^{IV} + x^{V})^5$$

and K can have no other unequal values of forms, as we have already seen, but those which arise from the 24 transpositions of the four first roots.

2. Thus far all is the same, as in the foregoing solution. But further, instead of forming the equation

$$K^4 - pK^3 + qK^2 - rK + s = 0$$

as heretofore, from the recurring period of the four first roots, I shall now assume that it has the four *following* values of forms for roots:

$$(ax' + a^2x'' + a^3x''' + a^4x^{IV} + x^{V})^5$$
$$(a^2x' + a^4x'' + ax''' + a^3x^{IV} + x^{V})^5$$
$$(a^3x' + ax'' + a^4x''' + a^2x^{IV} + x^{V})^5$$
$$(a^4x' + a^3x'' + a^2x''' + ax^{IV} + x^{V})^5$$

the three last of which are obtained from the first one, when in it we substitute successively a^2, a^3, a^4 for a.

3. In these four values of K, x^{IV} is always combined with another power of a, and $\therefore$ all the 24 unequal values of K may be derived from these, merely by permuting the roots x', x'', x'''. Thus, if the roots x^{IV}, x^{V}, retain their places, and we merely permute the three first roots, then each of the above four values of forms gives five new ones, and consequently all together give the 24 values of K.

4. Since the coefficients p, q, r, s, are symmetrical functions of the four above-mentioned values of K, $\therefore$ these functions undergo no change, either by the recurring transposition of all the roots, or by the substitution of a for a^2, a^3, a^4; and consequently they can have no more unequal values than those which arise exclusively from the transposition of the three first roots x', x'', x'''. Therefore these functions depend on equations of the sixth degree only.

5. Consequently, if we put $p = f:$ (12345), then $f:$ (12345), $f:$ (23145), $f:$ (31245), $f:$ (21345), $f:$ (13245), $f:$ (32145), are the six values of forms of p. Now if we assume that the three values of forms $f:$ (12345), $f:$ (23145), $f:$ (31245), which arise from the recurring transposition of the three first roots, are given by the equation

$$p^3 - p'p^2 + p'p - r' = 0,$$

then the coefficients p', q', r', are such functions of x', x'', x''', x^{IV}, x^{V}, as can only have the single value, which the substitution of x' for x'' gives. Therefore p' (and the

same obtains of q' and r') has no more unequal values of forms than the two f': (12345), f': (21345). Therefore p' depends on an equation of the second degree only.

6. Let

$$p'^2 - p''p' + q'' = 0$$

be this equation; then p'', q'', are symmetrical functions of x', x'', x''', x^{IV}, x^{V}, and consequently may be expressed rationally by the coefficients A, B, C, D, E of the given equation.

7. We now have the three following equations:

$$K^4 - qK^3 + qK^2 - rK + s = 0$$
$$p^3 - p'p^2 + q'p - r' = 0$$
$$p'^2 - p''p' + r'' = 0.$$

The last gives the value of p', from which, by the foregoing chapter, the coefficients q', r' may be determined. The solution of the second equation again will give the coefficient p; and from this again we may find the coefficients q, r, s. Now, by solving the first equation, we obtain four values of K.

8. Let K', K'' K''', K^{IV}, be these four values, we then have the four following equations (2):

$$(\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + x^{V})^5 = K'$$
$$(\alpha^2 x' + \alpha^4 x'' + \alpha x''' + \alpha^3 x^{IV} + x^{V})^5 = K''$$
$$(\alpha^3 x' + \alpha x'' + \alpha^4 x''' + \alpha^2 x^{IV} + x^{V})^5 = K'''$$
$$(\alpha^4 x' + \alpha^3 x'' + \alpha^2 x''' + \alpha x^{IV} + x^{V})^5 = K^{IV}.$$

If we extract the fifth root from both parts of these

equations, we then have, including the equation $x' + x'' + x''' + x^{IV} + x^{V} = A$,

$$x' + x'' + x''' + x^{IV} + x^{V} = A$$

$$\alpha x' + \alpha^2 x'' + \alpha^3 x''' + \alpha^4 x^{IV} + x^{V} = \sqrt[5]{K'}$$

$$\alpha^2 x' + \alpha^4 x'' + \alpha x''' + \alpha^3 x^{IV} + x^{V} = \sqrt[5]{K''}$$

$$\alpha^3 x' + \alpha x'' + \alpha^4 x''' + \alpha^2 x^{IV} + x^{V} = \sqrt[5]{K'''}$$

$$\alpha^4 x' + \alpha^3 x'' + \alpha^2 x''' + \alpha x^{IV} + x^{V} = \sqrt[5]{K^{IV}}.$$

9. If we add these equations together, we then obtain, since $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = {}'[1] = 0$

$$5x^{V} = A + \sqrt[5]{K'} + \sqrt[5]{K''} + \sqrt[5]{K'''} + \sqrt[5]{K^{IV}}.$$

If we multiply the second by α^4, the third by α^3, the fourth by α^2, the fifth by α, and then add them to the first, we obtain

$$5x' = A + \alpha^4\sqrt[5]{K'} + \alpha^3\sqrt[5]{K''} + \alpha^2\sqrt[5]{K'''} + \alpha\sqrt[5]{K^{IV}}.$$

If we multiply the second by α^3, the third by α, the fourth by α^4, the fifth by α^2, and then add them to the first, we obtain

$$5x'' = A + \alpha^3\sqrt[5]{K'} + \alpha\sqrt[5]{K''} + \alpha^4\sqrt[5]{K'''} + \alpha^2\sqrt[5]{K^{IV}}.$$

If we multiply the second by α^2, the third by α^4, the fourth by α, the fifth by α^3, and then add them to the first, we obtain

$$5x''' = A + \alpha^2\sqrt[5]{K'} + \alpha^4\sqrt[5]{K''} + \alpha\sqrt[5]{K'''} + \alpha^3\sqrt[5]{K^{IV}}.$$

Lastly, if we multiply the second by α, the third by α^2, the fourth by α^3, the fifth by α^4, and then add them to the first, we obtain

$$5x^{IV} = A + \alpha\sqrt[5]{K'} + \alpha^2\sqrt[5]{K''} + \alpha^3\sqrt[5]{K'''} + \alpha^4\sqrt[5]{K^{IV}}.$$

10. If we inspect the values of the roots x', x'', x''', x^{IV}, x^{V}, as they have been here found, we shall immediately observe, that if in any one of the four last, we substitute successively α^2, α^3, α^4, α^5, for α, we always obtain the other four. Therefore, all the roots of the given equation may be comprehended in the following expression:

$$x = \tfrac{1}{5}\left(A + \alpha\sqrt[5]{K'} + \alpha^2\sqrt[5]{K''} + \alpha^3\sqrt[5]{K'''} + \alpha^4\sqrt[5]{K^{IV}}\right)$$

when by α we suppose each root of the equation $x^5 - 1$.

11. I shall only further remark, that in this solution the root α must certainly vanish in the coefficients p, q, r, s. For, since the functions K', K'', K''', K^{IV}, in 8 are such, that in the substitution of α for α^2, α^3, α^4, consequently for the remaining roots β, γ, δ, of the equation $x^5 - 1 = 0$, they merely change places; $\therefore$ the coefficients p, q, r, s, as symmetrical functions of K', K'', K''', K^{IV}, must also remain the same in thiss substitution, and consequently must be symmetrical functions of α, β, γ, δ; $\therefore$, by the fifth chapter, are rational.

SECTION CLXVIII.

Prob. Solve the general equation of the undetermined nth degree

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$$

in such a way, that all the roots may be obtained at once, and yet under the supposition that n is a prime number.

Solution 1. As in § CLXVI, let

$$t = x' + ax'' + a^2x''' \ldots\ldots + a^{n-1}x^{(n)}$$

$\therefore$

$$K = t^n = (x' + ax'' + a^2x''' + \ldots\ldots + a^{n-1}x^{(n)})^n$$

or, which is the same,

$$K = (ax' + a^2x'' + a^3x''' + \ldots + a^{n-1}x^{(n-1)} + x^{(n)})^n.$$

The function K, as we have already seen in the above-named §, is then such, that it remains the same in the recurring transpositions of all the roots x', x'', x''',...$x^{(n)}$, and consequently can have no more unequal values than those which arise from the transposition of the $n-1$ first roots.

2. In order to find the $1.2.3\ldots n-1$ values of forms of the function K, which arise from the transposition of the $n-1$ first roots, we first substitute a^4, a^3, a^2,...a^{n-1} for a; hence there arise the following $n-1$ values:

$$(ax' + a^2x'' + a^3x''' + \ldots\ldots + a^{n-1}x^{(n-1)} + x^{(n)})^n$$
$$(a^2x' + a^4x'' + a^6x''' + \ldots\ldots + a^{n-2}x^{(n-1)} + x^{(n)})^n$$
$$(a^3x' + a^6x'' + a^9x''' + \ldots\ldots + a^{n-3}x^{(n-1)} + x^{(n)})^n$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(a^{n-1}x' + a^{n-2}x'' + a^{n-3}x''' + \ldots\ldots + ax^{(n-1)} + x^{(n)})^n.$$

Since in all these values of forms, because n is a prime number, the same powers of a occur, and in each the root $x^{(n-1)}$ is combined with another power of a; it is $\therefore$ evident, that we obtain all the values of K, when in these $n-1$ values, we permute the roots x', x'', x''',... ...$x^{(n-2)}$ in all possible ways, but let the root $x^{(n-1)}$ retain its place. Each of these values then gives (including the one under consideration) $1 . 2 . 3\ldots\ldots n-2$

values, and consequently all together give all the above $1 . 2 . 3 \ldots n-1$ values of K.

3. Now if we assume that the $n-1$ values in 2 are the roots of the equation

$$K^{n-1} - pK^{n-2} + qK^{n-3} - rK^{n-4} + \&c. = 0$$

consequently, by the fifth chapter, the root α must vanish in the coefficient sp, q, r, &c. and they ∴ remain the same when α is substituted for α^2, α^3, α^4, ... α^{n-1}. Hence, however, it necessarily follows, that these coefficients can have no more unequal values than those which arise from the transposition of the $n-2$ roots x', x'', x''',...$x^{(n-1)}$.

4. Since ∴ the coefficients p, q, r, &c. have no more unequal values than those which arise from the transposition of the $n-2$ first roots; consequently these are similarly circumstanced with the coefficients mentioned in § CLXVI. Thus the equation of the $1 . 2 . 3 \ldots$ $\ldots n-2$th degree, on which the coefficient p depends, may, by the union of those of its values which arise from the recurring transposition of the $n-2$ first roots, be reduced to an equation

$$p^{n-2} - p'p^{n-3} + q'p^{n-4} - r'p^{n-5} \ \&c. = 0$$

whose coefficients p', q', r', &c. only depend now on equations of the $1 . 2 . 3 \ldots n-3$th degree. Further, the equation for p', by uniting its values of forms, which arise from the recurring transposition of the $n-3$ first roots, may be reduced to an equation

$$p'^{n-3} - p''p'^{n-4} + q''p'^{n-5} - r''p'^{n-6} + \&c. = 0$$

whose coefficients p'', q'', r'', &c. only depend on equa-

tions of the $1 . 2 . 3 \ldots n-4$th degree, and so on, till we come to an equation of the second degree.

5. Having determined the values of the coefficients p, q, r, &c. by the solution of all these equations successively, and by means of the foregoing chapter, then the solution of the equation $K^{n-1} - pK^{n-2} + \&c. = 0$, gives $n-1$ values of K, which I shall denote by K', K'', $K''', \ldots\ldots K^{(n-1)}$. The $n-2$ values in 2 correspond to these values; we $\therefore$ have the following $n-1$ equations:

$$(ax' + a^2x'' + \ldots\ldots + a^{n-1}x^{(n-1)} + x^{(n)})^n = K'$$
$$(a^2x' + a^4x'' + \ldots\ldots + a^{n-1}x^{(n-1)} + x^{(n)})^n = K''$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$(a^{n-1}x' + a^{n-2}x'' + \ldots + ax^{(n-1)} + x^{(n)})^n = K^{(n-1)}.$$

6. By extracting the nth root there arise the following equations:

$$x' + x'' + x''' + \ldots\ldots + x^{(n)} = A$$
$$ax' + a^2x'' + a^3x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K'}$$
$$a^2x' + a^4x'' + a^6x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K''}$$
$$a^3x' + a^6x'' + a^9x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K'''}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$a^{n-1}x' + a^{n-2}x'' + a^{n-3}x''' + \ldots\ldots + x^{(n)} = \sqrt[n]{K^{(n-1)}}.$$

If we multiply the second equation by a^{n-1}, the third by a^{n-2}, the fourth by a^{n-4}, and so on, and then add them to the first, we get

$$nx' = A + a^{n-1}\sqrt{K'} + a^{n-2}\sqrt{K''} + \ldots\ldots + a\sqrt{K^{(n-1)}}.$$

If we multiply the second by a^{n-2}, the third by a^{n-4}, the

fourth by α^{n-6}, and so on, and then add them to the first, we get

$$nx'' = A + \alpha^{n-2}\sqrt[n]{K'} + \alpha^{n-4}\sqrt[n]{K''} + \ldots\ldots + \alpha^{2}\sqrt[n]{K^{(n-1)}}.$$

In a similar way, we further find

$$nx''' = A + \alpha^{n-3}\sqrt[n]{K'} + \alpha^{n-6}\sqrt[n]{K''} + \ldots\ldots + \alpha^{3}\sqrt[n]{K^{(n-1)}}$$

$$nx^{iv} = A + \alpha^{n-4}\sqrt[n]{K'} + \alpha^{n-8}K'' + \ldots\ldots + \alpha\sqrt[n]{K^{(n-1)}}$$

&c.

Lastly, if we add all the n equations together, we get

$$nx^{(n)} = A + \sqrt[n]{K'} + \sqrt[n]{K''} + \ldots + \sqrt[n]{K^{(n-1)}}.$$

It is readily seen, that we can derive the values of x'', x''', x^{iv},......$x^{(n)}$ from the value of x', by substituting in it α^2, α^3, α^4,......α^n $(=1)$ successively for α, consequently by substituting for α all the roots of the equation $x^n - 1 = 0$. We ∴ have the following general expression for the roots of the given equation:

$$x = \frac{1}{n}(A + \alpha\sqrt[n]{K'} + \alpha^2\sqrt[n]{K''} + \ldots\ldots + \alpha^{n-1}\sqrt[n]{K^{(n-1)}}.$$

Remark. When n is a compound number, then indeed, for the reasons given in 1, § CXXXVII, this method is not applicable; but for this case, I shall in the Third Part give a particular method, which has this advantage besides, that it is much shorter.

T. C. HANSARD, Paternoster-Row Press.

ERRATA.

The author has in the original designated the sum of the roots of an equation, the sum of their squares, cubes, and in general the sum of their μth powers by the symbols [1], [2], [3]....[μ], and has employed the *crotchet* for this particular purpose, instead of the *parenthesis*. By a mistake in the printing, which was not discovered till it was too late to be corrected, the parenthesis has been used indiscriminately for this and the usual purposes through the earlier part of the volume. This gives rise to a little confusion, the mere mention of which will be sufficient to prevent any obscurity which might otherwise have arisen. See, in particular, pages 5, 6, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 35, 36, 58, 66, 67, 69, 82, 83, 84, 85, 86, 87, 133, 134, where it will be most material to attend to this distinction.

Page 5, *line* 6, *for* function, *read* symmetrical function.

—— 11, —— 3, 4, *for* of the mth order, *read* taken m and m.

—— 13, —— 15, *for* letters, *read* roots.

——, —— 18 - - - ditto.

—— 21, —— 6, *for* $=(\beta-\alpha)$ *read* $-(\beta-\alpha)$.

—— 22, —— 23, *for* evolution, *read* development.

—— 41, —— 6, 7, *for* odd or even, *read* even or odd.

—— 49, —— 16, *for* number, *read* number of divisions.

——, —— 20, *for* numbers of a different kind, *read* numbers of different kinds of things.

—— 57, —— 18, The product in this line should be written thus: $(\mp m)^{\mu'}\times(\mp m')^{\mu'}\times(\mp m'')^{\mu''}$.

3 D

Page 59, *line* 9, 10, *for* in each transformation are changed, *read* when transformed are unchanged.

— 61, — 11, *for* the, *read* this.

— 68, — 14, *for* t^3, *read* x^3.

— 99, — 13, *for* B_{α}, *read* B_x.

— 106, — 8, 9, *for* § 41 and 42, *read* § 40 and 41.

— 109, — 1, 2, *for* when the roots are transformed and permuted, *read* by the substitution and permutation of the roots.

— 113, — 1, *dele* them.

— 130, — 23, *for* quite as, *read* as little.

— 132, — 6, *for* other, *read* others.

— 133, — 5, The coefficient of the second term should be x'.

— 135, — 12, 13, *for* transformation, *read* substitution.

— 143, — 23, *for* merely, *read* all.

— 144, — 4, *for* other, *read* higher.

— 149, — 3, The letters A', B', C', D', ought to have been old English capitals.

— 152, — 3, 4, The letters A', B', C', D', E', F' should have been old English capitals.

— 209, — 10, *for* referred, *read* in reference.

— 226, — 4, *dele* try to.

HIRSCH'S GEOMETRY;

OR

A SEQUEL

TO

EUCLID.

TRANSLATED FROM THE GERMAN

BY

THE REV. J. A. ROSS, A.M.

EDITED

BY

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AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS, &c. &c.

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INTRODUCTORY PREFACE

BY

THE EDITOR

FROM the encouragement bestowed upon our former labours—the Translations of Hirsch's Integral Tables, and of his First and Second Treatises upon Algebra—we are induced to lay before the Public another of the works of that most indefatigable and useful Author. Hirsch's Geometry, like his other productions, is vastly superior in every respect to any thing of the kind extant in our own language, and consequently will prove acceptable to every English Mathematician who shall look into it. In this country, Mathematicians who have ventured beyond the Elements of Euclid have, almost without exception, confined their speculations to "Deductions from Euclid," as the phrase goes, not daring, it would seem, to extend the Theories of that Father of Geometry. But Hirsch, being restrained by no such high feelings for antiquity, handles the subject in its generality, and strikes out a series of propositions, at once connected, and founded on views the most practical and beneficial. In all his works Hirsch is as elegant as he is useful, and perhaps none demands that eulogium more irresistibly than his Geometry. It is, indeed, both calculated to form the taste and direct the judg-

ment of the Mathematicians above alluded to; and indeed of almost all Geometers, without exception—at least in Britain.

Having thus spoken of the general merits of the work, we leave it to the reader to bear us out in particulars. We feel confident that all perusers will arrive at the same conclusion —viz. that *for those who have read Euclid, Hirsch's Geometry is by far the most useful and elegant production to be met with.*

Once for all, we must here explain a few difficulties. In page 29, Ex. 1. &c. it appears that the German measures are different from ours, not only in magnitude but in principle. They denote a certain measure of *Length*, viz. 1°, which means one measure of that kind; therefore, 325° means 325 of such lengths. Then the parts of these measures they subdivide into 10ths, 100ths, 1000ths, &c., or *decimally*, in such a way that

$$325^{\circ}.\ 7'.\ 9''.\ 6'''.\ 5^{\text{iv}}.\ \&c.$$

means

$$325{\cdot}7965,\ \&c.$$

of the lengths denoted by the *superior* $^{\circ}$.

In the first example, p. 29, for instance,

$$\frac{325{\cdot}79 \times 67{\cdot}83}{2} = 11049{\cdot}16785$$

which result is so many squares of which the side is 1°, and which is therefore represented symbolically by

$$11049{\cdot}16785\ \square^{\circ}.$$

This again, by similar notation, with respect to the decimal parts, is the same as

$$11049\ \square^{\circ}\ 16\ \square'\ 78\ \square''\ 50\ \square''',$$

for the squares evidently are formed by successive 100ths.

Care must be taken not to mistake these measures of length with angular measures having the same notation.

Thus in p. 32 we have for the area of a triangle

$$q = \frac{ab \ Sin. \ \alpha}{2}$$

and for a particular example in this same page

$$q = \frac{257'. \ 9'' \times 356'. \ 3'' \times Sin. \ 25^{\circ}. \ 13'}{2}$$

Here the angular minutes are different from the minutes of length, being 60th parts of a degree, whereas the minutes of lengths are only 10ths of the degree of length.

It may be supposed that error may hence arise; but that will not be the case, since all functions of angles, such as *Sin. Cos. Tan. Sec.* &c. are numbers perfectly *pure or abstract*. Thus $Sin. \ 45^{\circ} = \frac{1}{\sqrt{2}}$, $Sin. \ 30^{\circ} = \frac{1}{2}$, $Tan \ 45^{\circ} = 1$, and are therefore quite independent of all denominations of quantities, whether of length, weight, or any other kind.

Sometimes our author expresses his lengths wholly in minutes, sometimes wholly in degrees, which, of course, he is at liberty to do: in short the only thing for the student to bear in mind is the above distinction between lengths and angles. The necessity of this is very apparent in the examples down, p. 96.

J. M. F. W.

London, Oct. 1, 1827.

THE AUTHOR'S PREFACE.

I HERE give the public the Continuation announced in my Collection of Algebraical Problems and Formulæ, viz. the First Part of the Geometrical Collection, under the supposition that the First Part, at least, will not be received unfavourably. Were it not for the kind, and to me highly flattering, opinion of some persons I greatly value, I could not, for want of a critical judgment, presume to make this supposition, and to entertain the hope, that my labours, though they may not have entirely answered their expectations, yet have not altogether disappointed them. If, by omitting all that which is not necessary to the connection of the subject, brevity is an important requisite of a good mathematical book, since, agreeably to its design, it ought only to contain the fundamental *principles* and chief rules of the science; so it is equally necessary for the beginner to possess, also, another book which may afford him an opportunity of applying practically the rules he has already learnt, and by exercising himself in many different ways, prepare himself the better for reading a more comprehensive work. A book of this kind must, as it were, be intermediate between the first principles and such works as can be comprehended only by the more advanced, in order that the chasm between both may be filled up Thus, profound treatises on single subjects

are not so much required, because by their means no beginner can be grounded in those Elementary studies which they generally pre-suppose. Elementary treatises, interrupted here and there by occasional remarks, having a decided reference to practical application, appeared to me more desirable; by these the way is prepared to the former, and the foundation is laid for the future progress of the beginner.

That the geometrical object, if I may so express myself, requires nothing less than a facility of calculation in practice, every person will readily grant. This object, by which the experienced Geometrician as speedily sees the proportions and relations of the different parts of a figure, as the experienced calculator perceives the proportions and relations between numbers, can only be atained by a difference in the perspicuity and variety in the treatment. Geometry and Trigonometry, or, as the latter ought more properly to be called, Goniometry, furnishes us with ample materials for this purpose; arrangement and connection are only requisite to render them complete. Diffuse and *difficult* trigonometrical calculations, which only try the patience and not the head of the reader, cannot be absolutely considered as a means of improving the young geometrician; they are only so in reference to a higher object, and therefore only allowable in this view. It is calculations of this kind which have generally brought upon the inordinate admirers of the ancients, the unjust imputation, that their calculus does not tend to the improvement of the understanding, and that it lowers the sublime science of Geometry merely to mechanical purposes. But the entire separation of the geometrical method, so called, from the algebraical, has a favourable influence on the mind, since it accustoms it only to a certain form of thinking.

How far the present Collection fulfils all these negative conditions as it were, and whether the materials here made use of answer the above end, I cannot presume to decide. It only remained for me, therefore, in the above observations, to give the point of view in which I wish this treatise, on the whole, to be judged.

Respecting the problems themselves, they are partly taken from pure, partly from practical, Geometry; and, in the latter all technical terms are avoided both in the mode of treatment and expression. As in the algebraical collection, I have here strictly adhered to the fundamental rule, of proceeding from easy and simple examples, to more difficult and complicated ones, always mindful of the class of readers for whom I wrote. The treatises which I have made use of are mostly quoted, except those which, to avoid the imputation of plagiarism, I beg to mention in this Preface, viz. Schultz's Pocket Companion of Geometry; Schwab's Collection of 30 Geometrical Problems, as an Appendix to Euclid's Data; T. Simpson's Select Exercises for young Proficients in the Mathematics; Klügel's Mathematical Dictionary; also Mayer's Practical Geometry, which is not quoted in every place where it has been made use of. What is original will be readily perceived by the proficient.

The examples, of which there is an abundance, with the exception of two or three of the easiest, are all new; they are only omitted in the miscellaneous problems, because I no longer considered them necessary. In many of them, the final results only, together with a short notice of the mode of treatment, are given; in others, which involve long and complicated calculations, the intermediate results are also

adduced. This was done in order to save room; whether I have acted wisely in this respect, I shall, I trust, learn in time. However, I cannot be accused of doing it for the sake of any personal advantage or convenience. Should some of my readers not possess the information required for these cases, they must consult the tutor, or some proficient in the subject.

This First Part of the Geometrical Collection contains, besides, only Planimetrical problems, or such as may be classed under them, the Stereometrical are reserved for the Second Part. Much has been left out here rather unwillingly; to this belongs partly the doctrine of polygons in circles, which subject, since Gauss found the formula for the septemdecagon, appears to have excited some interest. The only excuse which I can offer for this omission is, that I had nothing new to say on the subject; however, my readers shall in due time be compensated for this.

The annexed Trigonometrical Tables of Formulæ contain such formulæ only as are used in the book. In quoting the Elements of Geometry, I have throughout referred to Lorenzini's Translation of Euclid, which probably is in the possession of most of my readers; the reference to it is denoted by Euc.; thus Euc. III. 27, implies the 27th Proposition of the Third Book of the Elements.

Berlin, Jan. 15, 1805.

CORRECTIONS.

Page 4, *line* 31, for $\triangle$ *ACD*, read $\triangle$ *AbD*
16, Head line, for *Algebra*, read *Analysis*
17, *line* 25, for *from I*, read *from T*
18, 5, for *CD*, read *AD*
32, 6, for *triangles*, read *angles*
32, last but one, for *Algebraically*, read *Geometrically*
40, 15, for *triangle*, read *quadrilateral*
47, 19, for *DE*, read *DF*
49, 18, for *BA*, read *BH*
56, 3, for $p = \infty$, read $x = \infty$
60, 2, for $e^2 = c^2 =$, read $e^2 = c^2 +$
62, 10, for *measured, from*, read *measured, in*
67, 20, for *radius*, read *diameter*
74, 2, for *Sin.* $(\gamma - \beta)$, read *Sin.* $(\gamma - \alpha)$
96, last but five, for *segment*, read *sector*
— last line, for *chord*, read *length*
97, *lines* 2, 3, 5, 10, 11, 12, 15, 19, 21, 26, for *segment*, read *sector*
98, 2, 14, 24, 26, for *segment*, read *sector*
99, 2, 10, 12, 15, 18, 20, 22, 24, 25, 27, 29, for *segment*, read *sector*
100, 3, 14, 17, 19, 21, 22, for *segment*, read *sector*
—— 14, 20, for *section*, read *segment*
103, *line* 17, for ϕ read *Sin.* ϕ
106, 28, 30, 33, for *segment*, read *sector*
107, 5, dele *curvilinear* $ACA' =$
110, 4, for a *Cos.* ϕ, read a *Cos.* α
—— 6, for *C*, read *D*

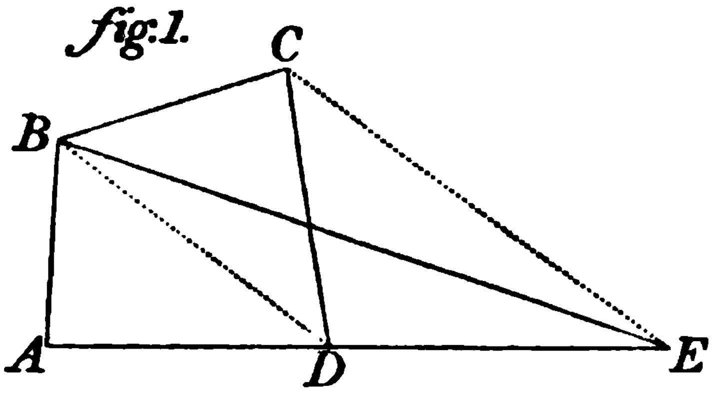
TRANSFORMATION.

I. TRANSFORMATION OF FIGURES.

SECTION I.

PROBLEM. *To transform a given quadrilateral figure into a triangle, whose vertex is in a given angle of the figure, and whose base is in one of the sides of the figure.*

CONSTRUCTION. Let $ABCD$ (*fig.* 1 & 2), be the given quadrilateral; the *fig.* 1. has all its angles outwards, and the



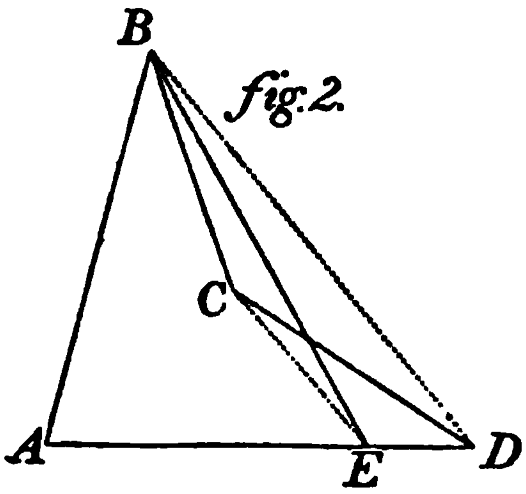


fig. 2 has one angle BCD inwards; let the vertex of the triangle, which is equal to it, fall in B.

1. Draw the diagonal BD (*fig.* 1 & 2), and, parallel to it, the line CE from C.

2. From E, where this line cuts AD (*fig.* 2), or its production (*fig.* 1), draw the line EB; then the $\triangle\, ABE =$ the quadrilateral figure $ABCD$.

DEMONSTRATION. Since $CE \parallel BD$, $\therefore$ (*fig.* 1 & 2) $\triangle BCD = \triangle BED$; consequently (*fig.* 1)

$$\triangle\, ABD + \triangle\, BCD = \triangle\, ABD + \triangle\, BED$$

or, the quadrilateral $ABCD = \triangle\, ABE$,

and (*fig.* 2)

$$\triangle ABD - \triangle BCD = \triangle ABD - \triangle BED$$

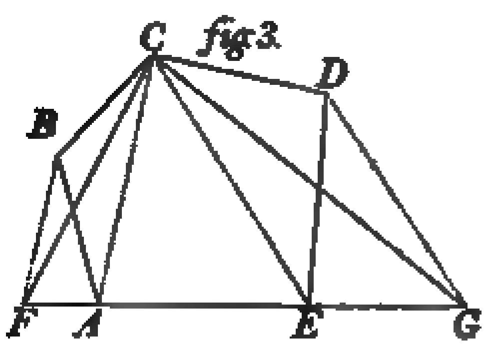
or, quadrilateral $ABCD = \triangle ABE$.

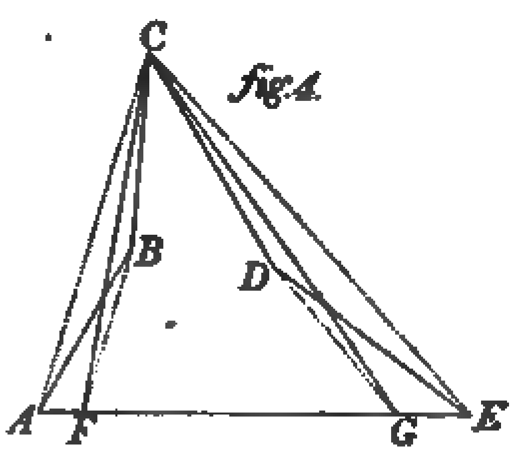
Corollary. If in the quadrilateral (*fig.* 1.), the side BC is parallel to its opposite side AD, then $BCED$ is a parallelogram; $\therefore BC = DE$, and consequently $AE = AD + DE = AD + BC$. In this case, the base AE of the triangle ABE is the sum of the two parallel sides. If $ABCD$ is a parallelogram, then $AD = BC = DE$, and $\therefore AE = 2\,AD$.

SECTION II.

Prob. *Transform a given pentagon into a triangle, whose vertex is in a given angle of the pentagon, and whose base is in one of its sides.*

Const. Let $ABCDE$ (*figs.* 3, 4, 5) be the given





pentagon; let the vertex of the triangle which is equal to it fall in C.

1. From C draw the diagonals CA, CE.

2. From B draw BF parallel to CA, and from D draw DG parallel to CE.

3. To F and G, where these parallels cut AE or its production, draw the lines CF, CG; then CFG will be the triangle sought.

Demon. In all the three figures,

$$\triangle\, CBA = \triangle\, CFA,\ \triangle\, CDE = \triangle\, CGE.$$

1. (*fig.* 3)

$\triangle CAE + \triangle CBA + \triangle CDE = \triangle CAE + \triangle CFA + \triangle CGE$;
or since $\triangle CAE + \triangle CBA + \triangle CDE =$ pentagon $ABCDE$,
and $\triangle CAE + \triangle CFA + \triangle CGE = \triangle CFG$; $\therefore$
the pentagon $ABCDE = \triangle CFG$.

2. (*fig.* 4)

$\triangle CAE - \triangle CBA - \triangle CDE = \triangle CAE - \triangle CFA - \triangle CGE$;
or since $\triangle CAE - \triangle CBA - \triangle CDE =$ pentagon $ABCDE$,
and $\triangle\, CAE - \triangle\, CFA - \triangle\, CGE = \triangle\, CFG$; $\therefore$
the pentagon $ABCDE = \triangle\, CFG$.

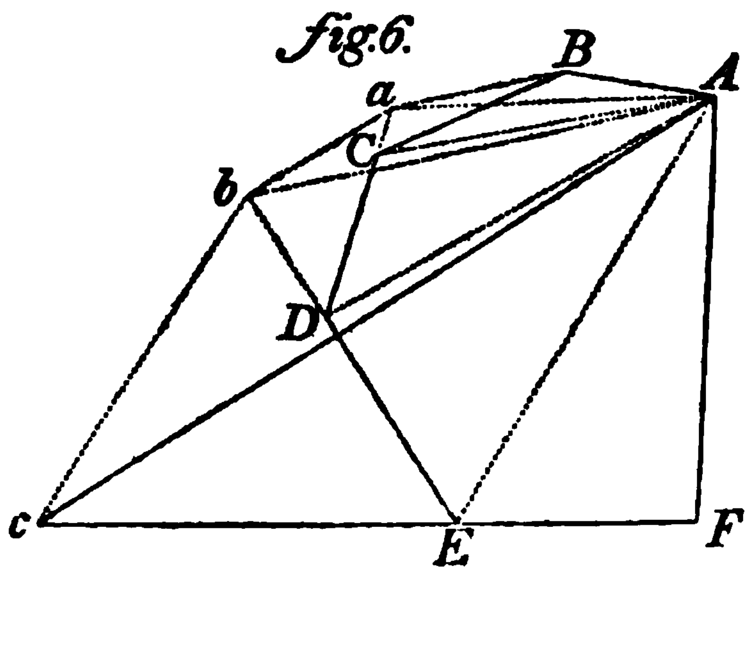
3. (*fig.* 5)

$\triangle CAE - \triangle CBA + \triangle CDE = \triangle CAE - \triangle CFA + \triangle CGE$;
or since $\triangle CAE - \triangle CBA + \triangle CDE =$ pentagon $ABCDE$,
and $\triangle CAE - \triangle CFA + \triangle CGE = \triangle CFG$; $\therefore$
the pentagon $ABCDE = \triangle\, CFG$.

SECTION III.

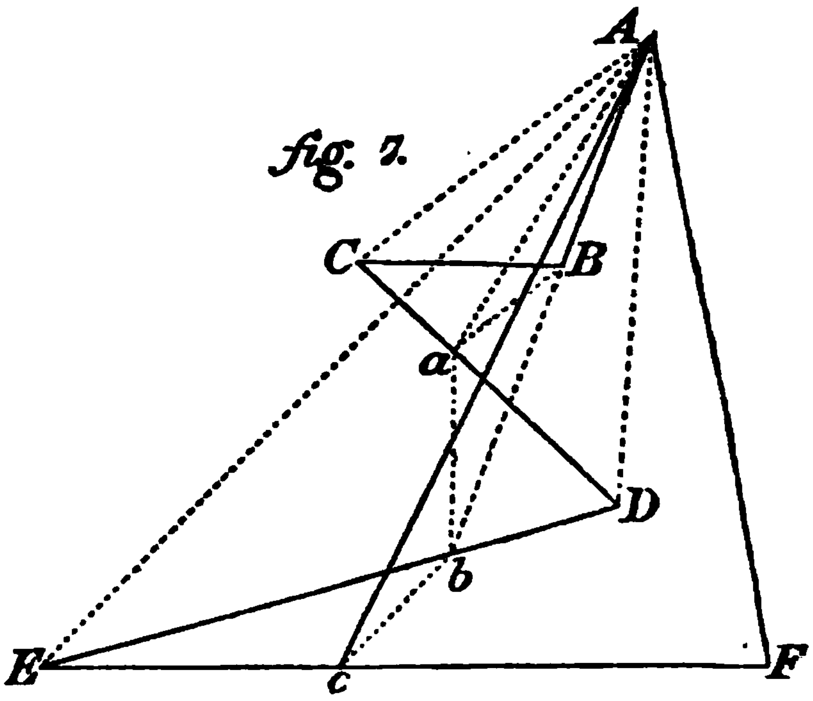
Prob. *To convert any given figure into a triangle, whose vertex shall be in a given angle of the figure, and whose base is in one of its sides.*

Const. Let $ABCDEF$ (*fig.* 6 & 7) be the given figure, in this case a hexagon, and A the angle in which the vertex of the given triangle is situated. For the sake of perspicuity, I shall enumerate the angles and sides of the figure from A, and call the first angle A, the second B, the third C, the fourth D, and so on; further,



AB the first side, *BC* the second, *CD* the third, *DE* the fourth, and so on. We shall then have the following general solution:

fig. 7.



1. From *A* to all the angles of the figure, draw the diagonals *AC*, *AD*, *AE*, which, according to the order in which they stand here, call the first, second, third diagonal.

2. Then draw from the second angle *B* a line parallel to the first diagonal *AC*; from the point *a*, where this parallel meets the third side *CD* (*fig.* 7), or its production (*fig.* 6), draw a line *ab* parallel to the second diagonal *AD*, and from the point *b*, where this meets the fourth side *DE* (*fig.* 7), or its production (*fig.* 6), draw another line *bc* parallel to the third diagonal *AE*.

3. Continue this till there are no more diagonals, and from the last point of section of the parallels and sides, (in this case *c*), draw the line *cA*, then *AcF* is the required triangle, whose vertex is in *A*, and whose base is in the side *EF*.

Demon.

1. (*fig.* 6). $\triangle\ ABC = \triangle\ AaC$ (because $Ba \parallel AC$); consequently

$$ACDEF + \triangle\ ABC = ACDEF + \triangle\ AaC;$$

or the hexagon $ABCDEF$ = pentagon $AaDEF$.

2. $\triangle AaD = \triangle ACD$ (because $ab \parallel AD$); consequently

$$ADEF + \triangle\ AaD = ADEF + \triangle\ AbD;$$

or the pentagon $AaDEF$ = quadrilateral $AbEF$.

3. $\triangle\ AbE = \triangle\ AcE$ (because $bc \parallel AE$); consequently

$$\triangle\ AEF + \triangle\ AbE = \triangle\ AEF + \triangle\ AcE;$$

or the quadrilateral $AbEF = \triangle\ AcF$.

1. (*fig.* 7). $\triangle ABC = \triangle AaC$ (because $Ba \parallel AC$); consequently

$$ACDEF - \triangle ABC = ACDEF - \triangle AaC;$$

or the hexagon $ABCDEF =$ the pentagon $AaDEF$.

2. $\triangle AaD = \triangle AbD$ (because $ab \parallel AD$); consequently

$$ADEF + \triangle AaD = ADEF + \triangle AbD;$$

or the pentagon $AaDEF =$ quadrilateral $ACEF$.

3. $\triangle AbE = \triangle AcE$ (because $bc \parallel AE$); consequently

$$\triangle AEF - \triangle AbE = \triangle AEF - \triangle AcE,$$

or the quadrilateral $AbEF = \triangle AcF$.

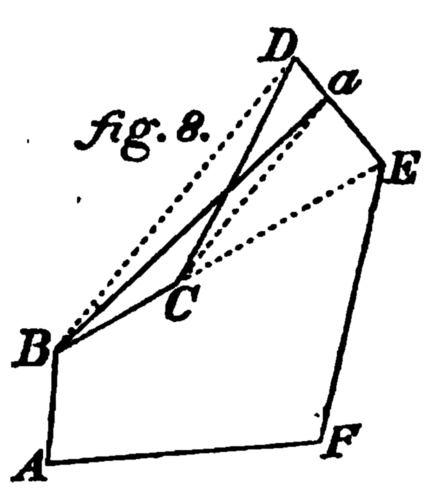
We ∴ have for both figures, the hexagon $ABCDEF =$ pentagon $AaDEF =$ quadrilateral $AbEF = \triangle AcF$.

First Remark. Although the solution here given is only intended for a hexagon, yet it may easily be applied to every other figure. All depends upon the substitution of one triangle for another by means of parallel lines, in which we have only to take care, that one side of the triangle substituted, be in one side of the figure, or in its production, because by these means the number of its sides will be diminished. Moreover, it is not absolutely necessary actually to draw the parallels; it is only requisite, for instance, to note the points at which they cut its sides or their productions, because all depends upon the determination of these points of section.

Second Remark. A cursory inspection of the constructions and demonstrations in §§ I, II, and III, will show a certain agreement between the figures with angles tending outwards and inwards, by means of which the constructions and demonstrations may be transferred almost literally from one figure to the other. The difference consists merely in the signs + and −, or in addition and subtraction.

Cor. 1. In the figure $ABCDEF$ (*fig.* 8) it happens, that when we assume E as the vertex of the required triangle, the side BC of the figure is in a straight line with the diagonal EC. If we wish to proceed according to the above directions, we must through D draw a line parallel to EC, which, however, would never meet BC, as is required for the further construction. But this circumstance may be easily remedied, by taking away, previously to the application of the given rules, the angle BCD which tends inwards. Thus if

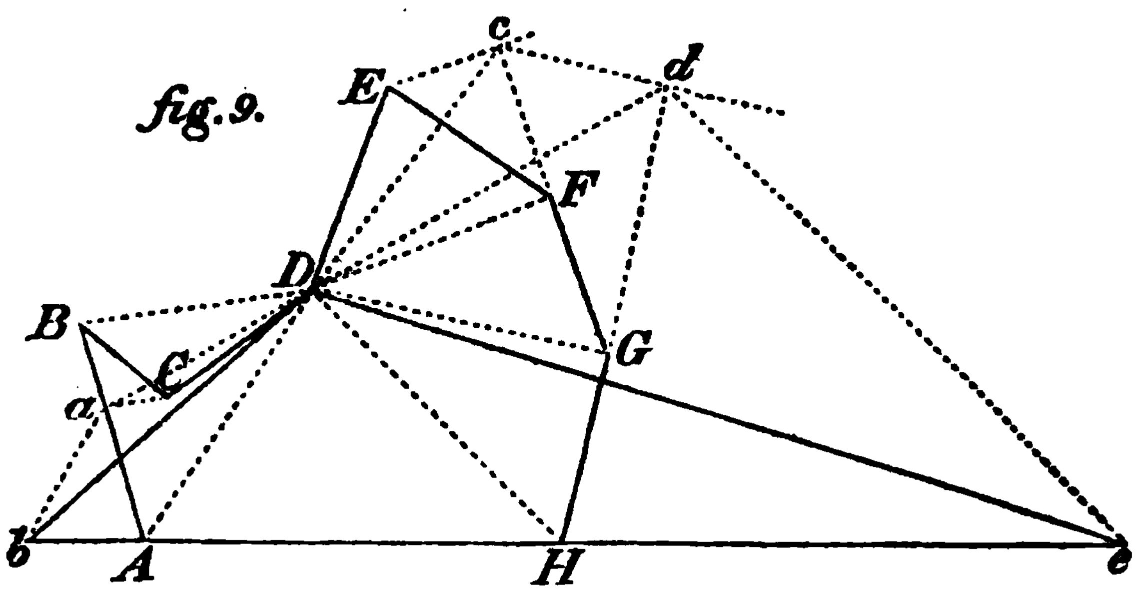
fig. 8.



we draw the diagonal BD, and Ca parallel to it, then, when Ba is drawn, $\triangle CBa = \triangle CDa$, and $\therefore$ the figure $ABaEF =$ figure $ABCDEF$. This taking away of the angle which inclines inwards, may also be made use of with advantage, in the case where the point of section of the parallels and sides falls far without the figure, and in which the construction would consequently occupy too much room.

COR. 2. If the vertex of the triangle is not only situated in a given angle of the figure, but the base of the triangle is in a certain side of the figure, as, for instance, when the octagon $ABCDEFGH$ is transformed (*fig.* 9) into

fig. 9.

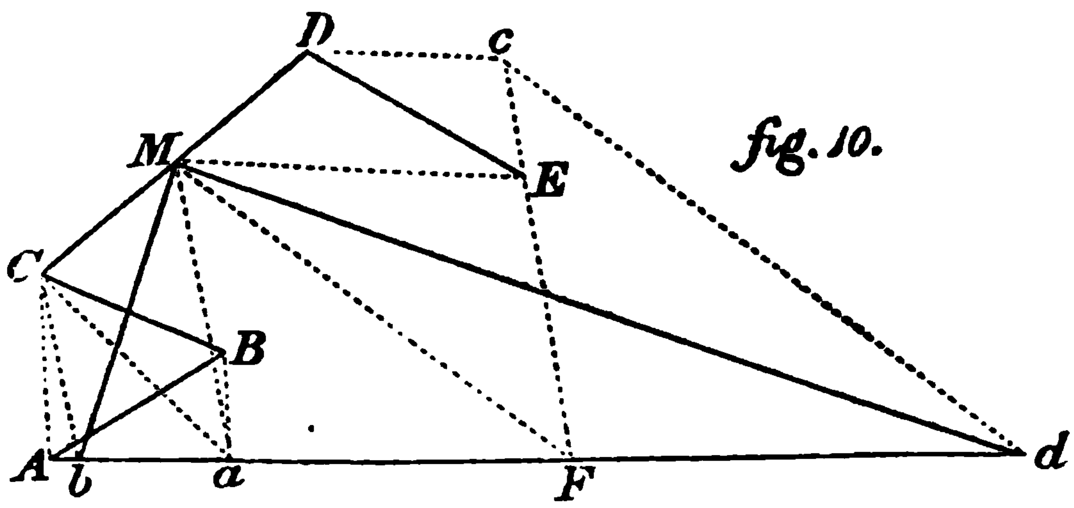


a triangle, whose vertex is in D, and whose base is in AH, we can then proceed in the following way: Draw the lines DA, DH; then the figure is divided into three parts, viz. into the triangle DAH, the quadrilateral figure $DCBA$ on the left, and the pentagon $DEFGH$ on the right. Now transform, according to the given rules, the quadrilateral figure $DCBA$ into the triangle DaA; draw from a the line ab parallel to DA, and to the point b, where it meets the production AH, the line Db, then triangle DbA = triangle DaA = quadrilateral figure $DCBA$. In the same manner, transform the pentagon $DEFGH$ into the triangle DdH; draw the line de parallel to DH, and from the point, where it cuts the production AH, the line De; then the triangle DeH = triangle DdH = pentagon $DEFGH$. Therefore triangle bDE = octagon $ABCDEFGH$; the vertex of the triangle is situated, as was required, in D, and the base is in the side AH. Further, to complete the proof, the lines Da, Dc are drawn.

SECTION IV.

Prob. *To transform any given figure into a triangle, whose vertex is in a certain point in one of the sides of the figure, or within it, and whose base is situated in a given side of the figure.*

Solution. *First Case.* Let (*fig.* 10) *ABCDEF* be the

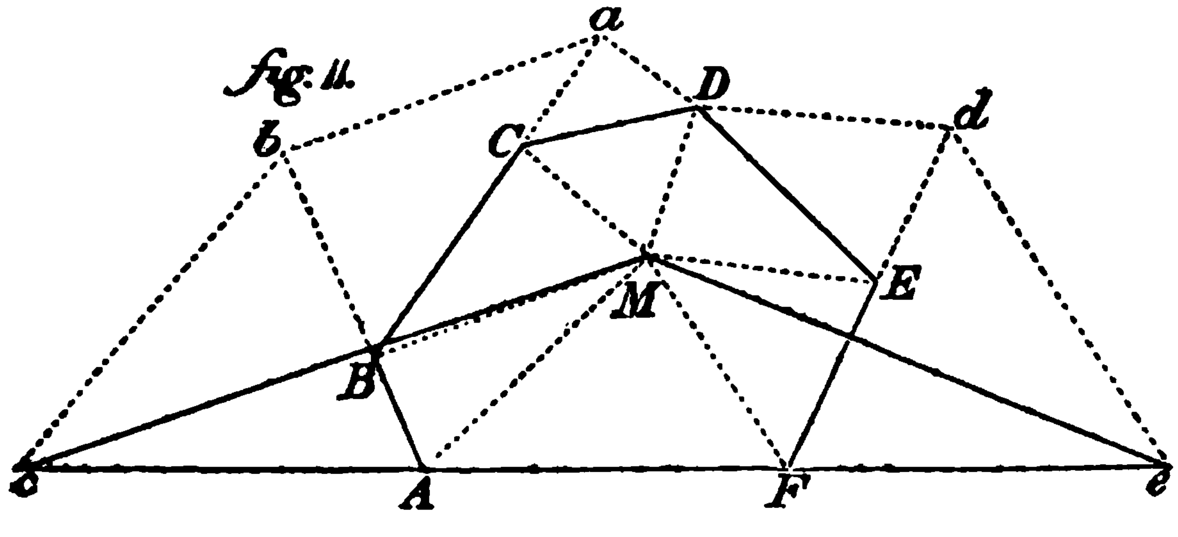


hexagon which is to be transformed into a triangle; let its vertex be in the point *M* in the side *CD*, and the base in *AF*.

1. In the first place, by § III, Cor. 1. omit the angle *ABC* which inclines inwards; since the triangle *BCa* is substituted for the triangle *BAa*, and by these means the hexagon *ABCDEF* is transformed into the pentagon *aCDEF*: then draw the lines *Ma*, *MF*, and the pentagon *aCDEF* is divided into the triangle *MaF*, the quadrilateral figure *MDEF* on the right, and the triangle *MCa* on the left.

2. Transform the quadrilateral figure *MDEF* and the triangle *MCa* into the triangles *MdF*, *Mba*, so that the bases may be in *AF*; then *Mbd* = hexagon *ABCDEF*.

Second Case. Let *ABCDEF* (*fig.* 11) be the given



figure; let the vertex of the required triangle be situated in the point *M* within the figure, and let the base be in *AF*.

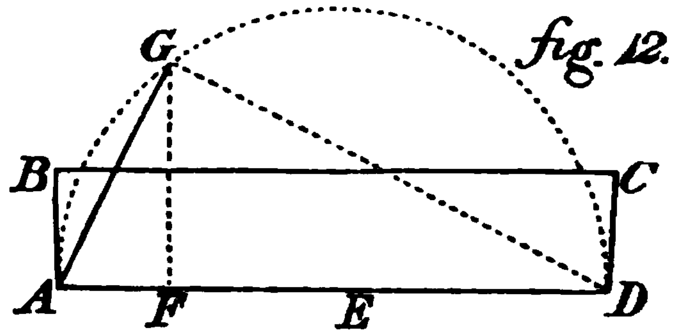
1. From *M* to any angle of the figure, say *D*, draw the line *MD*, and draw the lines *MA*, *MF*, by which means the figure *ABCDEF* is divided into the triangle *MAF*, and the figures *MDCBA*, *MDEF*.

2. Then transform *MDCBA* and *MDEF* into the triangles *McA*, *MeF*, whose bases are in *AF*; then triangle *cMe* = figure *ABCDEF*.

SECTION V.

PROB. *To transform a given rectangle into a square.*

CONST. Let *ABCD* (*fig.* 12) be the given rectangle, which is to be transformed into a square.



1. Bisect the longest side *AD* in *E*.

2. With a radius $EA = ED$, describe upon *AD* the semicircle *AGD*.

3. From *AD* cut off a part *AF* equal to the least side *AB* of the rectangle.

4. From *F* draw the perpendicular *FG*.

5. From the point *G*, where this perpendicular meets the circle, draw the straight line *GA*; then this is a side of the square sought.

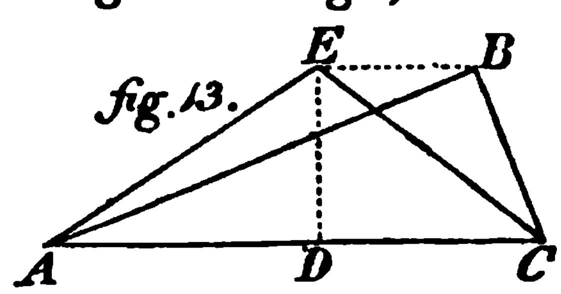
DEMON. Draw the line *GD*; then *AGD* is a right angle (*Euc.* III. 31), consequently *AG* is a mean proportional between *AD* and *AF* (*Euc.* VI. 8), ∴ (*Euc.* VI. 17) $AG^2 = AD \times AF = AD \times AB =$ the rectangle *ABCD*.

Remark. Any figure may be converted into a triangle, and any triangle into a rectangle (*Euc.* 1. 42); further, any rectangle, as appears from the above problem, may be converted into a square; consequently also any figure into a square.

SECTION VI.

Prob. *To convert any given triangle into an isosceles one.*

Const. Let ABC (*fig.* 13) be the given triangle, which is to be converted into an isosceles one.



1. Bisect the base AC in D, and from D draw the perpendicular DE.

2. From the vertex B of the given triangle, draw BE parallel to the base AC.

3. From the point E, where this parallel meets the perpendicular, draw the straight lines EA, EC; then EAC is the isosceles triangle sought.

The demonstration is very easily found.

SECTION VII.

Prob. *To convert a given isosceles triangle into an equilateral one.*

Const. Let ABC (*fig.* 14) be the given isosceles triangle, which is to be converted into an equilateral one.

1. Upon the base AC of the given triangle, draw the equilateral triangle AEC, and through the vertices E, B of both the triangles, draw the straight line EB, which evidently is perpendicular to AC, and bisects the last line in D.

2. Upon ED describe the semicircle EFD, and from B draw the perpendicular BF, which meets the semicircle in F.

3. From D with the radius DF, describe an arc FG, which cuts the line DE in G.

4. From G draw the lines GH, GI, parallel to the sides of the equilateral triangle AEC; then HGI will be the equilateral triangle sought.

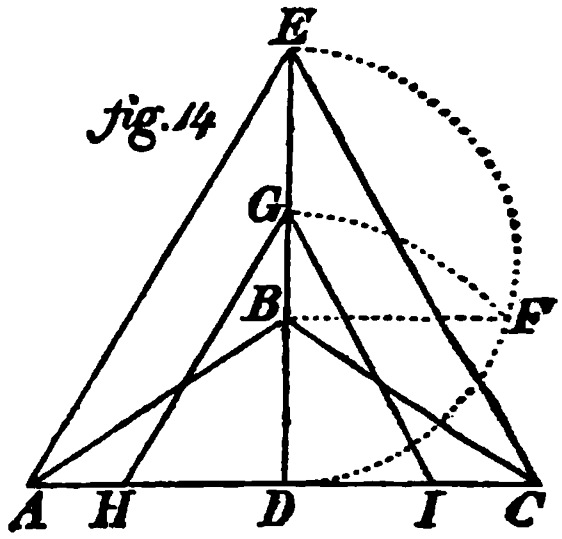


fig. 14

Demon. Since $GH \parallel EA$, and $GI \parallel EC$, the $\angle\, GHI = \angle\, EAC$, and $\angle\, GIH = \angle\, ECA$; consequently $\triangle\, AEC \backsim$ * $\triangle\, HGI$, and $\therefore$ the $\triangle\, HGI$ is equilateral.

Suppose the line DF drawn, then DF, consequently also DG is the mean proportional between DE and DB, and $\therefore$

$$DE : DG = DG : DB;$$

or also

$$DE : DG = AD : HD, \text{ (because } \triangle\, EAD \backsim \triangle\, GHD\text{)};$$

consequently

$$DG : DB = AD : HD,$$

and $\therefore$ because the $\angle\, GDA$ is common,

$$\triangle\, GHD = \triangle\, BAD \text{ (}Euc.\text{ VI. 15)}$$

But $\triangle\, GHD = \triangle\, GID$, and $\triangle\, BAD = \triangle\, BCD$ consequently also $\triangle\, ABC = \triangle\, HGI$.

Corol. If BD be greater than ED, then the perpendicular BF does not meet the semicircle. In this case it will merely be necessary to describe the semicircle on BD, and from E to draw the perpendicular; then in this case the points H, I will not be situated in the line AC, but in its production.

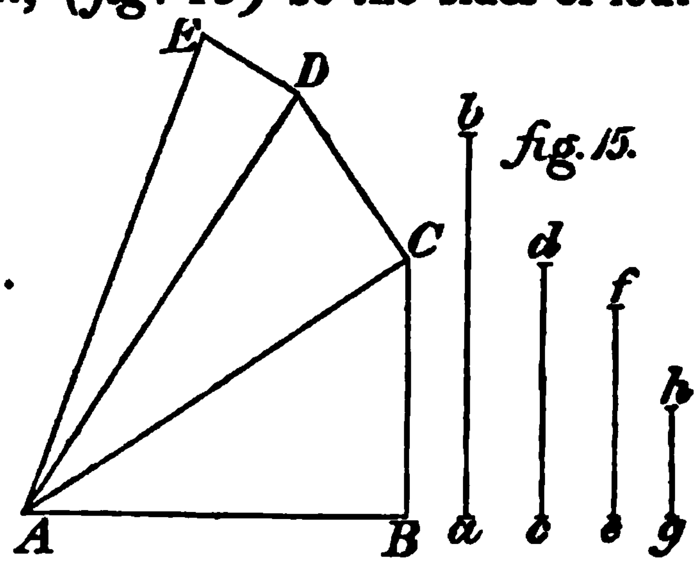
Remark. From this and the preceding §§, it appears, how any figure may be converted into an equilateral triangle; for it is only necessary first to convert the figure into a triangle, this triangle into an isosceles triangle, and the isosceles triangle again into an equilateral triangle.

* Wherever the symbol ($\backsim$) is used, it denotes "similar to."—*Translator.*

SECTION VIII.

Prob. *To describe a square which is equal to the sum of several given squares.*

Const. Let ab, cd, ef, gh, (*fig.* 15) be the sides of four squares: required to find a square which is equal to the sum of these four squares.



1. Make $AB = ab$, and from B draw the perpendicular $BC = cd$, and join AC.

2. From the point C in AC draw the perpendicular $CD = ef$, and join AD.

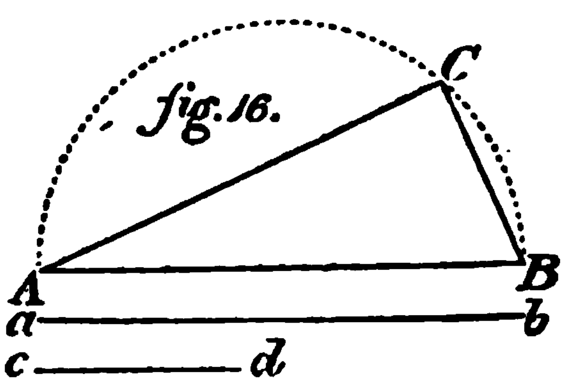
3. From the point D in AD draw the perpendicular $DE = gh$, and join AE, then $AE^2 = ab^2 + cd^2 + ef^2 + gh^2$. The demonstration is very easily deduced from *Euc.* I. 47.

Remark. Hence it appears how any number of squares may be converted into a single one. Since every figure may be converted into a square (§ V. Remark), consequently a square may always be found which is equal to the sum of several figures.

SECTION IX.

Prob. *To describe a square which is equal to the difference of two given squares.*

Const. Let ab, cd, (*fig.* 16) be the sides of two squares: required to find a square, which is equal to their difference.



1. Make $AB = ab$, and on AB describe a semicircle.

2. From B within the semi-

circle, draw the line $BC=cd$, and join AC; then AC is the side of the square sought.

The demonstration is deduced from *Euc.* III. 31, and I. 47.

SECTION X.

PROB. *Let several similar figures be given; construct a figure which is similar to each of them, and equal to their sum.*

CONST. Let *ab*, *cd*, *ef*, *gh*, (*fig.* 15) be the homologous sides of four similar figures, to which, collectively, one similar figure is required to be made.

Upon the lines *ab*, *cd*, *ef*, *gh*, construct right angles, exactly as in § VIII, and by *Euc.* VI. 18, describe upon *AE* a figure, which is similar to the given ones; then this will be equal to the sum of the four figures.

The demonstration is founded on *Euc.* VI. 31, and is easily derived from it.

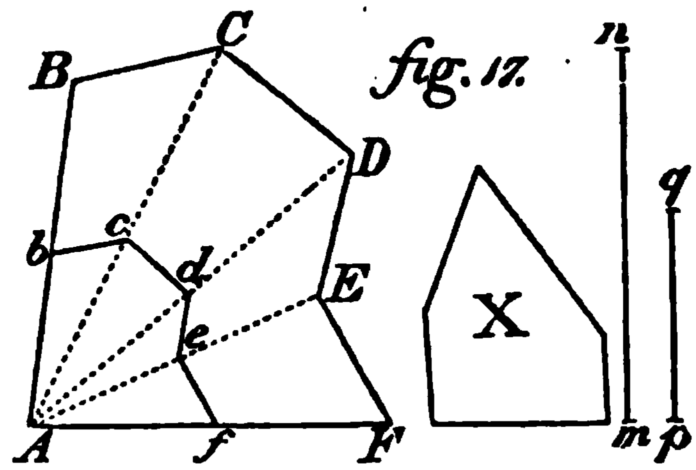
REMARK. By the above method, we are enabled to convert several similar triangles, quadrilateral figures, pentagons, and so on, into a triangle, quadrilateral figure, pentagon, and so on, similar to them. Since the areas of circles are as the squares of their diameters, consequently there always may be found a circle, which is equal to the sum of several given circles. Thus, let *ab*, *cd*, *ef*, *gh*, be the diameters of four circles, then a circle, whose diameter is *AE*, is equal to the sum of these four circles. If we proceed with the diameters of two given circles, as we did in § IX with the sides of the two given squares, we shall find the diameter of a circle, which is equal to the difference of these two circles.

SECTION XI.

PROB. *To transform a given figure in such a way, that it may be similar to another figure.*

CONST. Let X (*fig.* 17) be the given figure, and

ABCDEF the one to which it is to be similar.



1. Convert the figure *ABCDEF* into a square, and let its side be mn, so that $ABCDEF = mn^2$; convert also the figure X into a square, and let its side be pq, so that $X = pq^2$.

2. Take any side of the figure, say *AF*; to the three lines mn, pq, AF, find a fourth proportional (*Euc.* VI. 12), which cut off from *AF*; let *Af* be this fourth proportional, so that $mn : pq = AF : Af$.

3. Then draw the diagonals *AE*, *AD*, *AC*, and the lines *fe*, *ed*, *dc*, *cb*, parallel to the lines *FE*, *ED*, *DC*, *CB*; then *Abcdef* will be the required figure, viz. $= X$ and $\backsim$ *ABCDEF*.

Demon. It may be easily proved that $Abcdef \backsim ABCDEF$. Further, since by *Euc.* VI. 20.

$$ABCDEF : Abcdef = AF^2 : Af^2,$$

and by (2) $AF^2 : Af^2 = mn^2 : pq^2$,

then $ABCDEF : Abcdef = mn^2 : pq^2$;

But by (1) $ABCDEF = mn^2$,

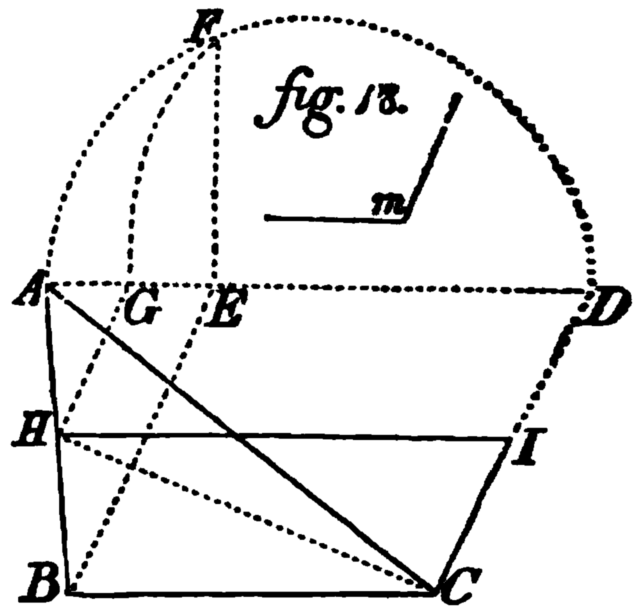
consequently also $Abcdef = pq^2 = X$.

SECTION XII.

Prob. *Upon the base of a given triangle, to describe a quadrilateral figure, equal to the given triangle, with two parallel sides, one of which is the base itself, and one of whose angles at the base is also one of the angles of the triangle, and the other angle is equal to a given angle.*

Const. Let *ABC* (*fig.* 18) be the given triangle, which

is required to be converted into a quadrilateral figure, one of whose sides is the base of the triangle, and another parallel to it; further, let the angle ABC of the triangle be also one of the angles of the quadrilateral figure, and the other angle at C equal to the given angle m.



1. On BC make the angle $BCD = m$.

2. From the vertex A of the triangle, draw AD parallel to BC, which meets CD in D, and from B draw the line BE parallel to CD, which meets AD in E.

3. Upon AD describe the semicircle AFD, and from E draw the perpendicular EF which meets the semicircle in F.

4. From D, with the radius DF, describe the arc FG, which meets AD in G.

5. From G draw the line GH parallel to DC, and from the point H, in which it meets AB, the line HI parallel to BC; then $BHIC$ is the quadrilateral figure sought.

DEMON. Suppose DF drawn, then, by *Euc.* III. 31 and VI. 8, because also $DF=DG$,

$$AD : DG = DG : DE,$$

consequently $AD - DG : DG - DE = DG : DE$,

or $AG : GE = HI : BC$; (because $DG = HI$, and $DE = BC$)

but $AG : GE = AH : HB$,

consequently $HI : BC = AH : HB$.

$$\left.\begin{array}{l} \text{Further, } HI : BC = \triangle\, HCI : \triangle\, CHB \\ \text{and } AH : HB = \triangle\, ACH : \triangle\, HCB \end{array}\right\} \textit{Euc. } \text{VI. 1.}$$

consequently $\triangle\, HCI : \triangle\, CHB = \triangle\, ACH : \triangle\, CHB$,

$\therefore \triangle\, HCI = \triangle\, ACH$,

and $\triangle\, HCI + \triangle\, HCB = \triangle\, ACH + \triangle\, HCB$;

or the quadrilateral figure $BHIC = \triangle\, ABC$.

SECTION XIII.

Prob. *To convert a given quadrilateral with two parallel sides, into another quadrilateral with two parallel sides, having one side and one of the angles adjacent to this side, common with the former, and whose angle at the other side is equal to a given one.*

Const. The quadrilateral $ABCD$ (*fig.* 19), in which the two sides AD, BC are parallel, is to be converted into another which has the angle A and line AD in common, but the other angle at D equal to the given one m.

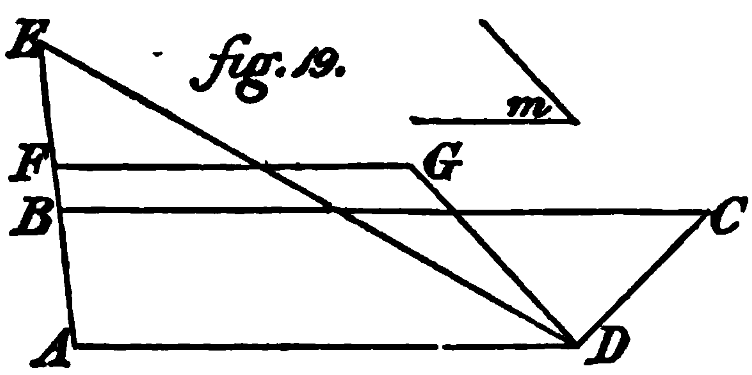


fig. 19.

Convert the quadrilateral $ABCD$, by § I. into the triangle DAE, and this again, by § XII., into the quadrilateral $AFGD$, so that the angle $ADG = m$, then $AFGD$ is the quadrilateral sought.

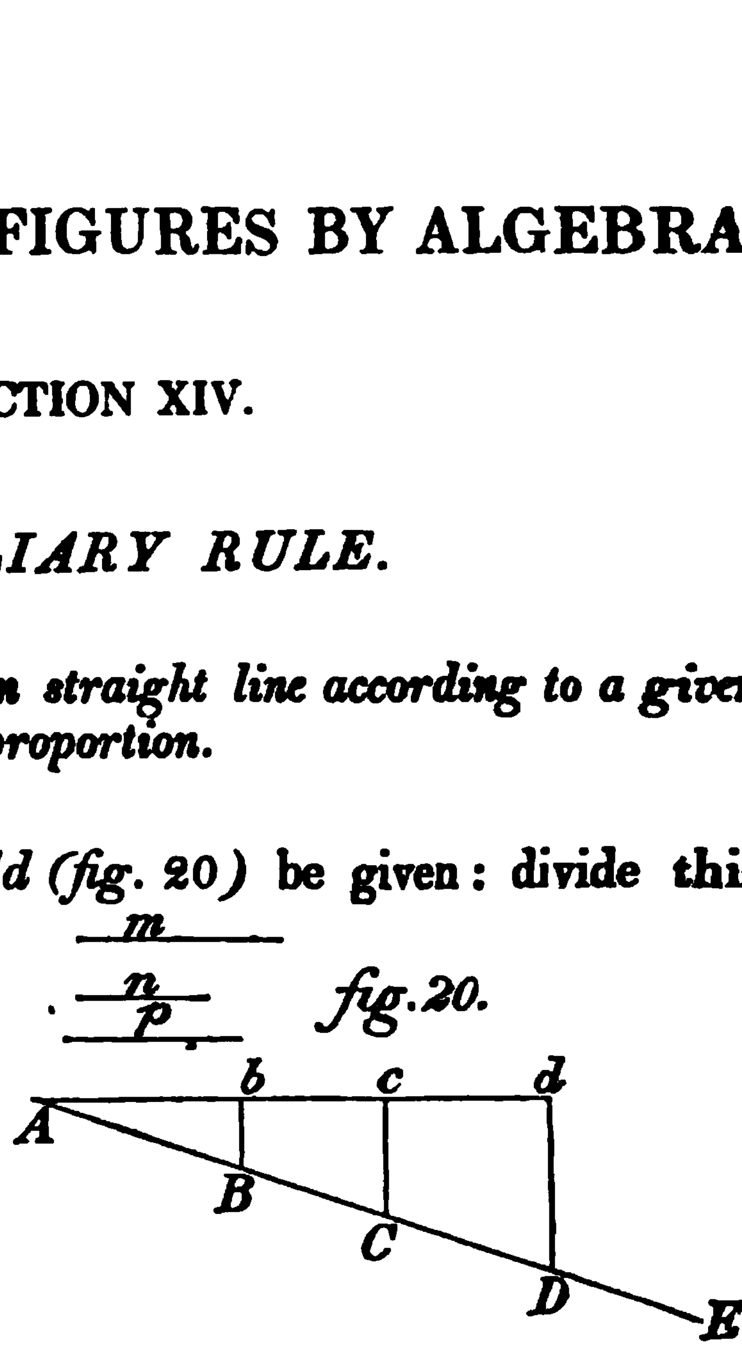
II. DIVISION OF FIGURES BY ALGEBRA.

SECTION XIV.

AUXILIARY RULE.

PROB. *To divide a given straight line according to a given proportion.*

CONST. Let the line Ad (*fig.* 20) be given: divide this line in such a way, that the parts, in the order in which they follow each other, may be in the same proportion as the lines m, n, p are.



1. Draw AE forming any angle with Ad, and from A towards E, draw the lines m, n, p, so that $AB = m$, $BC = n$, $CD = p$.

2. From the last point D draw the line Dd, and from B and C the lines Bb, Cc parallel to Dd, then b and c are the points of section of the line Ad; thus $Ab : bc : cd = m : n : p$.

The reason of this mode of treatment is easily seen.

COR. If, therefore, it is required to divide a straight line into a certain number of equal parts, it is only necessary upon AE to draw the same number of equal parts of arbitrary magnitudes, and then proceed according to 2.

SECTION XV.

Prob. *To divide a triangle from its vertex into a given number of equal parts.*

Const. Let ABC (*fig.* 21) be the given triangle, which is to be divided, say, into six equal parts: let A be the vertex, from which the lines of division are to be drawn.

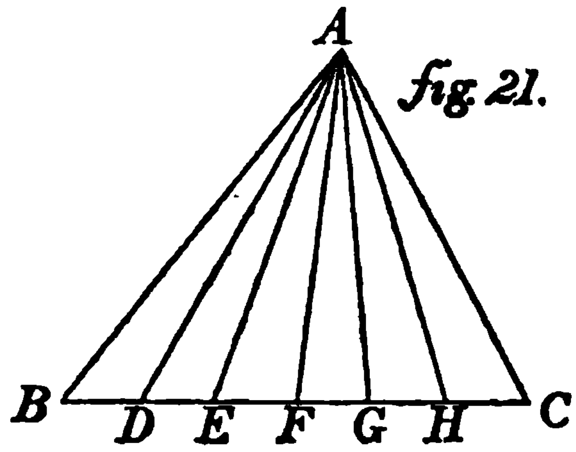


fig 21.

1. Divide the side BC opposite the vertex A into six equal parts BD, DE, EF, FG, GH, HC.

2. From A to the points of division D, E, F, G, H, draw the lines AD, AE, AF, AG, AH; then the triangle ABC is divided into the six equal triangles ABD, ADE, AEF, AFG, AGH, AHC.

Cor. If it is required to divide the triangle ABC according to a given proportion, it will only be necessary to divide the line BC in this proportion (§ XIV), and from A to draw lines to these points of division.

SECTION XVI.

Prob. *From a given point in one of the sides of a triangle to divide it into a given number of equal parts.*

Const. Let ABC (*fig.* 22) be the given triangle, which is to be divided into eight parts; the lines of division are to be drawn from I.

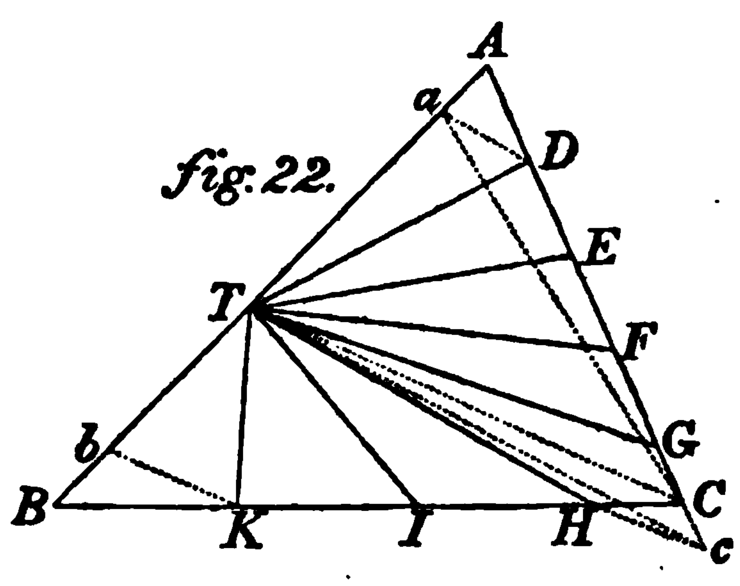


fig: 22.

1. Make $Aa = Bb = \frac{1}{8}AB$, and from T draw the line TC to the vertex of the triangle.

2. From a and b draw the lines aD, bK, pa-

rallel to TC, which meet the sides AC, BC, in D and K.

3. From AC, from A towards C, cut off as many lines equal to CD as possible, (in this case four), and by these means determine the points E, F, G; from BC also cut off as many lines equal to BK as possible, (here three), and by these means determine the points I, H.

4. From T draw the lines TD, TE, TF, TG, TH, TI, TK, then ATD, DTE, ETF, FTG, $GTHC$, HTI, ITK, KTB, are the eight equal parts of the triangle ABC.

DEMON. Draw Ca, then

$$\triangle ACB : \triangle ACa = AB : Aa = 8 : 1$$

$$\text{and } \therefore \triangle ACa = \tfrac{1}{8} \triangle ACB;$$

$$\text{but } \triangle ACa = ATD$$

because $\triangle AaD$ is common to both triangles, and $\triangle aTD = \triangle aCD$; consequently also

$$\triangle ATD = \triangle DTE = \triangle ETF = \triangle FTG = \tfrac{1}{8} \triangle ABC.$$

In like manner it may be proved, by drawing the line bC, that

$$\triangle BTK = \triangle KTI = \triangle ITH = \tfrac{1}{8} \triangle ABC.$$

Since $\therefore$ the triangles taken together compose $\frac{7}{8}$ of the triangle ABC, consequently the quadrilateral $CGTH$ must in like manner be the eighth part of the triangle ABC.

COR. 1 If after having cut off as many lines equal to AD as possible, we arrive at G, and wish to determine immediately the point H in the line BC, this may be very easily represented in the following simple manner. Produce AC, and from G make another line Gc equal to AD; then from c draw cH parallel to CT; then the point H is determined. For since $\triangle THC = \triangle TcC$ (because $cH \parallel CT$); then, when the $\triangle TCG$ is added, the quadrilateral $GTHC = \triangle GTc$; but $\triangle GTc = \triangle ATD = \frac{1}{8} \triangle ABC$; consequently also the quadrilateral $GTHC = \frac{1}{8} \triangle ABC$.

COR. 2. If it is required from a given point in one of the

sides of a triangle, not to divide it into equal parts, but according to a different proportion, then it will be best to proceed in the following way:—

Let ABC (*fig.* 23) be the triangle to be divided, and T the point, from which the lines of division are to be drawn.

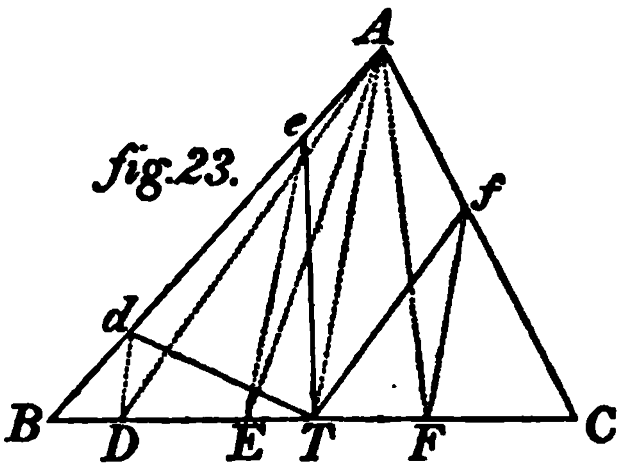


fig. 23.

Divide the line BC, in which the point T is situated, according to the proportion given in § XIV, say in D, E, F; draw AT, and parallel to it the lines Dd, Ee, Ff, which meet the sides AB, AC, in d, e, f; from these points draw the straight lines dT, eT, fT, then BTd, dTe, $TeAf$, fTC, are the parts sought.

For if we draw the lines AD, AE, AF, the triangles ABD, ADE, AEF, AFC are as their bases BD, DE, EF, FC (*Euc.* VI. 1), and since these triangles, as may be easily proved, are respectively equal to the parts BTd, dTe, $TeAf$, fTC; consequently also these last are as the given lines.

SECTION XVII.

Prob. *To divide a triangle, from a point within it, into a given number of equal parts.*

Const. Let ABC (*fig.* 24) be the given triangle, which is to be divided into five equal parts; T the point from which the lines of division are to be drawn.

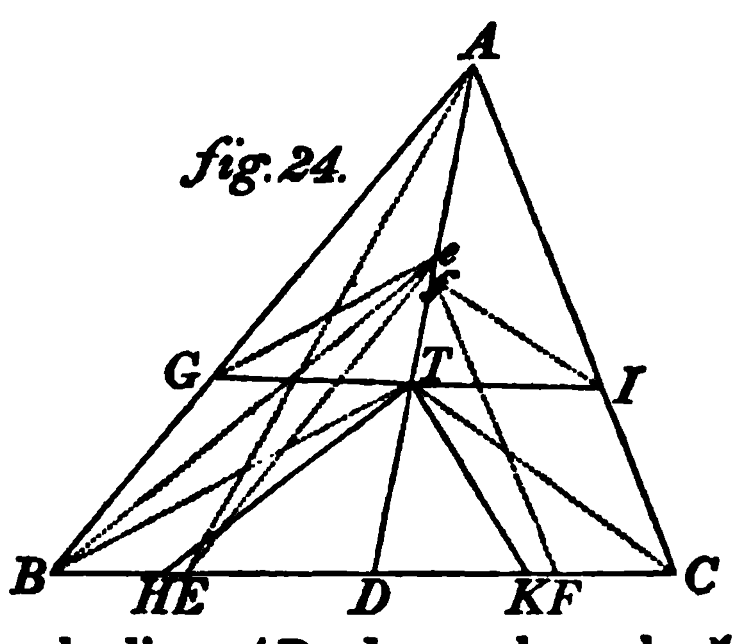


fig. 24.

1. Take any side of the triangle, say BC, and make, when, as here, the triangle is to be divided into five parts, $BE = CF = \frac{1}{5}BC$, and draw the lines Ee, Ff, parallel to the sides AB, AC, which meet the line AD, drawn through A and T.

2. From T draw the lines TB, TC, to the angular points B, C of the triangle ABC, and from e, f the parallels eG, fI; further, the lines TG, TI, then both the triangle ATG and the triangle $ATI = \frac{1}{3}$ triangle ABC.

3. In order to determine the other points of division, it is only necessary, as in the foregoing §, to cut off from the sides AB, AC, as many lines equal to AG, AI as possible, and in the case in which this can no longer be effected, or in which, as in the figure, this is impossible, the points H and K for the quadrilaterals $BGTH$, $CITK$ may be determined by § XVI, Cor. 1.

4. Since in this case the parts ATG, $BGTH$, ATI, $CITK$, collectively form $\frac{4}{5}$ of the triangle ABC, consequently the triangle HTK must be equal to the last part. If HTK be not the last part, then it is merely necessary, by § XV, to divide this triangle into as many equal parts as are necessary.

DEMON. Draw the auxiliary lines AE, eB, then

$$\triangle ABE = \tfrac{1}{5} \triangle ABC \text{ (because } BE = \tfrac{1}{5} BC),$$

further, $\triangle ABE = \triangle ABe$ (because $Ee \parallel AB$)

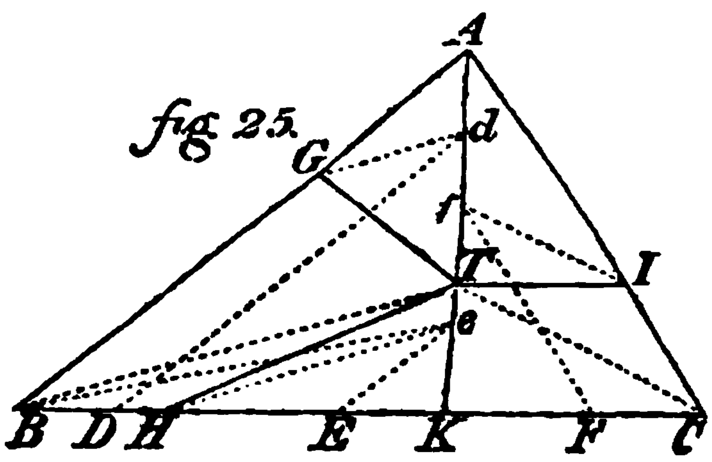
and $\triangle ABe = \triangle ATG$ (because $\triangle GBe = \triangle GTe$);

$$\text{consequently } \triangle ATG = \tfrac{1}{5} \triangle ABC.$$

The remainder of the demonstration is sufficiently clear of itself.

COR. If a triangle is not to be divided into equal parts, but according to a given proportion, then the following method will be the best.

Thus, let the triangle ABC (*fig.* 25) be divided into four parts according to a given proportion.



1 Divide any side of the triangle, say BC, according to this proportion; let the points of division be D, E, F; through and from T draw the lines AK, TB, TC.

2. From the points of division D, E, F, draw the lines Dd, Ee, Ff, parallel to the side AB, or AC, according as they are on one or the other side of the line AK.

3. From the points d, e, f, in which these parallels meet AK, draw the lines dG, eH, fI, parallel to TB or TC; then by these means, the points G, H, I, in the sides of the triangle, will be determined.

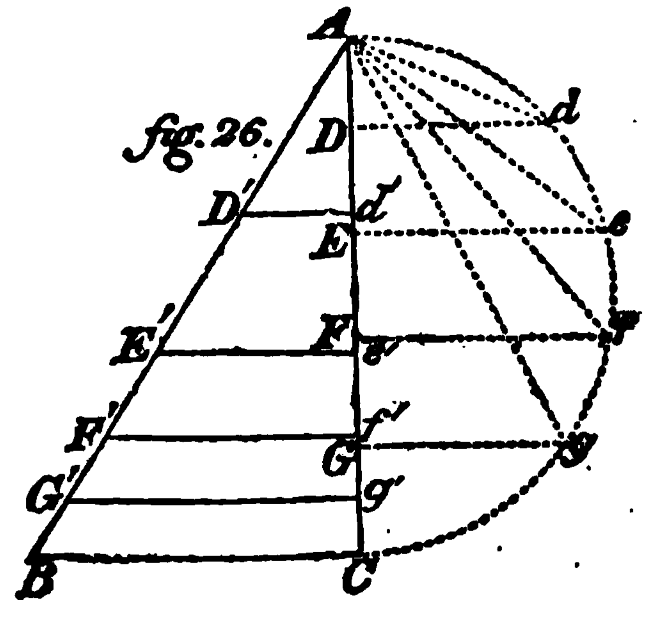
4. From T draw the lines TG, TH, TI; then the triangles ATG, ATI, and the quadrilateral figures $BGTH$, $CHTI$ are the parts of the triangle sought.

For if we suppose the lines AD, AE, AF drawn, then the triangles ABD, ADE, AEF, AFC, which collectively constitute the triangle ABC, have the required proportion, for they are to one another as their bases BD, DE, EF, FC. Now, as may in some measure be easily proved from the foregoing, the triangles ATG, ATI, are equal to the triangles ABD, AFC; and the quadrilaterals $BGTH$, $CHTI$, are equal to the triangles ADE, AEF; consequently also these parts have the required proportion.

SECTION XVIII.

Prob. *To divide a given triangle into a given number of equal parts, and in such a way, that the lines of division may be parallel to a particular side of the triangle.*

Const. Let ABC (*fig.* 26) be the given triangle; let the number of the parts into which it is required to be divided, be five, and BC the side to which the lines of division are to be parallel.



1. Upon one of the other two sides, say AC, describe the semicircle $AdefgC$, and divide the side AC in as many equal parts, as the triangle is

to be divided in, consequently in the present case into five; the points of section are D, E, F, G.

2. From these points of division, draw the perpendicular, Dd, Ee, Ff, Gg, which meet the semicircle in the points d, e, f, g.

3. From A in AC, draw Ad, Ae, Af, Ag, then make $Ad' = Ad$, $Ae' = Ae$, and so on, and by these means determine the points d', e', f', g'.

4. From these points draw the lines $d'D'$, $e'E'$, $f'F'$, $g'G'$, parallel to the side BC, then $AD'd'$, $D'd'e'E'$, $E'e'f'F'$, $F'f'g'G'$, $G'g'CB$ are the five equal parts of the triangle ABC which were sought.

Demon. Since $AC : Ad = Ad : AD$ (*Euc.* VI. 8);
then $Ad^2 = Ad'^2 = AC \times AD$ (*Euc.* VI. 17).
Further, since the triangles $AD'd'$, ABC are similar, consequently

$$\triangle ABC : \triangle AD'd' = AC^2 : Ad'^2 \text{ (Euc. VI. 19)}$$
$$= AC^2 : AC \,.\, AD = AC : AD.$$

Now, since $AD = \frac{1}{5} AC$, therefore

$$\triangle AD'd' = \tfrac{1}{5} \triangle ABC.$$

In like manner it may be proved, that $\triangle AE'e' = \frac{2}{5} \triangle ABC$; $\triangle AF'f' = \frac{3}{5} \triangle ABC$; $\triangle AG'g' = \frac{4}{5} \triangle ABC$, from which the rest follows of course.

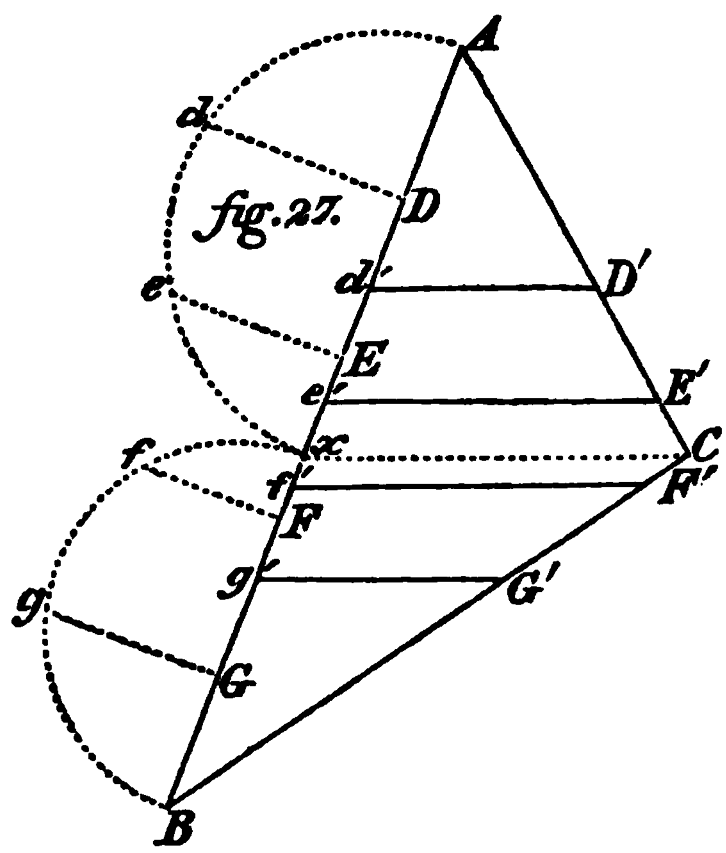
Cor. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will merely be necessary, as may be readily seen from the above, to divide the line AC according to this proportion, and then proceed as has been already shown.

SECTION XIX.

Prob. *To divide a triangle into a given number of equal parts, in such a way, that the lines of division may be parallel to a line given in position.*

Const. Let, for instance, ABC (*fig.* 27) be the given

triangle which is to be divided into five equal parts; the lines of division are to be parallel to the line Cx.



1. Upon Ax and Bx describe the semicircles $Adex$, $Bgfx$, and divide the line AB into five equal parts, in D, E, F, G.

2. From these points of division draw the perpendiculars Dd, Ee, Ff, Gg; from A draw the lines Ad', Ae', equal to Ad, Ae, and also the lines Bg', Bf', equal to Bg, Bf.

3. From the parts d', e', f', g' draw the lines $d'D'$, E', $f'F'$, $g'G'$, parallel to Cx, then $Ad'D'$, $d'D'E'e'$, $eE'CF'f'$, $f'F'G'g'$, $g'G'B$ are the five parts sought.

Demon. By this method, the triangle ACx, by § XVIII, Cor., is divided in the same proportion as the line Ax, and also BCx in the same proportion as the line Bx. Therefore we have

1. $\triangle\, ACx : \triangle\, AD'd' = Ax : AD$.
But $\triangle\, ABC : \triangle\, ACx = AB : Ax$ (*Euc.* VI. 1);
consequently $\triangle\, ABC : AD'd' = AB : AD = 5 : 1$.
In like manner it may be proved, that
$\triangle\, ABC : \triangle\, AE'e' = AB : AE = 5 : 2$.

2. $\triangle\, BCx : \triangle\, BG'g' = Bx : BG$.
But $\triangle\, ABC : \triangle\, BCx = AB : Bx$ (*Euc.* VI. 1);
consequently $\triangle\, ABC : \triangle\, BG'g' = AB : BG : = 5 : 1$;
and thus $\triangle\, ABC : \triangle\, BF'f' = AB : BF = 5 : 2$.

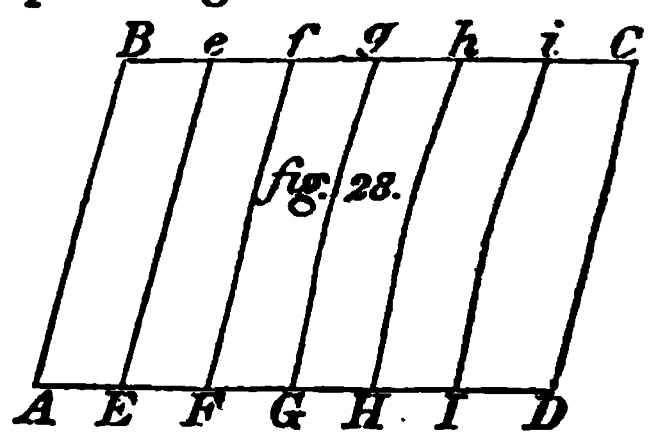
Since $\therefore \triangle AE'e' = \frac{2}{3} \triangle ABC$, and $\triangle BF'f' = \frac{2}{3} \triangle ABC$, consequently the pentagon $e'E'CF'f' = \frac{1}{3} \triangle ABC$.

Cor. If the triangle ABC is not to be divided into equal parts, but according to a given proportion, it will only be necessary to divide the line AB according to this proportion, and then proceed as has been shown. The reason of this is readily seen.

SECTION XX.

Prob. *To divide a parallelogram into a given number of equal parts, in such a way, that the lines of division may be parallel to two opposite sides of the parallelogram.*

Const. Let $ABCD$ be the parallelogram to be divided (*fig.* 28); let the number of parts be six, and let AB, CD, be the sides to which the lines of division are parallel.



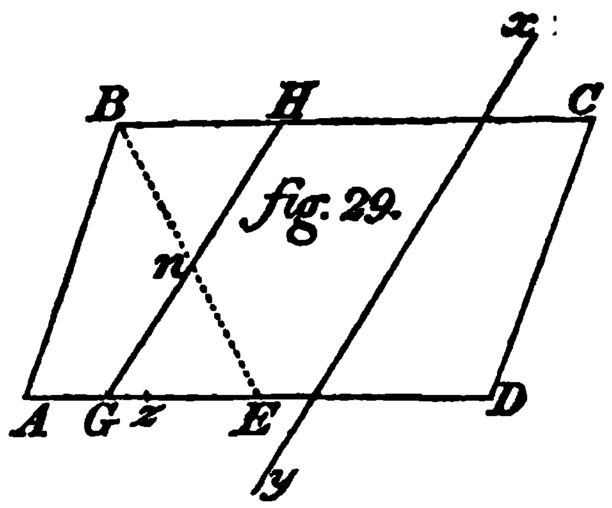
Divide one of the two other sides, say AD, into six equal parts, in E, F, G, H, I, and from these points draw the lines Ee, Ff, Gg, and so on, parallel to the sides AB, CD; then the division is done.

Cor. If it is required to divide the parallelogram according to a given proportion, it will merely be necessary, instead of dividing the line AB into equal parts, to divide it according to the given proportion, and then proceed as before.

SECTION XXI.

Prob. *To divide a parallelogram according to a given proportion, by a line given in position.*

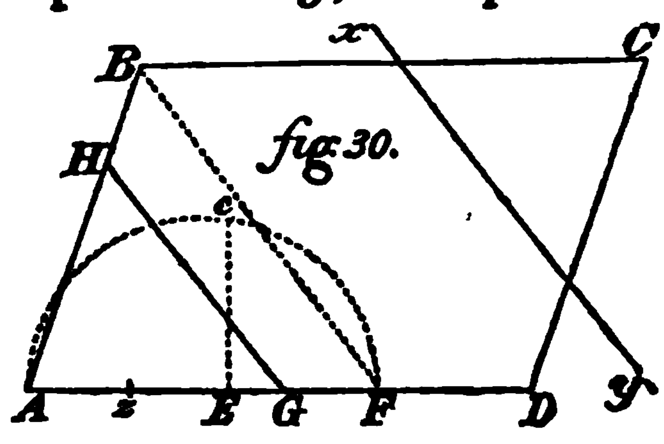
Const. Let $ABCD$ (*fig.* 29) be the parallelogram to be divided.



1. Divide one of its sides, say AD, according to the given proportion; let the point of division be z. Make $zE = Az$, and draw BE. Now if BE have the required position, then $\triangle ABE$, and quadrilateral $BCDE$ are the parts sought.

2. But if the line of division be parallel to xy, then bisect the line BE in n, and through this point draw the line GH parallel to xy, then $ABHG$ and $HCDG$ will be the required parts.

3. If the line xy have such a position, that one of the points G, H cannot fall upon the sides AD or BC, as in *fig.* 30; then draw the line BF parallel to xy, and upon AF describe the semicircle AeF; from E draw the perpendicular Ee, make $AG = Ae$, and draw GH parallel to xy; then AHG, $HBCDG$ will be the required divisions of the parallelogram.



DEMON. By *Euc.* I. 41, and VI. 1, it is evident, that

$$\text{Parall. } ABCD : \triangle ABE = AD : Az,$$

whence the accuracy of 1 is evident.

Further, since (*fig.* 29) HG is bisected in n, then $\triangle BnH = \triangle EnG$; and $\therefore$ also the method in 2 is proved.

In *fig.* 30 we have

$$\text{Parall. } ABCD : \triangle ABF = 2AD : AF,$$

and by reason of the triangles ABF, AHG being similar,

$$\triangle ABF : \triangle AHG = AF^2 : AG^2 \ (= Ae^2)$$
$$= AF^2 : AF \cdot AE,$$
$$= AF : AE;$$

consequently, by compounding both proportions,

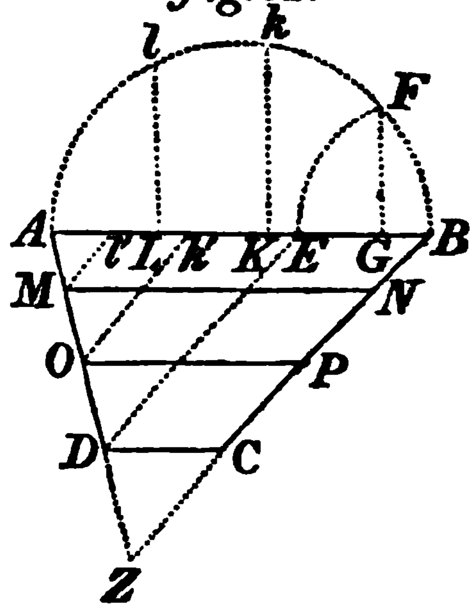
$$\text{Parall. } ABCD : \triangle\, AHG = 2AD : AE = AD : Az.$$

SECTION XXII.

Prob. *To divide a trapezium with two parallel sides into a given number of equal parts, so that the lines of division may be parallel to these sides.*

Const. Thus, let $ABCD$ (*fig.* 31) be the given trapezium, whose two sides AB, CD, are parallel, and which is to be divided into three equal parts.

fig. 31.



1. Upon AB, the greater of the two parallel sides, describe the semicircle AFB; draw DE parallel to BC; and from B, with the radius BE, describe the circle EF, which cuts the semicircle in F.

2. From F draw FG perpendicular to AB, and divide the line AG into three equal parts in K and L, and from these points draw the perpendiculars Kk, Ll.

3. Upon AB, from B towards A, draw the distances Bk', Bl', equal to Bk, Bl; from these points draw the lines $k'O$, $l'M$ parallel to BC; and from the points O, M, in which these parallels meet AD, draw the lines MN, OP, parallel to AB; then $ABNM$, $MNPO$, $OPCD$, will be the three required divisions of the trapezium $ABCD$.

Demon. Produce the lines AD, BC, till they meet in Z, then the triangles DZC, OZP, MZN, AZB, are similar to one another, also.

$$DC = BE = BF,\ OP = Bk' = Bk,\ MN = Bl' = Bl.$$

We have ∴

$$\triangle\, OZP : \triangle\, DZC = OP^2 : CD^2$$
$$= Bk^2 : BF^2 = BK : BG;$$

consequently $\triangle OZP - \triangle DZC : \triangle DZC = BK - BG : BG$

or Trapez. $DOPC : \triangle DZC = GK : BG = \frac{1}{3}AG : BG.$

In like manner it may be proved, that

Trapez. $DMNC : \triangle DZC = GL : BG = \frac{2}{3}AG : BG.$

Trapez. $DABC : \triangle DZC = AG : BG = \frac{3}{3}AG : BG.$

Hence it follows, that

Trapez. $DOPC : DMNC : DABC = \frac{1}{3} : \frac{2}{3} : \frac{3}{3}$
$= 1 : 2 : 3.$

Cor. If it is required not to divide the trapezium $ABCD$ into equal parts, but according to a given proportion, it will only be necessary to divide the line AG in this proportion, and then proceed as before.

SECTION XXIII.

Prob. *To divide a given figure into two parts according to a given proportion, and in such a way, that one of the parts may be similar to the whole figure.*

Const. Let $ABCDE$ (*fig.* 32) be the given figure.

1. Divide one side of the figure, say AB, according to the given proportion; let the point of division be Z.

2. Upon AB describe the semicircle AzB, and from Z draw the perpendicular Zz, which meets the semicircle in z.

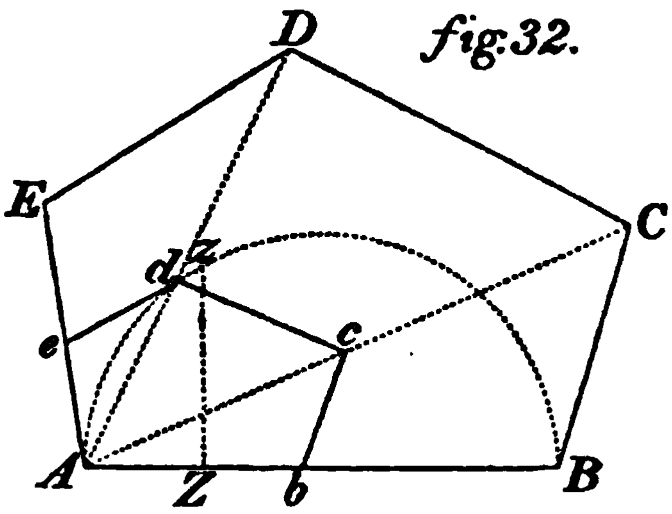


fig. 32.

3. Make $Ab = Az$, and upon Ab describe a figure $Abcde$,

which is similar to the given one $ABCDE$; then the line $bcde$ divides the figure in the manner required.

DEMON. $ABCDE : Abcde = AB^2 : Ab^2$

$$= AB^2 : Az^2 = AB : AZ,$$

consequently $ABCDE - Abcde : Abcde = AB - AZ : AZ$, or $BCDEedcb : Abcde = ZB : AZ$.

III. QUADRATURE OF RECTILINEAR FIGURES.

SECTION XXIV.

PROB. *From the given base and altitude of a triangle to find its area.*

SOLUT. If a denote the base, h the altitude, and q the area of a triangle; then

$$q = \frac{ah}{2}.$$

This rule is well known, and is given in all elementay books on Geometry.

COR. From this equation we obtain

$$h = \frac{2q}{a},\ a = \frac{2q}{h}.$$

By means of the first of these two formulæ, the altitde of a triangle may be determined, when the area and the ase are given, and by means of the second, the base of a riangle may be found, when its area and altitude are given.

EXAM. 1. The base of a triangle measures $325^{\circ}.\ '.\ 9''$, and its altitude $67^{\circ}.\ 8'.\ 3''$: what is its area? Ans. $11049.16785\ \square^{\circ}$, or $11049\ \square^{\circ}\ 16\ \square'\ 78\ \square''\ 50\ \square'''$.

EXAM. 2. The base of a triangle measures $763^{\circ}.\ '.\ 5''$, and its altitude $9'.\ 3''.\ 7'''$: what is its area? Ans. $357.488925\ \square^{\circ}$, or $357\ \square^{\circ}\ 48\ \square'\ 89\ \square''\ 25\ \square'''$.

EXAM. 3. The area of a triangle is 7325 □° 26 □′, and its base 58°. 9′. 7″: what is its altitude? Ans. 248°·4402..., or 248°. 4′. 4″. 0‴. 2ⁱᵛ...

EXAM. 4. The area of a triangle is 62583 □° 04 □′ 79 □″, and its altitude 127°. 5′: what is its base? Ans. 981°·6948..., or 981°. 6′. 9″. 4‴. 8ⁱᵛ.

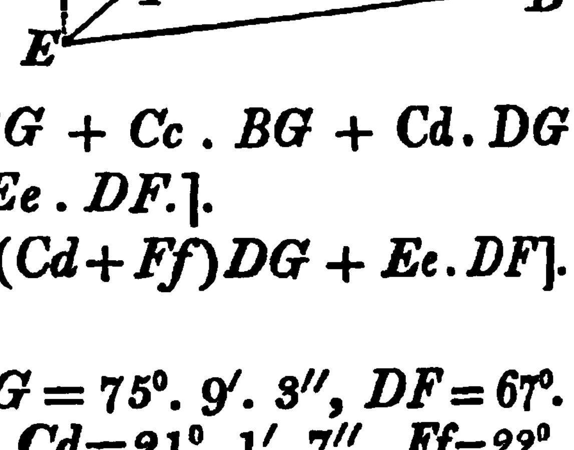
SECTION XXV.

PROB. *To calculate the area of any polygon.*

SOLUT. Divide the whole of the polygon into triangles by diagonals, calculate all these triangles, by multiplying the base of each by half its altitude, and by adding all the results thus obtained together; we then get the area of the polygon.

COR. In order to shorten the operation, a common base can always be given, where it is practicable, to two triangles. Thus the polygon $ABCDEFG$ (*fig.* 33) may be divided into the triangles ABG, BCG, DCG, DFG, DEF, of which the first and second have BG, and third and fourth DG, for a common base. When the necessary perpendiculars are drawn, we then have,

fig. 33.

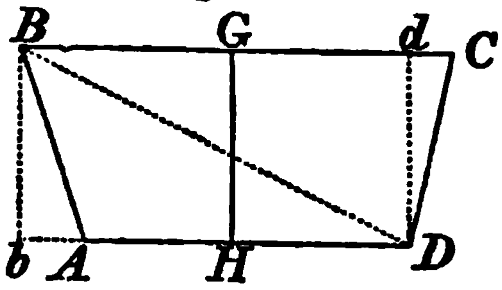


$$\text{Pol. } ABCDEFG = \tfrac{1}{2}[Aa \,.\, BG + Cc \,.\, BG + Cd \,.\, DG + Ff \,.\, DG + Ee \,.\, DF.]$$
$$= \tfrac{1}{2}[(Aa + Cc)BG + (Cd + Ff)DG + Ee \,.\, DF].$$

EXAM. Let $BG = 61^\circ. 5'$, $DG = 75^\circ. 9'. 3''$, $DF = 67^\circ. 3'$, $Aa = 15^\circ. 7'$, $Cc = 28^\circ. 0'. 9''$, $Cd = 21^\circ. 1'. 7''$, $Ff = 22^\circ$, $Ee = 11^\circ. 8'$: what is the area of the heptagon $ABCDEFG$? Ans. 350·81155 □°, or 3550 □° 81 □′ 15 □″ 50 □‴.

REMARK. A trapezium $ABCD$ (*fig.* 34) with two parallel sides AD, BC, may always be divided by the diagonal BD, into two triangles ABD, BDC, whose bases are these sides of the trapezium, and whose altitudes Bb, Dd, are equal to its altitude GH. Therefore in this case,

fig. 34.



$$\text{Trapez. } ABCD = \tfrac{1}{2}\, GH\,(AD + BC).$$

If the whole of a polygon can be divided into such trapeziums, then its area may more easily be calculated. This is the case in *fig.* 35, when the lines AB, CD, EF, GH, IK, are all parallel to one another. Thus then, when ae is perpendicular to these sides, the area of the polygon =

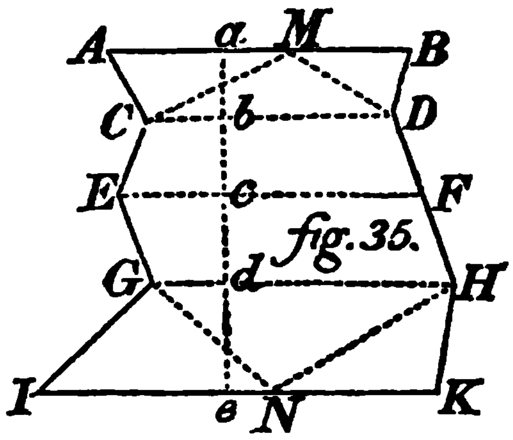


fig. 35.

$$\tfrac{1}{2}\left[\begin{array}{l} ab\,(AB + CD) + bc\,(CD + EF) \\ + cd\,(EF + GH) + de\,(GH + IK) \end{array}\right].$$

If the points A, B meet in M, as also the points I, K in N, so that, instead of the trapeziums $ABDC$, $GHKI$, we have the triangles CMD, GNH: then both AB and $IK=0$, and we find the area of the figure =

$$\tfrac{1}{2}\left[\begin{array}{l} ab\,.\,CD + bc\,(CD + EF) \\ + cd\,(EF + GH) + de\,.\,GH \end{array}\right].$$

If, besides, the distances between all these parallels are the same, or $ab = bc = cd = de$, then the area =

$$ab\,(CD + EF + GH),$$

Consequently we find the area of the figure, by multiplying the sum of all the parallels by the equal distances.

SECTION XXVI.

PROB. *From the two sides of a triangle, and the angle contained by them, to find the area of the triangle.*

CONST. Let ABC be the given triangle (*fig.* 36), of which the sides AB, AC, and the angle BAC, are given: find its area. Let $AC=a$, $AB=b$, $\angle\, BAC = \alpha$, and the required area of the triangle $= q$.

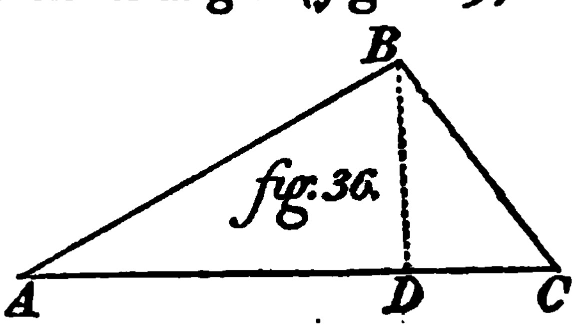


fig. 36.

1. Draw the perpendicular BD: then

$$BD = b\ Sin.\ \alpha.$$

2. Therefore $\triangle ABC = \frac{AC . BD}{2} = \frac{ab \; Sin. \alpha}{2}$

or $$q = \frac{ab \; Sin. \alpha}{2}$$

Cor. 1. Hence it follows: that two triangles, which contain one angle in the one equal to one angle in the other, are as the products of the sides which include these triangles.

Since every parallelogram is divided into two equal parts by its diagonal; consequently this rule likewise obtains for parallelograms which have one equal angle.

Cor. 2. From the equation $q = \frac{ab \; Sin. \alpha}{2}$, we get

$$a = \frac{2q}{b \; Sin. \alpha}, \; b = \frac{2q}{a \; Sin. \alpha}, \; Sin. \alpha = \frac{2q}{ab}.$$

The calculation of all these formulæ is most easily effected by means of logarithms.

Exam. 1. One side of a triangle measures 257′. 9″, the other 356′. 3″, and the angle contained by these sides 25°. 13′: what is the area of this triangle? Ans. 19574·46 □′, or 19574 □′ 46 □″.

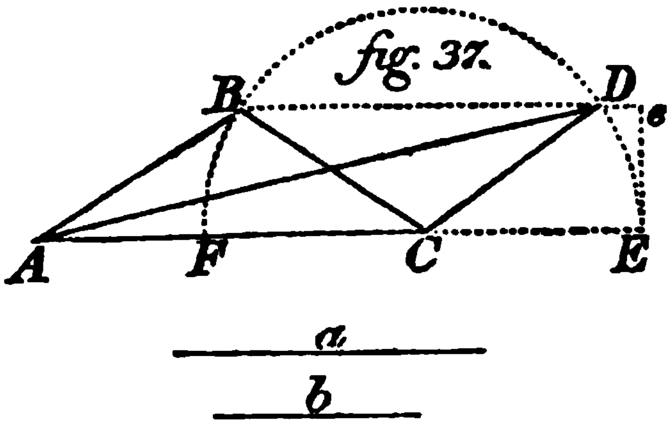
Exam. 2. The area of a triangle is 27534 □′, one of its sides 67′. 3″, and one of the two angles, which are adjacent to this side, 121°. 5′: what is the other side which is adjacent to this angle? Ans. 955′. 4″. 3″.

Exam. 3. The area of a triangle is 1254 □′ 26 □″, one of its sides = 138′, and the other = 59′: what is the angle included by these two sides? Ans. 17°. 56′. 40″. 3, or 162°. 3′. 19″, 7.

Remark. The double values of the angle in the last example arises from this; because two angles which together are equal to 180°, belong to the same sine. Consequently there are always two triangles, having a given area and two given sides, which may be easily represented algebraically.

Let a and b (*fig.* 37) be the given sides, and the given area of the triangle

$= q$. Assume one of these two lines, say a, as the base of the triangle, and make $AC = a$; with the radius $CF = b$, describe the semicircle $FBDE$, and from E, in which it meets AC produced, draw the perpendicular $Ee = \frac{2q}{a}$, then Ee is the altitude of the triangle (§ XXIV, Cor.). Then draw eB parallel to AE, and from the points B, D, in which these cut the semicircle, the lines CB, CD, AB, AD; then ABC, ADC are the required triangles, because they have the required altitude, and the given sides. Now, in the isosceles triangles BCD, the angles CBD, CDB, are equal; further, because eB is parallel to AE, $DBC = BCA$, and $BDC = DCE$; consequently also $BCA = DCE$, and $\therefore ACD + ACB = ACD + DCE = 2R$.



SECTION XXVII.

Prob. *From the two given angles of a triangle, and one of its sides, to find its area.*

Solut. In the triangle ABC (*fig.* 36) there are two angles given, consequently also the third, together with the side AC; required to find its area. Let $BAC = \alpha$, $ABC = \beta$, $ACB = \gamma$, $AC = a$, and the required area $= q$. Draw BD perpendicular to AC: then

1. $Sin.\ \beta : Sin.\ \gamma = a : AB$

$$\therefore AB = \frac{a\ Sin.\ \gamma}{Sin.\ \beta}$$

2. In the right-angled triangle ABD, whose right angle is at D,

$$BD = AB\ Sin.\ \alpha = \frac{a\ Sin.\ \alpha\ Sin.\ \gamma}{Sin.\ \beta}$$

3. Therefore $q = \frac{1}{2} AC \cdot BD = \dfrac{a^2\ Sin.\ \alpha\ Sin.\ \gamma}{2\ Sin.\ \beta}$

This formula is most easily calculated by means of logarithms.

Cor. If the triangle be an isosceles one, then we have

1. When $\beta = \gamma$, $q = \frac{1}{2} a^2 \; Sin. \; \alpha$.

2. When $\alpha = \gamma$, then $\beta = 180^0 - 2\alpha$, and therefore $Sin. \; \beta = Sin. \; 2\alpha = 2 \; Sin. \; \alpha \; Cos. \; \alpha$, consequently

$$q = \frac{a^2 \; Sin. \; \alpha}{2 \; Cos. \; \alpha} = \frac{1}{2} a^2 \; Tang. \; \alpha.$$

Exam. 1. When $\alpha = 38^0. \; 40'$, $\beta = 83^0. \; 30'$, consequently $\gamma = 57^0. \; 50'$, and $a = 120'$; then the area of the triangle is $3832.61 \; \Box'$.

Exam. 2. When $\alpha = 128^0. \; 25'$, $\beta = 27^0. \; 5'$, $\therefore \; \gamma = 24^0. \; 30'$, and $a = 135'$: then the area of the triangle is $6503{\cdot}19 \; \Box'$.

Exam. 3. When $\alpha = 37^0. \; 5'. \; 24''$, $\beta = 67^0. \; 45'. \; 23''$, $\therefore \; \gamma = 75^0. \; 9'. \; 13''$, and $a = 435''$: then the area of the triangle is $59587{\cdot}36 \; \Box'$.

SECTION XXVIII.

Prob. *To find the area of a triangle from its angles and altitude.*

Solut. In the triangle ABC (*fig.* 36) let the angle $BAC = \alpha$, $ABC = \beta$, $ACB = \gamma$, and the altitude $BD = h$ be given; let the required area $= q$: then

1. By § 27, $$q = \frac{AC^2 \; Sin. \; \alpha \; Sin. \; \gamma}{2 \; Sin. \; \beta}.$$

2. By § 24, Cor., $AC = \dfrac{2q}{h}$.

3. If we substitute 2 in 1, we then get

$$q = \frac{2q^2 \; Sin. \; \alpha \; Sin. \; \gamma}{h^2 \; Sin. \; \beta},$$

and $\therefore$, $$q = \frac{h^2 \; Sin. \; \beta}{2 \; Sin. \; \alpha \; Sin. \; \beta}.$$

Exam. When $h = 357\frac{1}{2}'$, $\alpha = 31^\circ.\ 13'.\ 7''$, $\beta = 106^\circ.\ 41'.\ 53''$, then $q = 176203{\cdot}6\ \square'$.

SECTION XXIX.

Prob. *To find the area of a triangle from its three sides.*

Solut. In the triangle ABC (*fig.* 36), the three sides AB, AC, BC are given, and its area is sought. Let $AB = a$, $AC = b$, $BC = c$, and the area $= q$.

1. We know from Trigonometry, that

$$c^2 = a^2 + b^2 - 2ab\ Cos.\ A$$

$$\therefore \qquad Cos.\ A = \frac{a^2 + b^2 - c^2}{2ab}.$$

2. Hence we obtain,

$$1 + Cos.\ A = 1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a + b)^2 - c^2}{2ab}$$

$$= \frac{(a + b + c)\,(a + b - c)}{2ab}$$

$$1 - Cos.\ A = 1 - \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2 - (a - b)^2}{2ab}$$

$$= \frac{(c + a - b)\,(c - a + b)}{2ab}$$

3. The multiplication of these equations gives

$$1 - Cos.^2 A = \frac{(a+b+c)\,(a+b-c)\,(c+a-b)\,(c-a+b)}{4a^2b^2}.$$

or, when we substitute $Sin.^2 A$ for $1 - Cos.^2 A$, and K for the number of the fraction on the right side of the equation,

$$Sin.^2 A = \frac{K}{4a^2b^2}, \text{ and } Sin.\ A = \frac{\sqrt{K}}{2ab}.$$

4. But by § 26, $q = \frac{ab \text{ Sin. } A}{2}$; we have therefore

$$q = \frac{\sqrt{K}}{4} = \tfrac{1}{4}\sqrt{(a+b+c)\ (a+b-c)\ (c+a-b)\ (c+b-a)}.$$

From this formula we deduce the following rules for finding the area of a triangle from its sides :

1. *Add all the sides together.*

2. *From the sum of every two sides subtract the third.*

3. *Multiply the sum obtained from* 1 *by the three remainders obtained from* 2.

4. *From the product subtract the square root, and divide this root by* 4.

The actual calculation, when a, b, c, are not very small numbers, is most easily performed by means of logarithms.

Cor. When the triangle is an equilateral one, $a = b = c$; consequently

$$q = \frac{\sqrt{3a^4}}{4} = \frac{a^2\sqrt{3}}{4}$$

When it is an isosceles triangle, let $c = b$, then

$$q = \frac{\sqrt{a^2\,(2b+a)\ (2b-a)}}{4} = \frac{a}{4}\sqrt{(2b+a)\ (2b-a)}$$

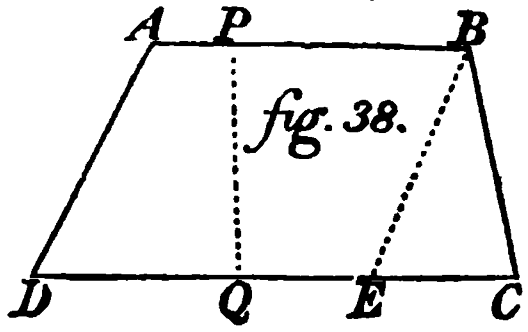
Exam. When $a = 563'$, $b = 295'$, $c = 387'$, then $q = 53447{\cdot}73\ \square'$.

SECTION XXX.

Prob. *From the four given sides of a trapezium, of which two are parallel to one another, to find its area.*

Solut. Let AB, DC (*fig.* 38) be the two parallel sides

of the trapezium $ABCD$, and $AB = a$, $BC = b$, $CD = c$, $DA = d$. Draw BE parallel to AD, and for shortness sake, put $CE = c - a = f$: then



1. In the triangle BEC, $BC = b$, $CE = f$, $BE = AD = d$; consequently, by the foregoing §,
$\triangle BEC = \frac{1}{4}\sqrt{(b+d+f)\ (b+d-f)\ (b+f-d(\ (d+f-b)}$.

2. Draw the perpendicular PQ: then this is the altitude both of the trapezium and the triangle; we have $\therefore$

Trapez. $ABCD = \frac{1}{2}PQ\ (AB + CD)$ (§ 25, Remark.)
and $\triangle BEC = \frac{1}{2}PQ \,.\, CE$, consequently
Trapez. $ABCD : \triangle BEC = \frac{1}{2}PQ\ (AB + CD) : \frac{1}{2}PQ \,.\, CE$
$= AB + CD : CE = a + c : f$
and $\therefore$

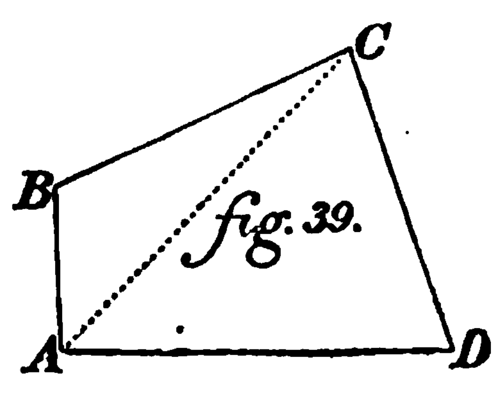
$$\text{Trapez. } ABCD = \frac{a+c}{f} \triangle BEC =$$
$$\frac{a+c}{4f}\sqrt{(b+d+f)\ (b+d-f)\ (b+f-d)\ (d+f-b)}$$

Exam. When $a = 324'$, $b = 137'$, $c = 431'$, $d = 122'$, then $q = 44079{\cdot}76\ \square'$.

SECTION XXXI.

Prob. *All the sides of a quadrilateral, whose opposite angles are together equal to two right angles (about which, consequently, a circle may be described), are given: required to find its area.*

Solut. Let $ABCD$ (*fig.* 39) be the given quadrilateral, in which, agreeably to the hypothesis, $ABC + ADC = 2R$. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$. The unknown angle $ADC = \phi$, and $\therefore$ $ABC = 180^0 - \phi$. Draw the diagonal AC; then



1. In the triangle ADC,

$$AC^2 = c^2 + d^2 - 2\,cd\ Cos.\ \phi$$

and in triangle ABC

$$AC^2 = a^2 + b^2 + 2\,ab\ Cos.\ \phi$$

(because $Cos.\ ABC = Cos.\ (180^0 - \phi) = -\ Cos.\ \phi$.) We have $\therefore$

$$a^2 + b^2 + 2\,ab\ Cos.\ \phi = c^2 + d^2 - 2\,cd\ Cos.\ \phi$$

and consequently $Cos.\ \phi = \dfrac{c^2 + d^2 - a^2 - b^2}{2\,ab + 2\,cd}$

2. Hence we get

$$1 + Cos.\ \phi = \frac{c^2 + 2\,cd + d^2 - a^2 + 2\,ab - b^2}{2\,ab + 2\,cd}$$

$$= \frac{(c + d)^2 - (a - b)^2}{2\,ab + 2\,cd}$$

$$= \frac{(c + d + a - b)\ (c + d - a + b)}{2\,ab + 2\,cd}$$

further $1 - Cos.\ \phi = \dfrac{a^2 + 2\,ab + b^2 - c^2 + 2\,cd - d^2}{2\,ab + 2\,cd}$

$$= \frac{(a + b)^2 - (c - d)^2}{2\,ab + 2\,cd}$$

$$= \frac{(a + b + c - d)\ (a + b - c + d)}{2\,ab + 2\,cd}$$

3. The multiplication of these two equations gives

$$1 - Cos.^2\ \phi = Sin.^2\ \phi =$$

$$\frac{(a+b+c-d)\ (a+b-c+d)\ (c+d-a+b)\ (c+d+a-b)}{(2\,ab + 2\,cd)^2}$$

or, when we substitute K for the numerator of the fraction and extract the root from both sides,

$$Sin.\ \phi = \frac{\sqrt{K}}{2\,ab + 2\,cd}$$

4. Now $\triangle\ ADC = \dfrac{cd\ Sin.\ \phi}{2}$

and $\quad \triangle ABC = \dfrac{ab\ Sin.\ (180^0 - \phi)}{2} = \dfrac{ab\ Sin.\ \phi}{2};$

consequently trapez. $ABCD = \dfrac{(ab + cd)\ Sin.\ \phi}{2},$

or, when for $Sin.\ \phi$ we substitute its value in 3,

$$\text{Trapez. } ABCD = \frac{\sqrt{K}}{4} =$$

$$\tfrac{1}{4}\sqrt{(a+b+c-d)\,(a+b+d-c)\,(c+d+a-b)\,(c+d+b-a)}$$

The rule obtained from this formula for the calculation of this kind of quadrilaterals, may be expressed in words in the following way:

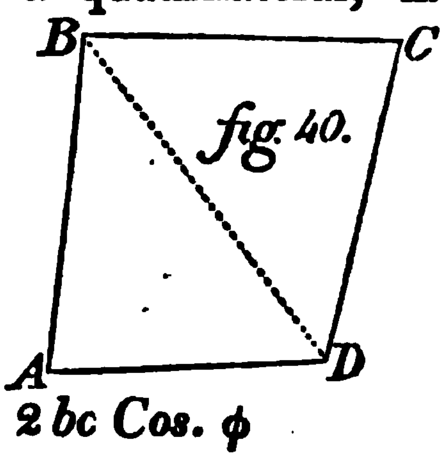
Add every three sides of the figure together, and always subtract the fourth from their sum; then multiply the four remainders together, extract the root from the product, and divide the root by 4.

SECTION XXXII.

PROB. *To find the area of a quadrilateral, in which two opposite angles are equal, from its four sides.*

CONST. Let $ABCD$ (*fig.* 40) be a quadrilateral, in which the angles BAD, BCD are equal, and $AB=a$, $BC=b$, $CD=c$, $DA=d$, the required area $= q$. Put the unknown angle $BAD=BCD=\phi$: then in the $\triangle\ BAD$

$$BD^2 = a^2 + d^2 - 2ad\ Cos.\ \phi$$



and in the $\triangle BCD$, $BD^2 = b^2 + c^2 - 2bc\ Cos.\ \phi$

consequently $a^2 + d^2 - 2ad\ Cos.\ \phi = b^2 + c^2 - 2bc\ Cos.\ \phi$

and $\therefore\ Cos.\ \phi = \dfrac{a^2 + d^2 - b^2 - c^2}{2ad - 2bc}.$

If we proceed now as in the foregoing §, we at last obtain

$$q = \tfrac{1}{4}\frac{ad+bc}{ad-bc}\sqrt{\begin{bmatrix}(a+b+c+d)\,(a+b-c-d)\\(a+d-b-c)\,(b+d-a-c)\end{bmatrix}}.$$

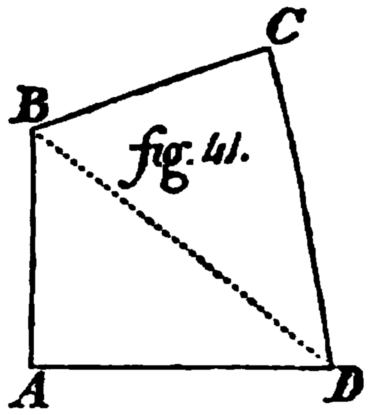
Exam. When $a=37^0$, $b=31^0.\ 7'$, $c=29^0$, $d=16^0.\ 14'$: then $q = 713{\cdot}0677\ \square^0$.

Remark. In this example, two of the factors, which are under the radical sign of the expression found for q, are negative; for we have $a+d-b-c=-8.\ 3$, and $b+d-a-c=-16.\ 9$. But since two negative magnitudes give a positive product, we consequently can omit entirely the sign $-$. Further, for this example, the denominator of the fraction $\frac{ad+bc}{ad-bc}$, is negative, and consequently the fraction itself; but in this case also it is not necessary to retain the negative sign, because the magnitude under the radical sign may be assumed to be either positive or negative. The problem may also be impossible, as, for instance, when we assume $a=17^\circ$, $b=23^\circ$, $c=27^\circ$, $d=40^\circ$.

SECTION XXXIII.

Prob. *To find the area of a ~~triangle~~ from its sides, when one of its angles is a right angle.*

Const. Let $ABCD$ (*fig.* 41) be the quadrilateral, and BAD a right angle. Further, let $AB=a$, $BC=b$, $CD=c$, $DA=d$, and the required area $=q$: then



1. $BD^2=a^2+d^2$ (*Euc.* I. 47), or $f^2=a^2+d^2$, when, for the sake of brevity, we put $BD=f$.

2. In the $\triangle\ BCD$ $\therefore$ all the three sides are known; we consequently have, by § XXIX,

$$\triangle BCD=\tfrac{1}{4}\sqrt{(b+c+f)\ (b+c-f)\ (b+f-c)\ (c+f-b)}$$

3. Since BA is perpendicular to AD, the

$$\triangle\ BAD=\tfrac{1}{2}\,ad,$$

consequently, because trapez. $ABCD=\triangle BAD+\triangle BCD$

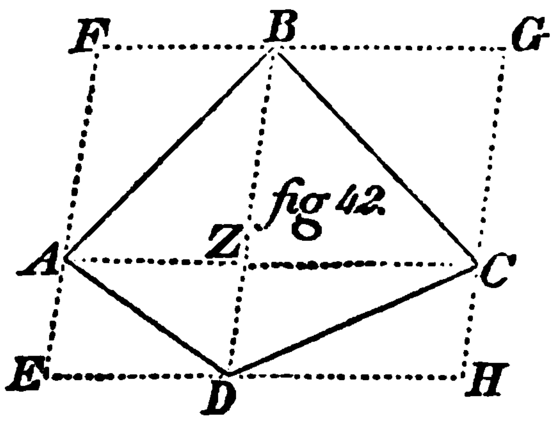
$$q=\tfrac{1}{2}ad+\tfrac{1}{4}\sqrt{(b+c+f)\ (b+c-f)\ (b+f-c)\ (c+f-b)}$$

Exam. When $a=28'$, $b=32'$, $c=41'$, $d=39'$, then $f=48{\cdot}0104$, and $q=1194{\cdot}3332\ \square'$.

SECTION XXXIV.

Prob. *To find the area of a quadrilateral from its two diagonals, and the angle which they include.*

Solut. Let $ABCD$ (*fig.* 42) be the given quadrilateral and $AC = a$, $BD = b$, $\angle AZD = \alpha$.



1. Through A, B, draw the lines EF, HG, parallel to the diagonal BD, and through B, D the lines FG, EH, parallel to AC; then $AFGC$, $AEHC$ are two parallelograms, and the triangles ABC, ADC are their halves. The quadrilateral $ABCD$ is $\therefore$ the half of the parallelogram $EFGH$, and consequently, if we suppose the line FH drawn, equal to the triangle FEH.

2. But by § XXVI, since $EH = AC = a$, $FE = BD = b$, and $\angle FEH = AZD = \alpha$,

$$\triangle FEH = \frac{ab \ Sin. \ \alpha}{2};$$

consequently also Trapez. $ABCD = \dfrac{ab \ Sin. \ \alpha}{2}$.

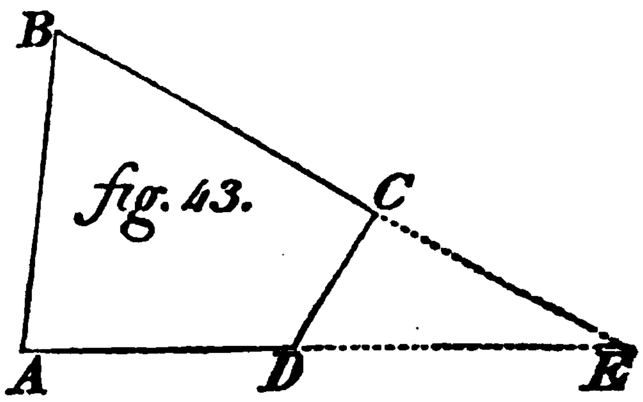
Cor. Consequently every quadrilateral is equal to a triangle, in which two sides are equal to the two diagonals of the quadrilateral and the angle included by them is equal to the angle of the quadrilateral opposite their intersection. In this case it is also immaterial, which of the two angles AZD, DZC is taken for the angle of the triangle, because both give equal triangles (§ XXVI, Remark).

SECTION XXXV.

Prob. *In a quadrilateral, three angles, consequently also the fourth, and two opposite sides, are given: required to find its area.*

Solut. Let $ABCD$ (*fig.* 43) be the quadrilateral figure,

and AB, DC, the given sides. Let $AB = a$, $CD = b$, $BAD = \alpha$, $ABC = \beta$, $BCD = \gamma$, and $ADC = 360^0 - (\alpha + \beta + \gamma) = \delta$; the required area $= q$. Produce the two unknown sides AD, BC, till they meet in E.



1. In the triangle ABE, the two angles $EAB = \alpha$, $EBA = \beta$, and the side $AB = a$ are given; we have $\therefore$ by § XXVII,

$$\triangle AEB = \frac{a^2 \, Sin. \, \alpha \, Sin. \, \beta}{2 \, Sin. \, (\alpha + \beta)},$$

(because $Sin. \, AEB = Sin. \, [180^0 - (\alpha + \beta)] = Sin. \, (\alpha + \beta)$.

2. In like manner, we find

$$\triangle CED = \frac{b^2 \, Sin. \, \gamma \, Sin. \, \delta}{2 \, Sin. \, (\alpha + \beta)},$$

(because $CD = b$, $Sin. \, DCE = Sin. \, (180^0 - \gamma) = Sin. \, \gamma$, $Sin. \, CDE = Sin. \, (180^0 - \delta) = Sin. \, \delta$).

3. Now since Trapez. $ABCD = \triangle AEB - \triangle CED$; therefore

$$q = \frac{a^2 \, Sin. \, \alpha \, Sin. \, \beta - b^2 \, Sin. \, \gamma \, Sin. \, \delta}{2 \, Sin. \, (\alpha + \beta)}$$

Remark. The calculation will be most easily performed by finding each of the triangles AEB, CED separately, and subtracting the areas thus found from one another.

Exam. Let $a = 536'$, $b = 379'$, $\alpha = 83^0. \, 28'$, $\beta = 69^0. \, 34'$, $\gamma = 102^0. \, 20'$: then $q = 145209{\cdot}1 \, \Box'$:

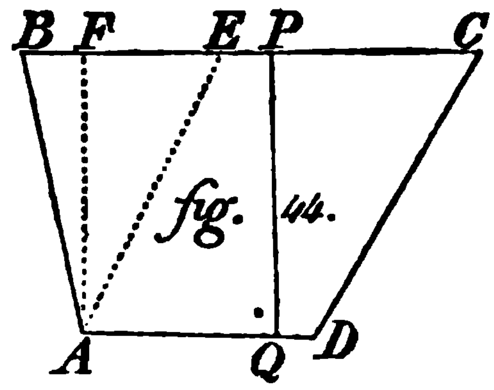
SECTION XXXVI.

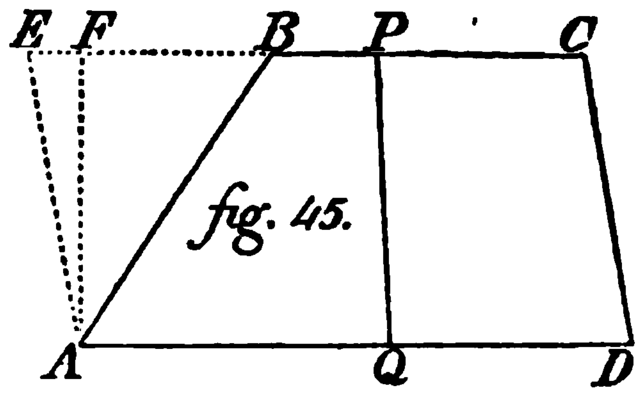
Prob. *To find the area of a trapezium with two parallel sides, when its altitude, one of the parallel sides, and the two angles adjacent to it are given.*

Solut. Let $ABCD$ (*figs.* 44, 45) be a trapezium, having

two parallel sides AD, BC, and PQ perpendicular to these sides. Let the side $AD = a$, the altitude $PQ = h$, and the $\angle\ BAD = \alpha$, $ADC = \delta$, the required area $= q$.

Draw AE parallel to DC, which intersects the line BC in E. This point falls either on the line BC itself (*fig.* 44), or upon BC produced (*fig.* 45). The first case obtains, when $\alpha + \delta > 2R$, the second, when $\alpha + \delta < 2R$.





First Case. 1. In the $\triangle\ ABE$ (*fig.* 44), the $\angle\ ABE = 180^0 - \alpha$, $AEB = BCD = 180^0 - \delta$, $BAE = BAD + ADC - (EAD + ADC) = \alpha + \delta - 180^0$; consequently $Sin.\ ABE = Sin.\ \alpha$, $Sin.\ AEB = Sin.\ \delta$, $Sin.\ BAE = -\ Sin.\ (\alpha + \delta)$, and $\therefore$ (§ XXVIII)

$$\triangle\ ABE = -\frac{h^2 Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

2. The area of the parallelogram $AECD = ah$. Now, since Trapez. $ABCD = AECD + \triangle\ ABE$; then

$$q = ah - \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

Second Case. 1. In the $\triangle\ ABE$ (*fig.* 45), $ABE = BAD = \alpha$, $AEB = ADC = \delta$, $BAE = 180^0 - (\alpha + \delta)$; consequently $Sin.\ ABE = Sin.\ \alpha$, $Sin.\ AEB = Sin.\ \delta$, $Sin.\ BAE = Sin.\ (\alpha + \delta)$ and therefore (§ XXVIII.)

$$\triangle\ ABE = \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

2. The area of the parallelogram $AECD = ah$. Now since Trapez. $ABCD = AECD - \triangle\ ABE$; therefore

$$q = ah - \frac{h^2\ Sin.\ (\alpha + \delta)}{2\ Sin.\ \alpha\ Sin.\ \delta}.$$

We find $\therefore$ for the area of the trapezium, onc and the

same expression, whether the point E be in the line BC, or this line produced, as might, indeed, have been expected from the generality of the trigonometrical and algebraical formulæ.

Cor. When $\alpha + \delta = 180^\circ$, then $Sin. (\alpha + \delta) = 0$, and $q = ah$, which indeed must be the case, because in this case the trapezium is transformed into a parallelogram.

When $\alpha + \delta = 90^\circ$, $Sin. (\alpha + \delta) = 1$, and $Sin. \delta = Sin. (90^\circ - \alpha) = Cos. \alpha$; consequently

$$q = ah - \frac{h^2}{2\, Sin.\, \alpha\, Cos.\, \alpha} = ah - \frac{h^2}{Sin.\, 2\, \alpha}$$

Exam. 1. Let $\alpha = 117^\circ. 36'$, $\delta = 135^\circ. 29'$, $a = 257'$, $h = 87'$. Here $\alpha + \delta = 253^\circ. 5'$, and $Sin. (\alpha + \delta) = - Sin. 73^\circ. 5'$; further $Sin. \alpha = Sin. 62^\circ. 24'$, $Sin. \delta = Sin. 44^\circ. 31$; $\therefore$

$$q = 257 \cdot 87 + \frac{87^2 . Sin.\, 73^\circ.\, 5'}{2\, Sin. 62^\circ.\, 24'. Sin. 44^\circ. 31'} = 28186 \cdot 38 \,\square'.$$

Exam. 2. When $\alpha = 37^\circ. 18'$, $\delta = 52^\circ. 42'$, $a = 350'$, $h = 34'$, then $q = 10700 \cdot 95 \,\square'$.

SECTION XXXVII.

Prob. *In a quadrilateral three of the sides in succession, and the angles included by them, are given: required to find its area.*

Solut. In the quadrilateral $ABCD$ (*fig.* 43) three sides are given, viz. $CB = a$, $BA = b$, $AD = c$, and the angles $DAB = \alpha$, $ABC = \beta$. Produce the sides AD, BC, till they meet in E.

1. In the triangle ABE, the angles DAB, ABC, and the side AB, are given; we have $\therefore$

$$BE = \frac{b\, Sin.\, \alpha}{Sin.\, (\alpha + \beta)},\; AE = \frac{b\, Sin.\, \beta}{Sin.\, (\alpha + \beta)}$$

$$\triangle\, ABE = \frac{b^2\, Sin.\, \alpha\, Sin.\, \beta}{2\, Sin.\, (\alpha + \beta)}\ (\S\ XXVII).$$

2. Hence we obtain

$$CE = BE - BC = \frac{b\ Sin.\ \alpha}{Sin.\ (\alpha + \beta)} - a$$

$$DE = AE - AD = \frac{b\ Sin.\ \beta}{Sin.\ (\alpha + \beta)} - c$$

and $\therefore$

$$\triangle\, CDE = \tfrac{1}{2}\left[\frac{b\ Sin.\ \alpha}{Sin.\ (\alpha+\beta)} - a\right]\left[\frac{b\ Sin.\ \beta}{Sin.\ (\alpha+\beta)} - c\right] Sin.(\alpha+\beta)$$
(§ XXVI).

3. Consequently
Trapez. $ABCD = \triangle\ ABE - \triangle CDE =$
$\tfrac{1}{2}\left[ab\ Sin.\ \beta + bc\ Sin.\ \alpha - ac\ Sin.\ (\alpha + \beta)\right]$.

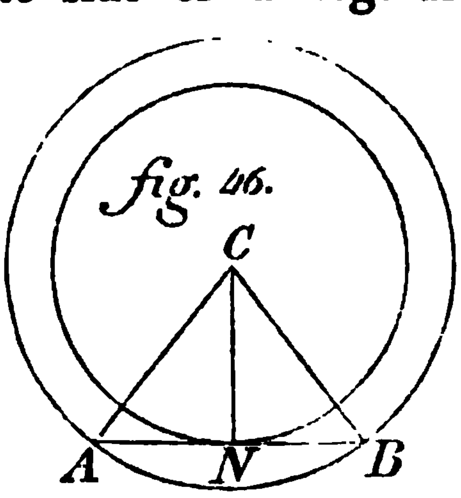
Cor. If AD be parallel to BC, then we have $\alpha+\beta = 180^0$, and $Sin.\ \beta = Sin.\ \alpha$, $Sin.\ (\alpha + \beta) = 0$; $\therefore$
Trapez. $ABCD = \tfrac{1}{2} b\ (a + c)\ Sin.\ \alpha$.

Exam. Let $a = 287'.\,3$, $b = 205'$, $c = 167'.\,4$, $\alpha = 75^0.\,13'$, $\beta = 49^0.\ 36'$. Here we find $\tfrac{1}{2}\ ab\ Sin.\ \beta = 22425{\cdot}96$; $\tfrac{1}{2}\ bc\ Sin.\ \alpha = 16590{\cdot}51$; $\tfrac{1}{2}\ ac\ Sin.\ (\alpha + \beta) = 19742{\cdot}19$; $\therefore$ Trapez. $ABCD = 19274{\cdot}28\ \square'$.

SECTION XXXVIII.

Prob. *To find the area of a regular polygon, from the number of its sides, and the radius of the circle described about it, or within it.*

Solut. Let AB (*fig.* 46) be the side of a regular polygon of n sides, the radius of a circle described about it, $CA = CB = r$, the radius of the circle described within it $CN = \rho$, and the area of the polygon $= P$.



1. Since the polygon has n sides the $\angle\ ACB = \frac{360^0}{n}$, and $\therefore$ (§ XXVI),

$$\triangle\ ACB = \tfrac{1}{2}\, r^2\, Sin.\, \frac{360^0}{n}$$

consequently $P = \tfrac{1}{2}\, nr^2\, Sin. \dfrac{360^0}{n}$.

2. Since $ACB = \dfrac{360^0}{n}$, therefore $ACN = \tfrac{1}{2}\, ACB \doteq \dfrac{180^0}{n}$,

$AN = \rho\, \text{Tan.}\, \dfrac{180^0}{n}$, $AB = 2\, AN = 2\rho\ \text{Tan.}\, \dfrac{180^0}{n}$; $\therefore$

$$\triangle\ ACB = \rho^2\ \text{Tan.}\, \frac{180^0}{n}$$

$$P = n\rho^2\, \text{Tan.} \frac{180^0}{n}.$$

EXAM. 1. What is the area of a Nonagon, when the radius of the circle described about it, is $5''.\ 8'''$? Ans. $97{\cdot}3052\ \Box''$.

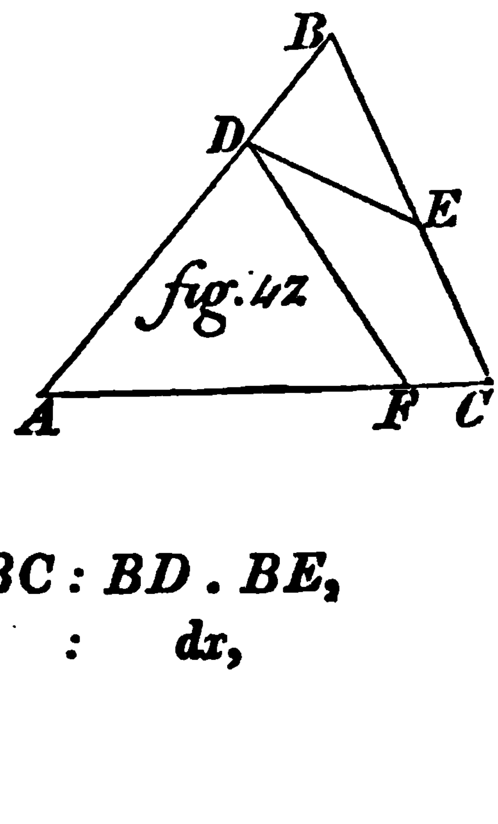
EXAM. 2. What is the area of a Quindecagon, when the radius of the circle described within it is $9''$? Ans. $252{\cdot}6127\ \Box''$, or $2{\cdot}526127\ \Box'$.

IV. PARTITION OF FIGURES BY ALGEBRA.

SECTION XXXIX.

Prob. *From a given point in one of the sides of a triangle, to divide it in a given proportion.*

Solut. Suppose the triangle ABC (*fig.* 47), from the point D, divided by a line DE in such a way, that the whole is to the part DBE, as $m : n$. Let $AB=a$, $BC=b$, $AC=c$, $BD=d$.



1. If we knew how to determine the point E, we could draw the line DE. Let $\therefore BE=x$. By § XXVI, Cor. 1,

$$\triangle ABC : \triangle BDE = AB \,.\, BC : BD \,.\, BE,$$

or
$$m : n = ab : dx,$$

$\therefore$
$$x = \frac{nab}{md}$$

2. If in the calculation of this expression, BE is found to be greater than BC, this indicates that the line of division does not meet the side BC. Let $\therefore DE$ be the line of division, and $AF=y$; then again, by § XXVI. Cor. 1.

$$\triangle ABC : \triangle ADF = AB \,.\, AC : AD \,.\, AF$$
$$= ac : (a-d)\,y$$

But according to the hypothesis,

$$\triangle ABC : BDFC = m : n$$

and $\therefore$
$$\triangle ABC : \triangle ADF = m : m-n$$

consequently
$$m : m-n = ac : (a-d)\,y$$

and
$$y = \frac{(m-n)\,ac}{m\,(a-d)}$$

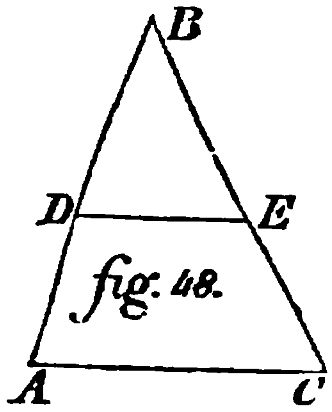
Exam. 1. Let $AB = 74'$, $BC = 47'$, $AC =$ tres $BD = 19'.\ 3''$: from the point D in the triangle from from B towards C, cut off a part, which is to the line triangle, as $7' : 48$. In this case the point E is in b, a $BE = 26'\cdot2802$.

Exam. 2. But if from this triangle the third part be cut off, then the point of section is in the line AC, and then we must assume $AF = 61'\cdot3284$.

SECTION XL.

Prob. *To divide a triangle in a given proportion by a line which is parallel to one of its sides.*

Solut. Let the triangle ABC (*fig.* 48) be divided by a line DE, which is parallel to AC, in such a way, that the whole triangle is to the section DBE, as $m : n$.



Since this is merely to determine the point D or E, from which the line DE is to be drawn; let $AB = a$, $BD = x$.

1. Because the triangles ABC, DBE, are similar,

$$\triangle ABC : \triangle DBE = AB^2 : BD^2$$

or

$$m : n = a^2 : x^2$$

consequently

$$x = \sqrt{\frac{na^2}{m}} = a\sqrt{\frac{n}{m}}$$

2. In the same way, when we put $BC = b$, $BE = y$, we find

$$y = b\sqrt{\frac{n}{m}}.$$

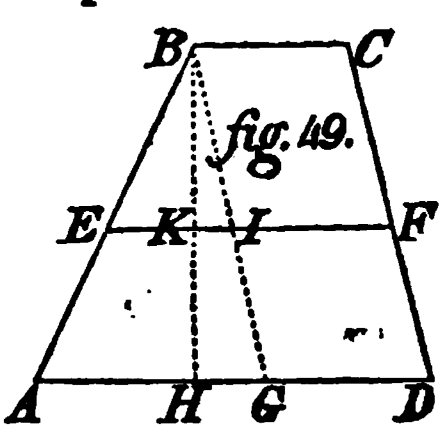
Exam. From the triangle ABC it is required to cut off the fifth part by the parallel line DE; the line AB contains 739 parts of a certain scale: how many of these parts must be taken from B towards D, in order to determine the point D? Ans. $330\frac{1}{2}$ nearly.

Exam. 2. From a field, which is of the form of the triangle ABC, and which contains 14356 □′, it is required to cut off a piece containing 3958 □′ by a parallel line DE. If the side $AB = 573'$: what is the size of BD? Ans. 30[illegible]·867.

SECTION XLI.

Prob. *From a given quadrilateral with two parallel sides, to cut off, by a line parallel to these sides, a part consisting of a given area.*

Solut. Let $ABCD$ (*fig.* 49) be the trapezium, with two parallel sides AD, BC, from which trapezium, by a line EF, parallel to these two sides, it is required to cut off a part $BCFE$, whose area $= q$.



1. Draw the perpendicular BH, and BG parallel to CD, and let $AD = a$, $BC = b$, and the altitude $BA = h$. If we knew how to determine the point K, in which the lines BH, EF intersect each other, we could then draw the line of division. Let $\therefore BK = x$, and $EF = y$.

2. Since EI is parallel to AG, therefore

$$AG : EI = BG : BI = BH : BK$$

or
$$a - b : y - b = h : x$$

consequently
$$(a - b)\,x = (y - b)\,h$$

3. $$\text{Trapez. } BCFE = \tfrac{1}{2}(y + b)\,x = q$$

consequently $(y + b)\,x = 2q$

4. Therefore the two equations 2 and 3, when solved, give:

$$y = \sqrt{\left[\frac{2q\,(a-b)}{h} + b^2\right]}$$

$$x = \frac{h}{a-b}\left[-b + \sqrt{\left(\frac{2q\,(a-b)}{h} + b^2\right)}\right]$$

Exam. Let $a = 76'$, $b = 36'$, $h = 23'$, and $\therefore$ the area of the trapezium $= 1288 \square'$; it is required to cut off from it a part containing $560 \square'$: what is the length of the line of section EF, and its distance from BC?

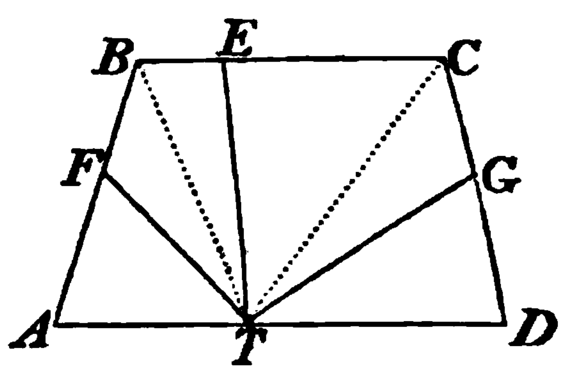
Ans. $EF = 56'\cdot954$, and $BK = 12\cdot048$.

SECTION XLII.

Prob. *To divide a trapezium with two parallel sides, in a given proportion, from a given point in one of its sides.*

Solut. Let $ABCD$ (*fig.* 50) be the trapezium, which from the point T is to be divided by a line TE in such a way, that the section $ABET$ is to the whole trapezium, as $n : m$. Let $AD = a$, $BC = b$, $AB = c$, $CD = d$, $AT = f$.

fig. 50.



1. Put $BE = x$; then, because the altitudes are equal, by § XXV, Remark,

$$\begin{array}{lcccc} & \text{Trapez. } ABCD : \text{Trapez. } ABET & = & a + b : f + x; \\ \text{but} & \text{Trapez. } ABCD : \text{Trapez. } ABET & = & m : n \\ \text{consequently} & a + b : f + x & = & m : n \end{array}$$

$$\therefore \qquad x = \frac{n(a+b)}{m} - f.$$

2. If x in the course of the operation be found to be negative, this indicates, that the point E is not situated in BC, but in AB. In this case, let TF be the line of section, and $AF = y$. Draw BT; then

$$\begin{array}{ll} & \text{Trapez. } ABCD : \triangle ATB = a + b : f \\ \text{and} & \triangle ATB : \triangle ATF = c : y; \\ \text{consequently} & \text{Trapez. } ABCD : \triangle ATF = c(a+b) : fy. \\ \text{But} & \text{Trapez. } ABCD : \triangle ATF = m : n \\ \text{consequently} & c(a+b) : fy = m : n \end{array}$$

$$\therefore \quad y = \frac{nc(a+b)}{mf}.$$

3. If in 1 x be found greater than b, this indicates that the line of section must fall in CD. Let $\therefore$ TG be the line of section, and $DG = z$. Draw CT; then

$$\text{Trapez. } ABCD : \triangle\, CTD = a + b : a - f$$

and
$$\triangle\, CDT : \triangle\, GTD = \quad d \ : \ z$$

consequently

$$\text{Trapez. } ABCD : \triangle\, GTD = d\,(a + b) : (a - f)\, z.$$

But according to the hypothesis,

$$\text{Trapez. } ABCD : ABCGT = m : n$$

and $\therefore$ $\quad$ Trapez. $ABCD : \triangle\, GTD = m : m - n.$

We have $\therefore$

$$d\,(a + b) : (a - f)\, z = m : m - n$$

and
$$z = \frac{(m - n)\,(a + b)\,d}{m\,(a - f)}$$

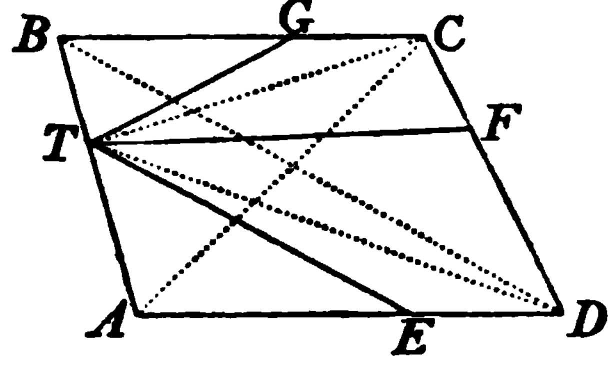
Exam. Let $a = 112'$, $b = 80'$, $c = 45'$, $d = 40'$, $f = 30'$. If it is required to cut off the third part from the trapezium, assume $BE = x = 34'$, and draw TE. If the tenth part is to be cut off, assume $AF = y = 28'{\cdot}8$, and draw TF; but if $\frac{5}{8}$ths of the trapezium are to be cut off, assume $DG = z = 35'{\cdot}112$, or thereabouts, and draw TG.

SECTION XLIII.

Prob. *To divide a trapezium having two parallel sides in a given proportion, from a given point not in the parallel sides.*

Solut. Let $ABCD$ (*fig.* 51) be the trapezium; AD, BC the parallel sides, and T the point from which the line of section is drawn.

fig. 51.



First calculate the triangles ATD, CTD, BTC, with reference to the trapezium; then from the magnitude of these triangles, and from the magnitude of the part to be cut off, we may easily judge whether

the point E in the line of section TE, falls in AD, CD, or BC. Let $\therefore$ $AD = a$, $BC = b$, $TA = c$, $TB = d$, and the area of the trapezium $= A$.

1. If we draw the lines AC, BD, then we find, by similar conclusions to those made in the foregoing §,

$$\triangle ATD = \frac{ac}{(a+b)(c+d)} . A$$

$$\triangle BTC = \frac{bd}{(a+b)(c+d)} . A$$

$$\triangle CTD = \text{Trapez. } ABCD - \triangle ATD - \triangle BTC$$

$$= \frac{ad+bc}{(a+b)(c+d)} . A$$

2. If it is required to cut off a given part from the trapezium, it is only necessary to divide one of these triangles, from its vertex T, in a given proportion, which, by § X, is done by dividing its base in this proportion. The following example will elucidate this.

Exam. Let $a = 120'$, $b = 98'$, $c = 46'$, $d = 31'$; also $\triangle ATD = 0{\cdot}3288.A$, $\triangle BTC = 0{\cdot}1810.A$, $\triangle CTD = 0{\cdot}4902.A$. If it is required to cut off the fourth part of the trapezium, or $\frac{1}{4} A = 0{\cdot}25.A$, we must then divide AD in E, so that $AD : AE = 3288 : 2500$, and then, if we draw TE, TAE is the fourth part. If it is required to cut off $\frac{2}{3}$rds of the trapezium, or $\frac{2}{3} A = 0{\cdot}6666.A$, we must in this case add a $\triangle DTF = 0{\cdot}3378$ to ATD, and consequently divide DC in F, in such a way, that $DC : DF = 4902 : 3378$; then $ATFD$ will be the part required. If we wish to cut off the $\frac{8}{9}$th part, or $\frac{8}{9} A = 0{\cdot}8888.A$, we must, because $\triangle ATD + \triangle DTC = 0{\cdot}8190.A$, add a $\triangle CTG = 0{\cdot}0698.A$ to the quadrilateral $ATCD$, and consequently divide BC in G, so that $BC : CG = 1810 : 698$; then $ATGCD$ will be the part required.

SECTION XLIV.

Prob. *From a given point to divide any trapezium in a given proportion.*

Solut. Let $ABCD$ (*fig.* 52) be the given trapezium,

and T the point from which the line of division TE is so drawn that trapez. $ABCD$: trapez. $DCET = m : n$.

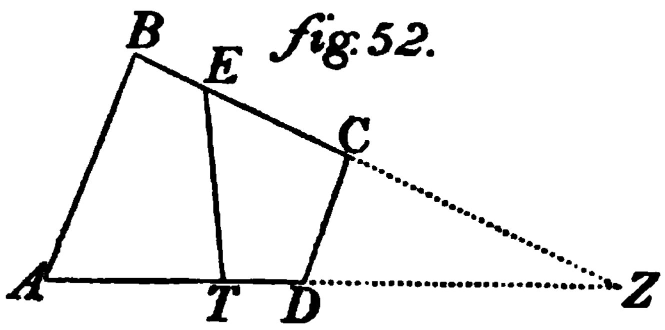


fig. 52.

1. Produce the sides BC, AD till they meet in Z. Since the trapez. $ABCD$ is given, the lines AZ, BZ, CZ, DZ, may also be determined. Let $\therefore$ $AZ=a$, $BZ=b$, $DZ=c$, $CZ=d$. Since the point T is also given, let $ZT=f$. In order now to determine the point E, we put $ZE = x$.

2. By § XXVI, Cor. 1,

$$\triangle BZA : \triangle CZD = ab : cd$$

$$\therefore \triangle BZA - \triangle CZD : \triangle CZD = ab - cd : cd$$

or trapez. $ABCD : \triangle CZD = ab - cd : cd$

3. In like manner

$$\triangle EZT : \triangle CZD = fx : cd$$

$$\therefore \triangle EZT - \triangle CZD : \triangle CZD = fx - cd : cd$$

or trapez. $DCET : \triangle CZD = fx - cd : cd$.

4. From 2 and 3 we obtain

Trapez. $ABCD$: trapez. $DCET = ab - cd : fx - cd$

But trapez. $ABCD$: trapez. $DCET = m : n$

consequently $ab - cd : fx - cd = m : n$

and
$$x = \frac{n\,(ab - cd)}{mf} + \frac{cd}{f}.$$

Exam. Let $a = 200'$, $b = 178'$, $c = 112'$, $d = 120'$, $f = 140'$, and it is required to cut off $\frac{5}{8}$ths of the trapezium. Assume $ZE = 155\frac{5}{14}$, and draw TE; then $DCET$ is the part required.

SECTION XLV.

PROB. *To divide a quadrilateral, which is already divided by a straight line into two other quadrilaterals, by another straight line, in such a way, that from each of the two quadrilaterals, into which the whole figure is divided, parts may be cut off containing given areas.*

SOLUT. The quadrilateral $ABCD$ (*fig.* 53), which is divided into the quadrilaterals $EBCF$, $EADF$, by the line EF, is required to be divided by the line IL, so that the quadrilaterals, $BEKI$, $EALK$, may have given areas.

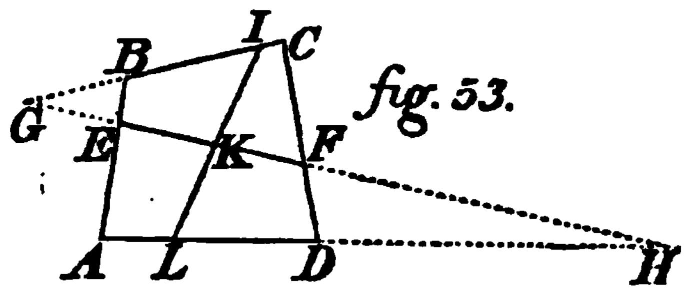


fig. 53.

1. Produce the sides BC, AD, till they meet EF produced in G and H. Since the trapez. $ABCD$, and one of its lines EF, are given, we can $\therefore$ assume the areas of the triangles BEG, AEH, also the angles BGE, AHE, and the line GH as known. Now, since the areas of the trapeziums $BEKI$, $EALK$ are known, consequently also the areas of the triangles GKI, LKH are known. Put therefore $\triangle\, GKI = p$, $\triangle\, LKH = q$, $\angle\, IGK = \alpha$, $\angle\, KHL = \beta$, and $GH = a$.

2. If the line GK, and the angle GKI are known, we can draw the line of division IL. Put $\therefore$ $GK = x$, $\angle\, GKI = \phi$.

3. By § XXVII,

$$\triangle\, GKI = \frac{x^2 \text{ Sin. } \alpha \text{ Sin. } \phi}{2 \text{ Sin. } (\alpha + \phi)},$$

$$\triangle\, LKH = \frac{(a-x)^2 \text{ Sin. } \beta \text{ Sin. } \phi}{2 \text{ Sin. } (\beta + \phi)}.$$

Now since $\triangle\, GKI = p$, $\triangle\, LKH = q$, we have the two equations,

$$x^2 \, Sin.\, \alpha \; Sin.\, \phi = 2\, p \; Sin.\, (\alpha + \phi)$$
$$(a-x)^2 \, Sin.\, \beta \; Sin.\, \phi = 2\, q \; Sin.\, (\beta + \phi).$$

4. Expand $Sin.\, (\alpha + \phi)$, $Sin.\, (\beta + \phi)$ divide the first equation by $Sin.\, \alpha \;\, Sin.\, \phi$, and the second by $Sin.\, \beta \; Sin.\, \phi$, and put $Cot.\, \alpha$, $Cot.\, \beta$, $Cot.\, \phi$, for $\frac{Cos.\, \alpha}{Sin.\, \alpha}$, $\frac{Cos.\, \beta}{Sin.\, \beta}$, $\frac{Cos.\, \phi}{Sin.\, \phi}$. By these means the foregoing equations are transformed into the following ones:

$$x^2 = 2\, p\, (Cot.\, \phi + Cot.\, \alpha)$$
$$(a-x)^2 = 2\, q\, (Cot.\, \phi + Cot.\, \beta).$$

5. Hence by eliminating $Cot.\, \phi$, we obtain

$$\frac{(a-x)^2}{2\, q} - \frac{x^2}{2p} = Cot.\, \beta - Cot.\, \alpha$$

or $\frac{(a-x)^2}{2q} - \frac{x^2}{2p} = \frac{Sin.\, (\alpha - \beta)}{Sin.\, \alpha \; Sin.\, \beta}$,

or $x^2 - \frac{2ap}{p-q}\, x = \frac{2pq \; Sin.\, (\alpha - \beta)}{(p-q) \; Sin.\, \alpha \; Sin.\, \beta} - \frac{a^2 p}{p-q}$

6. The solution of this equation gives

$$x = \frac{ap}{p-q} \pm \sqrt{\left[\frac{a^2pq}{(p-q)^2} + \frac{2pq \; Sin.\, (\alpha - \beta)}{(p-q)\; Sin.\, \alpha \; Sin.\, \beta}\right]},$$

or $x = \frac{a}{p-q}\left[p \pm \sqrt{\left(pq + \frac{2\,pq\,(p-q)\; Sin.\, (\alpha - \beta)}{a^2 \; Sin.\, \alpha \; Sin.\, \beta}\right)}\right]$.

7. Having found x, ϕ is also known, for

$$Cot.\, \phi = \frac{x^2}{2p} - Cot.\, \alpha.$$

Cor. If $\alpha = \beta$, then $Sin.\, (\alpha - \beta) = 0$, and we obtain

$$x = \frac{a}{p-q}\, [p \pm \sqrt{pq}].$$

If we put $p = q$, from the formula in 6, we obtain

$$x = \frac{a}{p-p}[p \pm \sqrt{p^2}],$$

consequently, either $p = \infty$, or $x = \frac{0}{0}$. The first of these two values cannot be used here; the second is indeterminate. But for this particular case, we have from 4, the two following equations:

$$x^2 = 2p\,(Cot.\,\phi + Cot.\,\alpha)$$
$$(a-x)^2 = 2p\,(Cot.\,\phi + Cot.\,\beta).$$

If the second be subtracted from the first, we then obtain an equation of the first degree only, viz.

$$2ax - a^2 = 2p\,(Cot.\,\alpha - Cot.\,\beta) = \frac{2p\,Sin.\,(\alpha-\beta)}{Sin.\,\alpha\,Sin.\,\beta}$$

and $\therefore$ $x = \dfrac{p\,Sin.\,(\alpha-\beta)}{a\,Sin.\,\alpha\,Sin.\,\beta} + \frac{1}{2}a.$

If besides in this case $\beta = \alpha$, we obtain $x = \frac{1}{2}a$, which is also easily inferred from the figure, because by reason of the equal angles, and the equal areas of the triangles GKI, LKH, it necessarily follows that $GK = KH$.

Exam. A square field $ABCD$, which consists of two parts, the part $EBCF$ of pasture, and the part $EADF$ of arable land, is required to be divided by a line IL, so that the part $BEKI$, which is cut off from the pasture-land, has an area of 2600 □°, and the part $EALK$, which is cut off from the arable land, has an area of 2900 □°.

Let $CGF = \alpha = 40°.\,40'$, $AHE = \beta = 18°.\,26'$, $GH = a = 228°$, $\triangle\,BEG = 480$ □°, $\triangle\,AEH = 6488$ □°; $\therefore$ $\triangle\,GKI = p = 3080$ □°, $\triangle\,LKH = q = 3588$ □°. The calculation is effected in the following way:

$$\frac{2pq\,(q-p)\,Sin.\,(\alpha-\beta)}{a^2\,Sin.\,\alpha\,Sin\,\beta} = u:$$

then

$$x = \frac{a}{q-p}[\pm\sqrt{(pq-u)} - p].$$

The calculation by means of logarithms gives

$$log.\ u = [log.\ 2p + log.\ q + log.\ (q-p) + log.\ Sin.\ (\alpha-\beta)] - [2\,log.\ a + log.\ Sin.\ \alpha + log.\ Sin.\ \beta]$$

$$= [log. 6160 + log. 3588 + log. 508 + log. Sin. 22^\circ 14'] - [2\,log.\ 228 + log.\ Sin.\ 40^\circ 41' + log.\ Sin.\ 18^\circ 26']$$

$$= 9{\cdot}6282243 - 4{\cdot}0298522 = 5{\cdot}5983721,$$

$\therefore$

$$u = 396617{\cdot}63;\ \sqrt{(pq - u)} = 3264{\cdot}11$$

and $x = \frac{228}{508}[3264{\cdot}11 - 3080] = 82{\cdot}63.$

Of the two values found for x, only the first can be used here, because the second is negative.

Hence we further obtain

$$Cot.\ \phi = \frac{x^2}{2q} - Cot.\ \alpha = 1{\cdot}10839 - 1{\cdot}16397 = -0{\cdot}05558,$$

and $\therefore$ $\phi = 93^\circ\ 10'$.

If $\therefore$ we make $GK = 82^\circ. 63'$, and through K draw the line IL, forming an angle $GKI = 93^\circ. 10'$, then the required part is cut off.

REMARK. The problem here solved is of the greatest importance to the practical surveyor. A similar problem is to be found in Lambert's learned German Correspondence, published by J. Bernoulli, 2nd vol. (1782), p. 412; also in Von Tempelhof's Supplement to Clairault's Rudiments of Algebra, (Second Edition, 1797), p. 225.

V. GEOMETRICAL DETERMINATION OF HEIGHTS AND DISTANCES.*

SECTION XLVI.

PROB. *To determine the distance between two objects, when there is an obstacle between them, as a sea, a morass, or mountain, the distance between them not being known, under the supposition that there is a station, from which we can measure the distance to these objects in a straight line.*

First Solution.

1. Let A and B (*fig.* 54) be the two objects, whose distance from each other is required to be found. Take a station C, from which the distances to A and B may be measured; measure the distances CA, CB, and from these last backwards measure off the same spaces towards D and E, make $CD = CA$, $CE = CB$. If, after this, we measure the distance DE, so that $AB = DE$, or, make $CD' = CA$, $CE' = CB$, then also $AB = D'E'$.

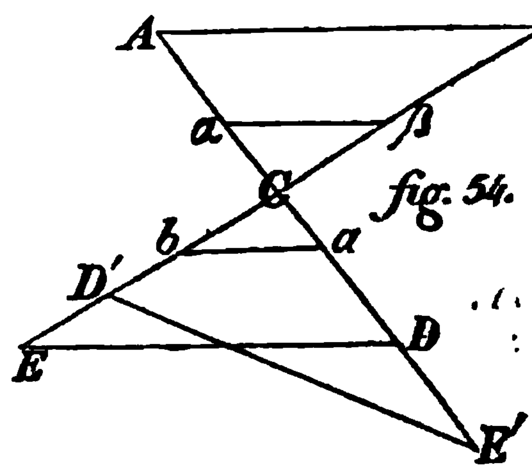


fig. 54.

2. If the distances CA, CB are very great, and if it be impracticable to measure them backwards by reason of impediments, take merely an equal part of the two distances, the half, the third, fourth, or in general the nth part; measure off these parts backwards from C towards a, b, or forwards from C towards α, β; then ab, or $\alpha\beta$, is the same part of the

* The problems in this chapter must properly be considered merely as geometrical exercises (as appears from the mode of treatment already adopted), and consequently are not determined for surveyors only.

unknown distance AB, as Ca, or $C\alpha$ is of CA, and Cb, or $C\beta$ is of CB.

The reason of this is easily seen.

Second Solution.

Again, A and B, (*fig.* 55) are the two objects, and C the chosen station. In the direction CB, make the distance $CD = CA$, and if it be practicable, measure AD. Let $AC = CD = a$, $BC = b$, $AD = c$, $BD = b - a = d$: then in the triangle ACB,

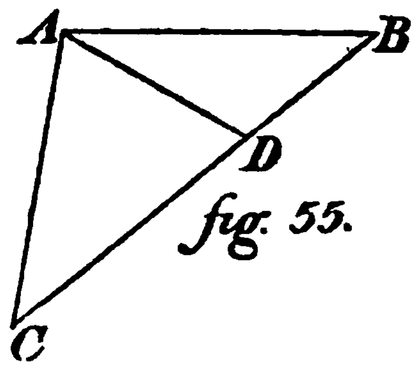


fig. 55.

$$AB^2 = a^2 + b^2 - 2ab\ Cos.\ C,$$

and in the triangle ACD,

$$c^2 = 2a^2 - 2a^2\ Cos.\ C,$$

because $AC = CD = a$. If we subtract the value of $Cos.\ C$ from the second equation, and substitute it in the first, we then obtain

$$AB^2 = a^2 + b^2 - 2ab + \frac{bc^2}{a}$$

$$= (a - b)^2 + \frac{bc^2}{a}$$

$$= d^2 + \frac{bc^2}{a},$$

and $\therefore\ AB = \sqrt{\left(d^2 + \frac{bc^2}{a}\right)}$.

Exam. When $a = 50'$, $b = 76'$, $c = 32'$, $AB = 47' \cdot 249$.

Third Solution.

If in the directions CA, CB (*fig.* 56), there are two points D, E, whose distance DE can be measured, then, when the distances CA, CB, CD, CE are also measured, the distance of the objects A, B may in like manner be determined. Let $CA = a$, $CB = b$, $CD = c$, $CE = d$, $DE = e$; then in the triangle ACB,

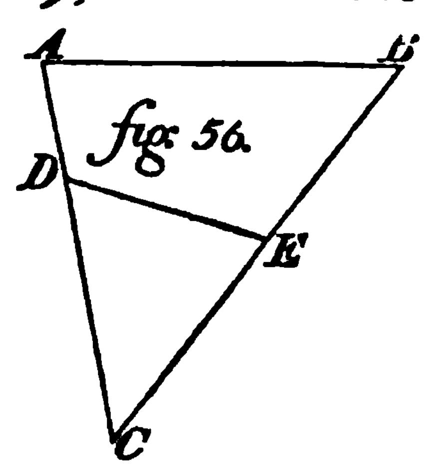


fig. 56.

$$AB^2 = a^2 + b^2 - 2ab\ Cos.\ C,$$

and in the triangle DCE,

$$e^2 = c^2 = d^2 - 2cd \text{ Cos. } C,$$

and when we substitute in the first equation the value of *Cos. C* from the second equation

$$AB = \sqrt{[a^2 + b^2 - \frac{ab}{cd}(c^2 + d^2 - e^2)]}$$

EXAM. When $a = 30^0$, $b = 35^0$, $c = 20^0$, $d = 15^0$, $e = 13^0$, $AB = 23^0$.

Fourth Solution.

If we are provided with instruments for measuring angles, it will only be necessary (*fig.* 56) to measure the angle ACB, and the distances CA, CB. Let $CB = a$, $CA = b$, $\angle ACB = \alpha$; then

$$AB = \sqrt{(a^2 + b^2 - 2ab \text{ Cos. } \alpha)}.$$

But we can also first determine the angles CAB, CBA. For since $CAB + CBA = 180^0 - \alpha$, $\therefore$, by means of the proportion,

$$a + b : a - b = \text{Tan.} \frac{CAB + CBA}{2} : \text{ Tan.} \frac{CAB - CBA}{2}$$

$$= \text{Tan.} \frac{180^0 - \alpha}{2} : \text{ Tan.} \frac{CAB - CBA}{2}$$

it will merely be necessary to find the difference between the two angles CAB, CBA. If the angles CAB, CBA are determined, we then have

$$AB = \frac{b \text{ Sin. } \alpha}{\text{Sin. } CAB} = \frac{a \text{ Sin. } \alpha}{\text{Sin. } CBA}.$$

COR. If $AC = BC$, we then find

$$AB = \sqrt{(2a^2 - 2a^2 \text{ Cos. } \alpha)} = a\sqrt{2(1 - \text{Cos. } \alpha)} = 2a \text{ Sin. } \tfrac{1}{2}\alpha$$

If ACB be a right angle, then $\text{Cos. } \alpha = 0$, and $\therefore$.

$$AB = \sqrt{(a^2 + b^2)}.$$

If $ACB = 45^0$, then because $\text{Cos. } 45^0. = \sqrt{\tfrac{1}{2}}$,

$$AB = \sqrt{(a^2 + b^2 - ab\sqrt{2})}$$

Exam. 1. When $a = 168'$, $b = 102'$, $\alpha = 49^{\circ}.\,25'$, $AB = 127'{\cdot}797$.

Exam. 2. When $a = 189'$, $b = 114\frac{3}{4}'$, $\alpha = 107^{\circ}.\,48'$, $AB = 249'{\cdot}295$.

Exam. 3. When $a = b = 250'$, $\alpha = 43^{\circ}.\,50'$, $AB = 186'{\cdot}629$.

SECTION XLVII.

Prob. *To determine the distance between two objects, when only one of them is accessible.*

First Solution.

Suppose A (*fig.* 57) is an inaccessible object, and B another, whose distance from A it is required to find.

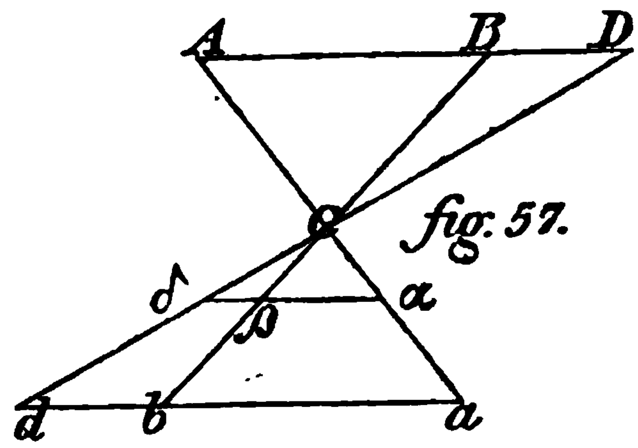


fig: 57.

1. In AB produced, assume any point D, and find a station C, from which both B and D are accessible; measure the distances CB, CD, and measure these off backwards on CB, CD, produced; then make $Cd = CD$, and $Cb = CB$; then proceed in the direction db till a point a is arrived at, which lies in a straight line with A and C. If we now measure the distance ba, we then also have the distance AB, because $AB = ab$.

2. If there be not sufficient space to measure off the whole of the distances CB, CD backwards, it will only be necessary to take equal parts of them, $C\delta = \frac{1}{n}CD$, $C\beta = \frac{1}{n}CB$, then to proceed in the direction $\delta\beta$, till we arrive at a point α, which lies in a straight line with A, C. If after this we measure $\beta\alpha$, we shall then find AB from the proportion

$$AB : \beta\alpha = CB : C\beta = CD : C\delta.$$

The reason of this method is easily discovered.

Second Solution.

If we can measure from C towards A and B (*fig.* 57), also from B towards A and C, and likewise the angles ACB, ABC, and besides these the distance BC, then

$$AB = \frac{BC \; Sin.\, ACB}{Sin.\, (ACB + ABC)}.$$

Exam. When $BC = 738'$, $ACB = 24^\circ.\ 16'.\ 13''$, $ABC = 31^\circ.\ 5$, $AB = 368'\cdot734$.

Third Solution.

Let AB (*fig.* 58) be the distance to be measured, from which the point B is only accessible; C a point beyond it, and D a point in AB produced.

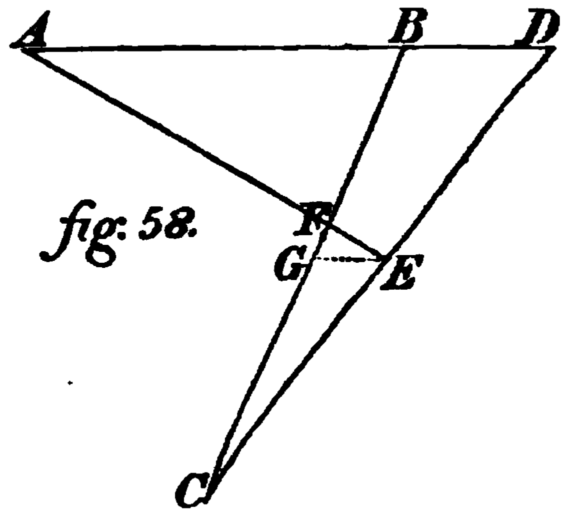


fig. 58.

1. In CD take any point E, and proceed in the direction EA, till a point F is arrived at, which is in a straight line with B, C; then measure the distances BD, BF, FC, CE, ED: from hence the distance AB may be determined. Let $BD=a$, $BF=b$, $CF = c$, $DE = d$, $EC = e$.

2. Draw EG parallel to BD; then because the triangles CGE, CBD are similar,

$$CD : CE = BD : GE$$

or $\qquad d + e : e = a : GE$

and $\qquad CD : CE = BC : CG$

or $\qquad d + e : e = b + c : CG,$

consequently $GE = \dfrac{ae}{d+e}$, $CG = \dfrac{(b+c)e}{d+e}$

$$FG = CF - CG = \frac{cd - be}{d+e}$$

3. The triangles EFG, AFB, are in like manner similar; we $\therefore$ have

$$FG : GE = BF : AB,$$

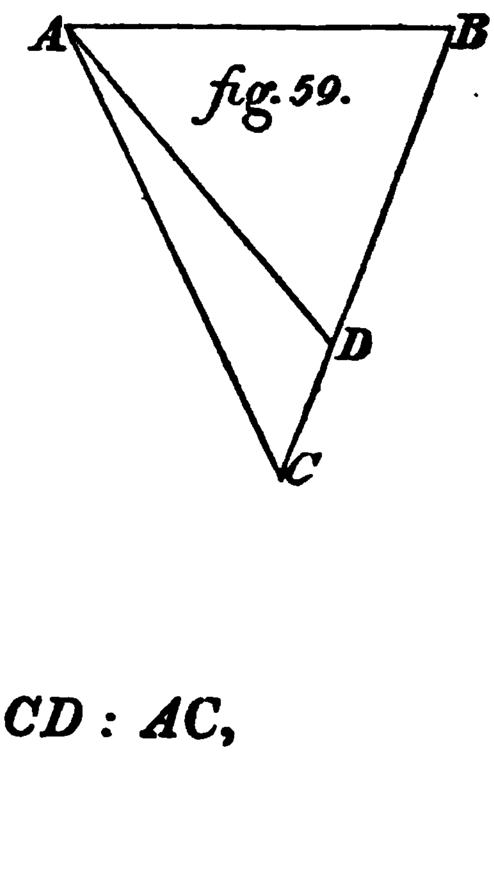
or $\frac{cd - be}{d + e} : \frac{ae}{d + e} = b : AB$,

consequently $AB = \frac{abe}{cd - be}$.

Exam. When $a = 250'$, $b = 76'$, $c = 132'$, $d = 140'$, $e = 80'$, $AB = 122\frac{18}{31}'$.

Fourth Solution.

Again, let AB (*fig.* 59) be a distance which is accessible only in B, but which cannot be produced in any direction.



1. Take two stations C, D, which are in a straight line with B, from which also both A and B may be seen; measure the distances CD, CB, and the angles ACB, ADB. Let $BC = a$, $CD = b$, $ADB = \alpha$, $ACB = \beta$.

2. In the triangle ACD

$$\text{Sin. } CAD : \text{Sin. } ADC = CD : AC,$$

or $\text{Sin. } (\alpha - \beta) : \text{Sin. } \alpha = b : AC$,

$$\therefore AC = \frac{b \text{ Sin. } \alpha}{\text{Sin. } (\alpha - \beta)}.$$

3. But in the triangle ACB,

$$AB^2 = BC^2 + AC^2 - 2BC \cdot AC \cdot \text{Cos. } \beta$$

or $$AB^2 = a^2 + \left(\frac{b \text{ Sin. } \alpha}{\text{Sin. } (\alpha - \beta)}\right)^2 - \frac{2\, ab \text{ Sin. } \alpha \text{ Cos. } \beta}{\text{Sin. } (\alpha - \beta)},$$

consequently $AB = \sqrt{\left[a^2 + \left(\frac{b \text{ Sin. } \alpha}{\text{Sin. } (\alpha-\beta)}\right)^2 - \frac{2ab\text{Sin.}\alpha\text{Cos.}\beta}{\text{Sin. } (\alpha-\beta)}\right]}$

Exam. Let $a = 500'$, $b = 67'$, $\alpha = 38^\circ.\ 7'$, $\beta = 21^\circ.\ 43'$.

Here

$$a^2 = 250000$$

$$\left(\frac{b \ Sin. \ \alpha}{Sin. \ (\alpha - \beta)}\right)^2 = 21455{\cdot}72$$

$$\frac{2 \ ab \ Sin. \ \alpha \ Cos. \ \beta}{Sin. \ (\alpha - \beta)} = 186081{\cdot}46$$

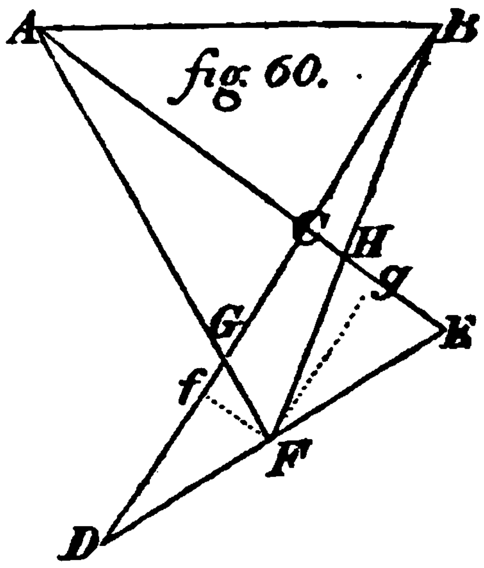
consequently $AB = \sqrt{135374{\cdot}26} = 367'{\cdot}932.$

SECTION XLVIII.

Prob. *To find the distance of two objects from one another, when neither of them is accessible.*

First Solution.

Let AB (*fig.* 60) be the distance to be measured.



1. Take a position C, extend the directions AC, BC, indefinitely towards D and E, and bisect DE in F. Measure from F towards A and B, and determine the points G, H, in which the sight-lines FA, FB, cut the lines CD, CE. Then measure the three sides of the triangle DCE, and the distances CG, CH. Let $CD = a$, $CE = b$, $DE = c$, $CG = d$, $CH = e$.

2. Draw Ff parallel to CE; then, because DE is bisected in F, $Cf = \frac{1}{2}a$, $Gf = \frac{1}{2}a - d$, and because the triangles AGC, FGf are similar,

$$Gf : Ff = CG : AC$$

or $$\tfrac{1}{2}a - d : \tfrac{1}{2}b = d : AC$$

consequently $$AC = \frac{bd}{a - 2d}$$

3. In like manner, when Fg is parallel to CD, we find

$$BC = \frac{ae}{b - 2e}.$$

4. After the lines AC, BC have been found, it will only be necessary, by the first solution, § XLVI, to measure these lines themselves, or proportional parts of them, backwards on the lines produced; then the distance AB will be deduced from hence.

COR. If we wish to determine AB by arithmetic, it may be done in the following way.

Since the three sides of the triangle DEC are given, we have

$$Cos.\ DCE = \frac{a^2 + b^2 - c^2}{2ab}.$$

But in the triangle ACB, we have

$$AB^2 = AC^2 + BC^2 - 2\,AC\,.\,BC\,.\,Cos.\ ACB.$$

If ∴ for AC, BC, and $Cos.\ ACB = Cos.\ DCE$, we substitute their values already found, we then obtain

$$AB = \sqrt{\left[\frac{b^2 d^2}{(a-2d)^2} + \frac{a^2 e^2}{(b-2e)^2} - \frac{de\,(a^2 + b^2 - c^2)}{(a-2d)\,(b-2e)}\right]};$$

to which expression we can also give the following form, which is more convenient for calculation by logarithms:

$$AB = \sqrt{\left[\left(\frac{bd}{a-2d} - \frac{ae}{b-2e}\right)^2 - \frac{de\,(a-b+c)\,(a-b-c)}{(a-2d)\,(b-2e)}\right]}.$$

EXAM. When $a = 156'$, $b = 98'$, $c = 187'$, $d = 63'$, $e = 34'$, $AB = 275'{\cdot}791$.

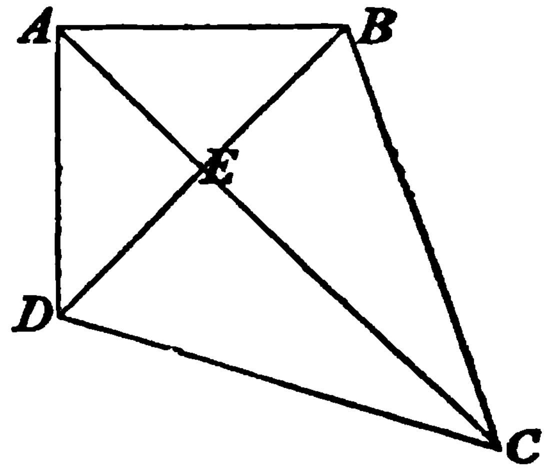
Second Solution.

Let AB (*fig.* 61) be the inaccessible distance.

1. Take any point whatever, C, measure towards A and B, and by these means determine the angle ACB; make

$ACD = ACB$, and proceed in the direction CD, till we find a point D, where the angle $BDC = 90^{\circ} - BCA$, and measure the distance CD, and the angle ADC. Let $BCA = ACD = \alpha$, $ADC = \beta$, $CD = a$.

fig. 61.



2. Since $ACD = \alpha$, and $BDC = 90^{\circ} - \alpha$, CED is a right angle, consequently $\triangle\ CED$ is similar to $\triangle\ BEC$, and $\therefore BC = CD$. Consequently also $\triangle\ ACD$ is similar to $\triangle\ ACB$, and $\therefore AD = AB$.

3. But in the triangle ACD

$$Sin.\ DAC : Sin.\ DCA = CD : AD$$

or $$Sin.\ (\alpha + \beta) : Sin.\ \alpha = a : AD\ (= AB),$$

consequently $$AB = \frac{a\ Sin.\ \alpha}{Sin.\ (\alpha + \beta)}.$$

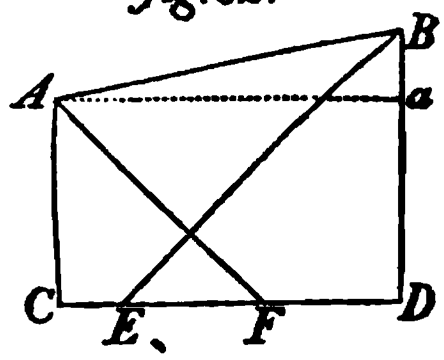
Exam. When $\alpha = 31^{\circ}.\ 5'$, $\beta = 113^{\circ}.\ 17'$, $a = 567'$, $AB = 502'{\cdot}463$.

Third Solution.

Let AB (*fig.* 62) be the inaccessible distance.

1. Take any position C, make $ACD = 90^{\circ}$, proceed in the direction CD, till the point D is arrived at, likewise $BDC = 90^{\circ}$, and in the line CD find two points E, F, so situated, that $AFC = BED = 45^{\circ}$.

fig. 62.



2. Having determined these points, measure the distances CD, CF, DE. Let $CD = a$, $CF = b$, $DE = c$. Draw Aa parallel to CD.

3. Then $aD = AC = CF = b$, and $BD = DE = c$; consequently $Ba = c - b$.

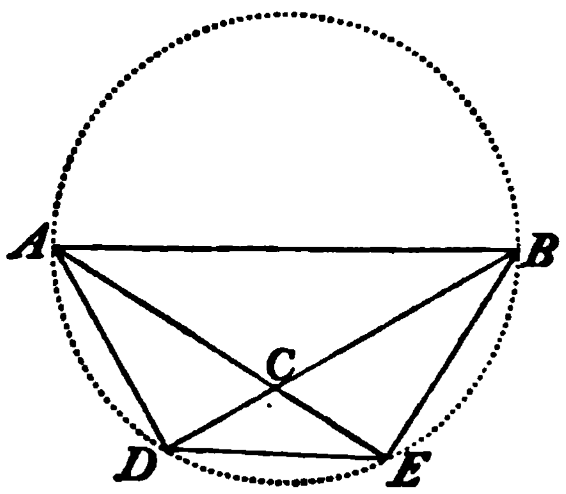
4. But in the right-angled triangle ABa, $AB^2 = Aa^2 + Ba^2 = CD^2 + Ba^2$; $\therefore$

$$AB = \sqrt{[a^2 + (c - b)^2]}$$

Fourth Solution.

Let AB (*fig*. 63) be the distance to be measured.

fig. 63.



1. Take any position C, where ACB is an obtuse angle, measure this angle in measuring towards A and B, then its adjacent angle ACD is also known. Now, on the sight-lines AC, BC, fix on two points D, E, in such a position, that $ADB = AEB = 90^\circ$, and measure the distance DE, then from hence AB may be determined. Let $DE = a$, $ACD = \alpha$.

2. Since ADB, AEB, by the construction are right angles, the points D, E are in a circle, whose radius is AB. If we suppose this circle actually described, then DAE is an angle at the circumference, and DE is the chord of the arc, upon which it stands; consequently, from known trigonometrical principles

$$DE = a = AB \text{ Sin. } DAE.$$

But $\text{Sin. } DAE = \text{Cos. } \alpha$, (because ADC is a right angle), $\therefore a = AB \text{ Cos. } \alpha$, and

$$AB = \frac{a}{\text{Cos. } \alpha}.$$

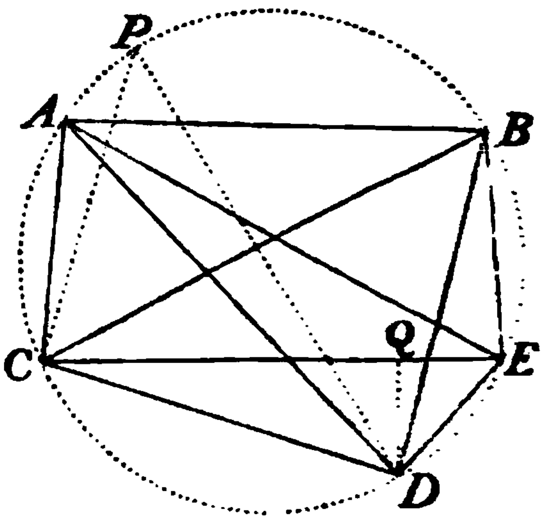
EXAM. When $a = 568'{\cdot}7$, $\alpha = 42^\circ.\ 19'{\cdot}7''$, $AB = 762'.362$.

Fifth Solution.

Let AB (*fig*. 64) be the distance to be measured.

1. Fix upon three stations C, D, E, so that the three angles ACB, ADB, AEB, which, in measuring towards A and B, include the sight-lines, are equal to one another; let each of these angles $= \alpha$. Then measure the three sides of the triangle CDE, and from these we can determine the distance AB. Let $CD = a$, $DE = b$, $CE = c$.

fig. 64.



2. Upon AB describe an arc, which includes the given angle α, and complete the circle; then this circle will be given by the three points C, D, E. Draw the diameter DP, the line PC, and the perpendicular DQ.

3. The triangles DCP, DQE are similar; for $DQE = DCP = R$, and $DEC = DPC$; consequently

$$DQ : DE = CD : DP,$$

or, because DQ, is the altitude of the triangle CDE, $= \frac{2\,\triangle\,CDE}{c}$,

$$\frac{2\,\triangle\,CDE}{c} : b = a : DP,$$

and $\therefore$

$$DP = \frac{abc}{2\,\triangle\,CDE}.$$

4. If in this expression we substitute for the triangle CDE its value from § XXIX, we then obtain

$$DP = \frac{2\,abc}{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}}.$$

5. When the diameter of the circle is found, it is easy to determine the chord AB. For since $AB = DP\ Sin.\ \alpha$, when

for DP its value is substituted,

$$AB = \frac{2\,abc\,Sin.\,\alpha}{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}},$$

which expression readily admits of being calculated by logarithms.

Exam. When $a = 197'$, $b = 113'$, $c = 235'$, $\alpha = 56^{\circ}.\ 29'$, $AB = 196'{\cdot}536$.

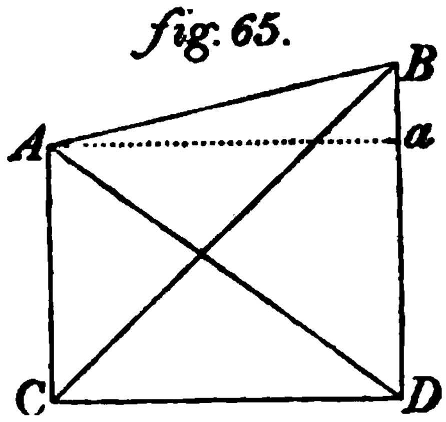
Cor. From 4 it appears at once, that when a, b, c, are the three sides of a triangle, the radius of the circle described about it =

$$\frac{abc}{\sqrt{(a+b+c)\,(a+b-c)\,(a+c-b)\,(b+c-a)}}.$$

Sixth Solution.

Let AB (*fig.* 65) be the distance to be measured.

fig. 65.



1. Take any position C, make the right angle ACD, and proceed in the direction CD, as far as the point D, where also BDC is a right angle; measure from C towards B and D, likewise from D towards A and C, and determine by these means the angles BCD, ADC; measure also the line CD. Let $BCD = \alpha$, $ADC = \beta$, $CD = a$.

2. From the right-angled triangles ACD, BDC, we obtain,

$$BD = a\ Tan.\ \alpha,\ AC = a\ Tan.\ \beta.$$

If Aa is drawn parallel to CD; then

$$Aa = a,\ Ba = a\,(Tan.\ \alpha - Tan.\ \beta).$$

3. Therefore in the right-angled triangle AaB

$$AB^2 = a^2 + a^2\,(Tan.\ \alpha - Tan.\ \beta)^2$$

and consequently $AB = a\sqrt{[1 + (Tan.\ \alpha - Tan.\ \beta)^2]}$.

4. In order the more easily to calculate the expression found for AB, put $Tan.\ \alpha - Tan.\ \beta = Tan\ \phi$, $\therefore$ find an angle ϕ such, that its tangent is equal to the difference of the tangents of the two angles α, β. Having found this angle, then

$$AB = a\sqrt{(1 + Tan.^2\ \phi)} = a\ Sec.\ \phi$$

EXAM. Let $a = 1375'$, $\alpha = 65^\circ.\ 17'$, $\beta = 39^\circ.\ 48'$. Here

$$Tan.\ \phi = Tan.\ \alpha - Tan.\ \beta = 2{\cdot}1724911 - 0{\cdot}8331686$$
$$= 1{\cdot}3393225$$

consequently $\phi = 53^\circ.\ 15'\ 11''$.

Hence we obtain,

$$log.\ Sec.\ \phi = log.\ Sec.\ 53^\circ.\ 15'.\ 11'' = 0{\cdot}2230941$$
$$log.\ a\ Sec.\ \phi = 3{\cdot}3613968$$
$$\therefore AB = 2298'{\cdot}247.$$

Seventh Solution.

Again let AB (*fig.* 65) be the distance to be measured.

1. Take any two positions C, D; measure the angles ACD, BCD, BDC, ADC also the line of vision CD. Let $CD = a$, $ACD = \alpha$, $BCD = \beta$, $BDC = \gamma$, $ADC = \delta$; then in the triangle ACD, $\angle\ CAD = 180^\circ - \alpha - \delta$, and in the triangle BCD, $\angle\ CBD = 180^\circ - \gamma - \beta$. Since $\therefore$ these last angles are also known, for shortness' sake, put $CAD = A$, $CBD = B$.

2. In the triangle CAD we $\therefore$ have

$$Sin.\ A : Sin.\ \delta = a : AC,$$

and in the triangle CBD

$$Sin.\ B : Sin.\ \gamma = a : BC,$$

consequently $AC = \dfrac{a\ Sin.\ \delta}{Sin.\ A}$, $BC = \dfrac{a\ Sin.\ \gamma}{Sin.\ B}$.

3. Now, since in the triangle ACB, both the two sides AC, BC, and also the angle included by them $ACB = \alpha - \beta$,

are known, we $\therefore$ obtain

$$AB = a\sqrt{\left[\left(\frac{Sin.\ \delta}{Sin. A}\right)^2 + \left(\frac{Sin.\ \gamma}{Sin.\ B}\right)^2 - \frac{2\ Sin.\ \delta\ Sin.\ \gamma\ Cos.\ (\alpha-\beta)}{Sin.\ A\ Sin.\ B}\right]}.$$

In this expression we must calculate each part of the magnitudes under the radical sign separately; but the calculation will be essentially shortened, by using in the third part the logarithms of $\frac{Sin.\ \delta}{Sin. A}$ and $\frac{Sin.\ \gamma}{Sin.\ B}$, which must be calculated for the two first parts

EXAM. Let $\alpha = 110^0$, $\beta = 37^0.\ 40'$, $\gamma = 117^0.\ 30'$, $\delta = 38^0.\ 20'$, $a = 750'$. Here $A = 31^0.\ 40'$, $B = 24^0.\ 50'$,

$$\left(\frac{Sin.\ \delta}{Sin. A}\right)^2 = 1{\cdot}395832$$

$$\left(\frac{Sin.\ \gamma}{Sin.\ B}\right)^2 = 4{\cdot}460672$$

$$\frac{2\ Sin.\ \delta\ Sin.\ \gamma\ Cos.\ (\alpha - \beta)}{Sin.\ A\ Sin.\ B} = 1{\cdot}514520;$$

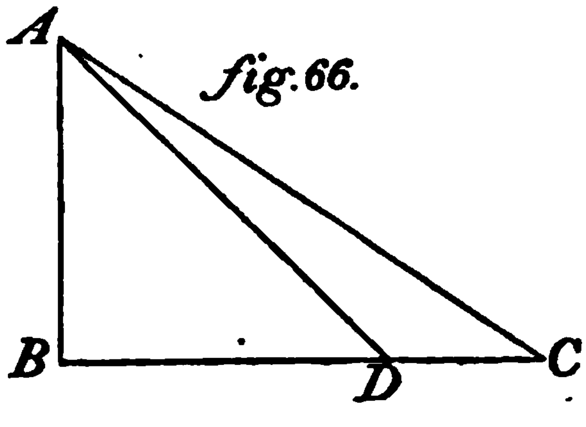
consequently $AB = 750\sqrt{4{\cdot}341984} = 1562'{\cdot}80$.

SECTION XLIX.

PROB. *To find the altitude of an object, for instance of a tower, when its base lies in the same horizontal plane with a chosen or given station, under the supposition, that the distance from this station to the object can be measured.*

First Solution.

Let AB (*fig.* 66) be the altitude of the object to be measured, or, more properly, the vertical line, which is drawn from the highest point of the object to its lowest, in the horizontal plane; let C be the chosen or given station, and $\therefore$, according to the hypothesis, BC is a horizontal line.



Measure CB, and the angle of elevation ACB. Let $CB = a$, $ACB = \alpha$; then in the right-angled triangle ABC,

$$AB = a \; Tan. \; \alpha$$

Exam. When $a = 367'$, $\alpha = 32^\circ. \; 17'.23''$, $AB = 231' \cdot 915$.

Second Solution.

If the distance from C to B cannot be measured in a straight line, measure only a part of the line CB, say CD; also in C, D, the angles of elevation ACB, ADB. Let $CD = a$, $ACB = \alpha$, $ADB = \beta$.

In the triangle ADC, if the side CD, and the angles ACD, DAC $(= \beta - \alpha)$ be given; therefore

$$AD = \frac{a \; Sin. \; \alpha}{Sin. \; (\beta - \alpha)}.$$

In the right-angled triangle ABD we $\therefore$ have the side AD, and the angle ADB; consequently

$$AB = AD \; Sin. \; \beta = \frac{a \; Sin. \; \alpha \; Sin. \; \beta}{Sin. \; (\beta - \alpha)}.$$

Also BD, and consequently BC, may be determined; for

$$BD = AD \; Cos. \; \beta = \frac{a \; Sin. \; \alpha \; Cos. \; \beta}{Sin. \; (\beta - \alpha)}.$$

Exam. When $a = 967'$, $\alpha = 7^\circ. \; 5'. \; 13''$, $\beta = 16^\circ. \; 48'. \; 5''$, $AB = 205' \cdot 131$, and $BD = 682' \cdot 955$.

SECTION L.

Prob. *To find the altitude of an object, when its lowest point is not in the same horizontal plane with the given station, on the supposition, that the distance from this station to the object can be measured.*

First Solution.

Let AB (*fig.* 67) be the altitude to be measured, and C the given station.

1. Arrange the telescope of the protractor to the horizontal direction Cc, then direct it towards A and B, and determine by these means the angle of elevation ACc, and the angle of depression BCc. If now the side CB is measured, then AB may be determined.

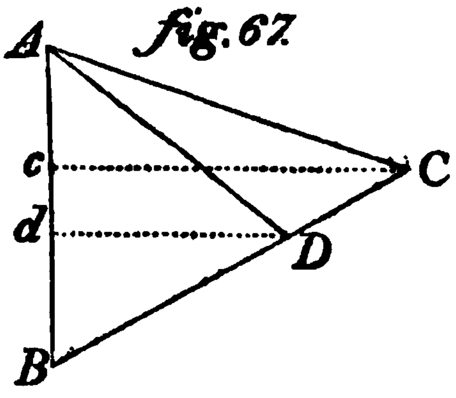


fig. 67

2. For let $ACc = \alpha$, $BCc = \beta$; then $ACB = \alpha + \beta$, $BAC = 90^0 - \alpha$; consequently in the triangle BAC, one side and two angles are given; $\therefore$, because $Sin. (90^0 - \alpha) = Cos. \alpha$,

$$AB = \frac{a \; Sin. (\alpha + \beta)}{Cos. \alpha}$$

Cor. If, as in *fig.* 68, C is lower than B, it will merely be necessary to assume the angle $BCc = \beta$ to be negative, and we then obtain

$$AB = \frac{a \; Sin. (\alpha - \beta)}{Cos. \alpha}.$$

which may also be very easily proved from the figure itself.

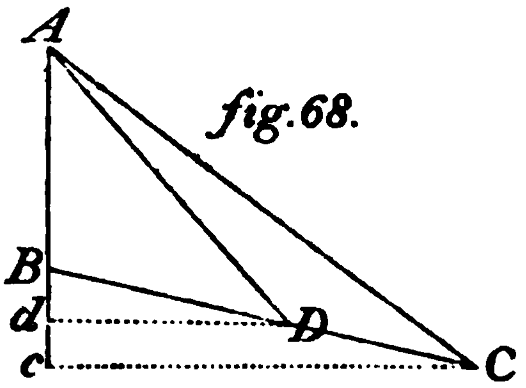


fig. 68.

Exam. Let C be higher than B, and $\alpha = 19^{\circ}. 7'$, $\beta = 25^0. 18'$, $a = 1852'{\cdot}7$; then $AB = 1000'{\cdot}509$.

Second Solution.

1. If the distance from C to B cannot be measured, from C measure a part CD only; determine at C the angle of elevation ACc and angle of depression BCc, also at D the angle of elevation ADd. It is not necessary to measure the angle BDd, because $BDd = BCc$.

2. Let $ACc = \alpha$, $BCc = \beta$, $ADd = \gamma$; then $ACB = \alpha + \beta$, $ADB = \gamma + \beta$; and $\therefore$ $CAD = ADB - ACB = \gamma - \alpha$. Consequently in the triangle CAD all the angles and the side CD are known; consequently

$$AD = \frac{a \, Sin. \, (\alpha + \beta)}{Sin. \, (\gamma - \beta)}.$$

3. Now in the triangle BAD, the angles ADB, ABD ($= 90^0 - \beta$), and the side AD are known; consequently also AB. Thus

$$AB = \frac{AD \, Sin. \, (\gamma + \beta)}{Cos. \, \beta} = \frac{a \, Sin. \, (\alpha + \beta) \, Sin. \, (\gamma + \beta)}{Sin. \, (\gamma - \alpha) \, Cos. \, \beta}.$$

Cor. In *fig.* 67 it was assumed, that C was higher than B. But if (*fig.* 68) C be lower than B, it will only be necessary in the expression found for AB, to substitute $-\beta$ for β; we then get, because $Cos. -\beta = Cos. \, \beta$,

$$AB = \frac{a \, Sin. \, (\alpha - \beta) \, Sin. \, (\gamma - \beta)}{Sin. \, (\gamma - \alpha) \, Cos. \, \beta};$$

which may also be very easily proved immediately from the figure itself.

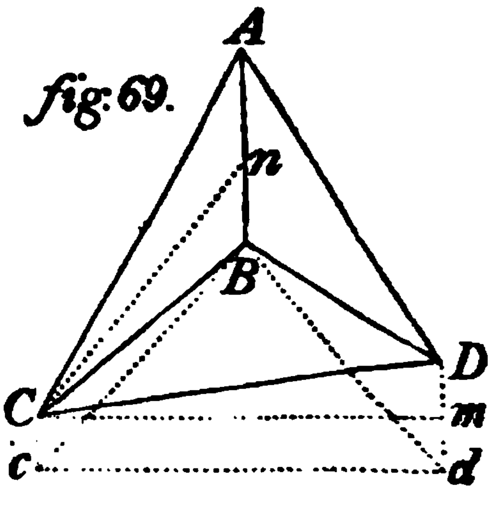
Exam. In *fig.* 68, let $\alpha = 29^0$, $\beta = 17^0. \, 6'$, $\gamma = 32^0. \, 49'{\cdot}8''$, $a = 1152'$; then $AB = 1010'{\cdot}953$.

SECTION LI.

Prob. *To find the altitude of an object, when it is impracticable, from the chosen or given station in the direction of the object, to measure either forward or backward.*

First Solution.

Let AB (*fig.* 69) be the altitude to be measured, and C the given station.



1. Measure any horizontal station $CD = a$. Now, if this line be in the same horizontal plane BCD, as the lowest point B of the altitude AB; measure the horizontal angles BCD, BDC, also one of the angles of elevation ACB, ADB, viz. ACB. Let $BCD = \alpha$, $BDC = \beta$, $ACB = \gamma$.

2. Then in the triangle CBD there are two angles BCD, BDC, and the side CD given; consequently

$$CB = \frac{a\ Sin.\ \beta}{Sin.\ (\alpha + \beta)}.$$

3. But in the right-angled triangle ABC, having the right angle at B,

$$AB = BC\ Tan.\ \gamma;$$

if for BC its value be substituted from 2, we obtain

$$AB = \frac{a\ Sin.\ \beta\ Tan.\ \gamma}{Sin.\ (\alpha + \beta)}.$$

EXAM. When $\alpha = 79^{\circ}.\ 45'$, $B = 61^{\circ}.\ 4'$, $\gamma = 14^{\circ}.\ 19'.27''$, $a = 857'$, $AB = 303'{\cdot}128$.

Second Solution.

1. If no station can be found which is in the same horizontal plane with B, let CD be any other line in any position whatever; respecting which, for the sake of greater generality, I shall assume, that neither C nor D is in the same horizontal plane with B.

2. Suppose a horizontal plane cBd drawn through B, which cuts the vertical lines Cc, Dd, drawn from C, D, in c, d. It is well known that at C, D the horizontal angles Bcd, Bdc, may be measured, although these points are higher than c, d. Measure the said angle, the station CD, likewise the angle of elevation ACn, and the angle of depression BCn, of the altitude to be measured, Cn being an horizontal line. Let $CD = a$, $Bcd = \alpha$, $Bdc = \beta$, $ACn = \gamma$, $BCn = \delta$.

3. If now we suppose the horizontal line Cm drawn, then also the vertical angle DCm may be measured; let this angle $= \varepsilon$. Now since Cm is horizontal, and Dd vertical, consequently DCm is a right-angled triangle, and $\therefore$

$$Cm = a\ Cos.\ \varepsilon.$$

Likewise $cd = Cm$, because $Ccdm$ is a parallelogram.

4. Consequently in the triangle Bcd, the two angles Bcd,

Bdc, and the side *cd* are known; ∴

$$cB = \frac{cd \; Sin. \; \beta}{Sin. \; (\alpha + \beta)} = \frac{a \; Cos. \; \varepsilon \; Sin. \; \beta}{Sin. \; (\alpha + \beta)}.$$

Also $cB = Cn$, because $CcBn$ is a parallelogram.

5. Since AB is vertical, and Cn horizontal; ∴ ACn, BCn, are right-angled triangles; consequently

$$An = Cn \; . \; Tan. \; \gamma, \; Bn = Cn \; . \; Tan. \; \delta; \; ∴$$

$$AB = Cn \; (Tan. \; \gamma + Tan. \; \delta) = \frac{Cn \; . \; Sin. \; (\gamma + \delta)}{Cos. \; \gamma \; Cos. \; \delta},$$

or, when we substitute for $Cn = cB$ its value taken from 4,

$$AB = \frac{a \; Cos. \; \varepsilon \; Sin. \; \beta \; Sin. \; (\gamma + \delta)}{Sin. \; (\alpha + \beta) \; Cos. \; \gamma \; Cos. \; \delta}.$$

Cor. By this method, we at the same time find the altitudes of the stations C, D above the horizontal plane cBd; for $Cc = Bn$, $Dd = Dm + md = Dm + Cc$; consequently

$$Cc = \frac{a \; Cos. \; \varepsilon \; Sin. \; \beta \; Tan. \; \delta}{Sin. \; (\alpha + \beta)}$$

$$Dd = \frac{a \; Cos. \; \varepsilon \; Sin. \; \beta \; Tan. \; \delta}{Sin. \; (\alpha + \beta)} + a \; Sin. \; \varepsilon.$$

The measured angles may also be negative; the mode of proceeding in this case is given in the following example.

Exam. Let AB represent a church-steeple in a valley, whose height is required to be determined from a measured station CD on a neighbouring hill. I assume, that the station C is higher than the top of the spire A, and D lower than C, and that the following are known, viz. $a = 357' \cdot 3$, $\alpha = 85^0. \; 37'. \; 14''$, $\beta = 79^0. \; 13'. \; 12''$, $\gamma = -13^0. \; 5'. \; 49''$, $\delta = 20^0. \; 18'. \; 9''$, $\varepsilon = -3^0. \; 48'. \; 10''$. Now, since whatever angle ϕ may denote, $Cos. -\phi = Cos. \; \phi$, and $Sin. -\phi = -Sin. \; \phi$; we obtain

$$AB = \frac{357' \cdot 3 \; Cos. \; 3^0. \; 48'. \; 10'' \; Sin. \; 79^0. \; 13'. \; 12'' \; Sin. \; 7^0. \; 12'. \; 20''}{Sin. \; 164^0. \; 50'. \; 26'' \; Cos. \; 13^0. \; 5'. \; 49'' \; Cos. \; 20^0. \; 18'. \; 9''}$$

$$= 183' \cdot 892$$

$$Cc = \frac{357'{\cdot}3 \; Cos. 3^{\circ}. 48'. 10'' \; Sin. 79^{\circ}. 13'. 12'' \; Tan. 20^{\circ}. 18'. 9''}{Sin. \; 164^{\circ}. \; 50'. \; 26}$$

$$= 495'{\cdot}470$$

$$Dd = Cc - 357'{\cdot}3 \; Sin. \; 3^{\circ}. \; 48'. \; 10'' = 471'{\cdot}773.$$

SECTION LII.

PROB. *A person is standing upon a tower, whose height above the horizontal plane upon which it stands is known: the person wishes, without moving from the spot, to determine the distance between two objects which are in the above-mentioned horizontal plane.*

SOLUT. Let AB (*fig.* 69) represent the tower, whose altitude $= h$; a person standing at A, wishes from it to measure the distance CD, which is in the same horizontal plane CBD with B.

Measure the angles $CAB = \alpha$, $DAB = \beta$, $CAD = \gamma$; then CD may be found. For, since ABC, ABD are two right-angled triangles, and the line AB, together with the angles CAB, DAB are known, therefore AC, AD may be determined: thus

$$AC = h \; sec. \; \alpha, \quad AD = h \; sec. \; \beta.$$

Now, since in the triangle ACD, the sides AC, AD, and the angle CAD, are known, CD may be found: thus

$$CD = h \sqrt{(sec. \; \alpha^2 + sec. \; \beta^2 - 2 \, sec. \; \alpha \; sec. \; \beta \; Cos. \; \gamma)}.$$

EXAM. When $\alpha = 56^{\circ}. \; 34'$, $\beta = 69^{\circ}. \; 12'$, $\gamma = 81^{\circ}. \; 20'$, $h = 214'$, $CD = 665'{\cdot}949$.

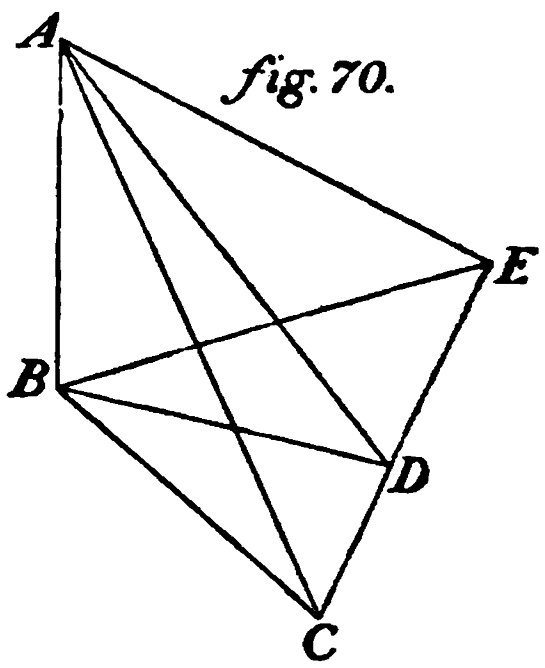
SECTION LIII.

PROB. *An object standing vertically, for instance a tower, is seen from three stations, which are in a straight line, and in the same horizontal plane upon which the object stands; the distance of these three*

stations from one another, also the angles at which we see the object from each of these stations, are given: find the height of the object, and its distance from each of the three points.

Solut. Let AB (*fig.* 70) be the object, CE any straight line, respecting which it is assumed, that it is in the same horizontal plane with B, and C, D, E are the three stations in this plane. The angles $ACB=\alpha$, $ADB=\beta$, $AEB=\gamma$, likewise the distances $CD=a$, $DE=b$ are given: find the altitude AB, and the distances BC, BD, BE.

fig. 70.



1. If the line BD can be found, then, from the right-angled triangle ABD, in which both the line BD, and the angle ADB are known, the altitude AB may be calculated. Having found this, then the right-angled triangles ABC, ABE, in which the angles ACB, AEB are known, also give the distances BE, BC. Let $\therefore BD = x$.

2. Then the right-angled triangles ABC, ABD, ABE, give

$$AB = x \text{ Tan. } \beta = BC \text{ Tan. } \alpha = BE \text{ Tan. } \gamma,$$

consequently $BC = \dfrac{x \text{ Tan. } \beta}{\text{Tan. } \alpha}$, $BE = \dfrac{x \text{ Tan. } \beta}{\text{Tan. } \gamma}$

3. But in the triangle BCD,

$$\text{Cos. } BDC = \frac{BD^2 + CD^2 - BC^2}{2\,BD\,.\,CD} = \frac{x^2 + a^2 - \dfrac{x^2 \text{ Tan}^2\, \beta}{\text{Tan.}^2\, \alpha}}{2\,ax}$$

and in the triangle BED,

$$\text{Cos. } BDE = \frac{BD^2 + DE^2 - BE^2}{2\,BD\,.\,DE} = \frac{x^2 + b^2 - \dfrac{x^2 \text{ Tan.}^2\, \beta}{\text{Tan.}^2\, \gamma}}{2\,bx}.$$

Now since

$BDC + BDE = 180^0$: $Cos.\ BDC = -Cos.\ BDE$, $\therefore$

$$\frac{x^2 + a^2 - \dfrac{x^2\ Tan.^2\ \beta}{Tan.^2\ \alpha}}{2\ ax} = -\frac{x^2 + b^2 - \dfrac{x^2\ Tan.^2\ \beta}{Tan.^2\ \gamma}}{2\ bx},$$

or $bx^2 + a^2b - \dfrac{bx^2\ Tan.^2\ \beta}{Tan.^2\ \alpha} = -ax^2 - ab^2 + \dfrac{ax^2\ Tan.^2\ \beta}{Tan.^2\ \gamma}$,

or $\dfrac{ax^2\ (Tan.^2\ \beta - Tan.^2\ \gamma)}{Tan.^2\ \gamma} + \dfrac{bx^2\ (Tan.^2\ \beta - Tan.^2\ \alpha)}{Tan.^2\ \alpha}$

$$= (a + b)\ ab.$$

4. Now

$$Tan.^2\ \beta - Tan.^2\ \gamma = (Tan.\ \beta + Tan.\ \gamma)\ (Tan.\ \beta - Tan.\ \gamma)$$

$$= \frac{Sin.\ (\beta + \gamma)\ Sin.\ (\beta - \gamma)}{Cos.^2\ \beta\ Cos.^2\ \gamma}$$

and in like manner

$$Tan.^2\ \beta - Tan.^2\ \alpha = (Tan.\ \beta + Tan.\ \alpha)\ (Tan.\ \beta - Tan.\ \alpha)$$

$$= \frac{Sin.\ (\beta + \alpha)\ Sin.\ (\beta - \alpha)}{Cos.^2\ \beta\ Cos.^2\ \alpha}:$$

also, $Tan.\ \alpha = \dfrac{Sin.\ \alpha}{Cos.\ \alpha}$, $Tan.\ \gamma = \dfrac{Sin.\ \gamma}{Cos.\ \gamma}$. If these values be substituted in the foregoing equation, after the requisite reductions we obtain

$$\frac{ax^2\ Sin.\ (\beta + \gamma)\ Sin.\ (\beta - \gamma)}{Sin.^2\ \gamma\ Cos.^2\ \beta} + \frac{bx^2\ Sin.\ (\beta + \alpha)\ Sin.\ (\beta - \alpha)}{Sin.^2\ \alpha\ Cos.^2\ \beta}$$

$$= (a + b)\ ab$$

and hence

$$x = Sin.\ \alpha Sin.\ \gamma Cos\ \beta \sqrt{\frac{(a + b)\ ab}{\begin{bmatrix} a\ Sin.(\beta + \gamma) Sin.(\beta - \gamma) Sin.^2 \alpha \\ + b\ Sin.(\beta + \alpha) Sin.(\beta - \alpha) Sin.^2 \gamma \end{bmatrix}}}$$

or when, for the sake of brevity, we substitute

$$a\ Sin.\ (\beta + \gamma)\ Sin.\ (\beta - \gamma)\ Sin.^2\ \alpha = A,$$

$$b\ Sin.\ (\beta + \alpha)\ Sin.\ (\beta - \alpha)\ Sin.^2\ \gamma = B,$$

$$x = Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \beta \sqrt{\frac{(a + b) ab}{A + B}} = BD$$

5. Hence we further obtain

$$AB = x \, Tan.\, \beta = Sin.\, \alpha \, Sin\, \beta \, Sin.\, \gamma \sqrt{\frac{(a+b)\,ab}{A+B}}$$

$$BC = \frac{AB}{Tan.\, \alpha} = Sin.\, \beta \, Sin.\, \gamma \, Cos.\, \alpha \sqrt{\frac{(a+b)\,ab}{A+B}}$$

$$BE = \frac{AB}{Tan.\, \gamma} = Sin.\, \alpha \, Sin.\, \beta \, Cos.\, \gamma \sqrt{\frac{(a+b)\,ab}{A+B}}$$

Exam. Let $\alpha = 19^\circ.\, 27'.\, 15''$, $\beta = 13^\circ.\, 4'.\, 7''$, $\gamma = 10^\circ.\, 48'.\, 25''$, $a = 1750'$, $b = 1047'$. Here $\beta - \alpha = -6^\circ.\, 23'.\, 8''$, and $Sin.\, (\beta - \alpha) = -Sin.\, 6^\circ.\, 23'.\, 8''$: consequently B is negative, and we have

$$\begin{aligned} A &= \quad a \, Sin.\, (\beta+\gamma)\, Sin.\, (\beta-\gamma)\, Sin.^2 \alpha = \quad 3{\cdot}100596 \\ B &= -\, b \, Sin.\, (\beta+\alpha)\, Sin.\, (\alpha-\beta)\, Sin.^2 \gamma = -2{\cdot}200971 \\ & \qquad A + B = 0{\cdot}899625 \\ & \qquad log.\, (A + B) = 0{\cdot}9540615 - 1 \\ & \qquad log.\, \sqrt{\frac{(a+b)\,ab}{A+B}} = 4{\cdot}8778078, \end{aligned}$$

hence

$$\begin{aligned} AB &= 1065'{\cdot}754 \\ BC &= 3017'{\cdot}266 \\ BD &= 4591'{\cdot}201 \\ BE &= 5583'{\cdot}320 \end{aligned}$$

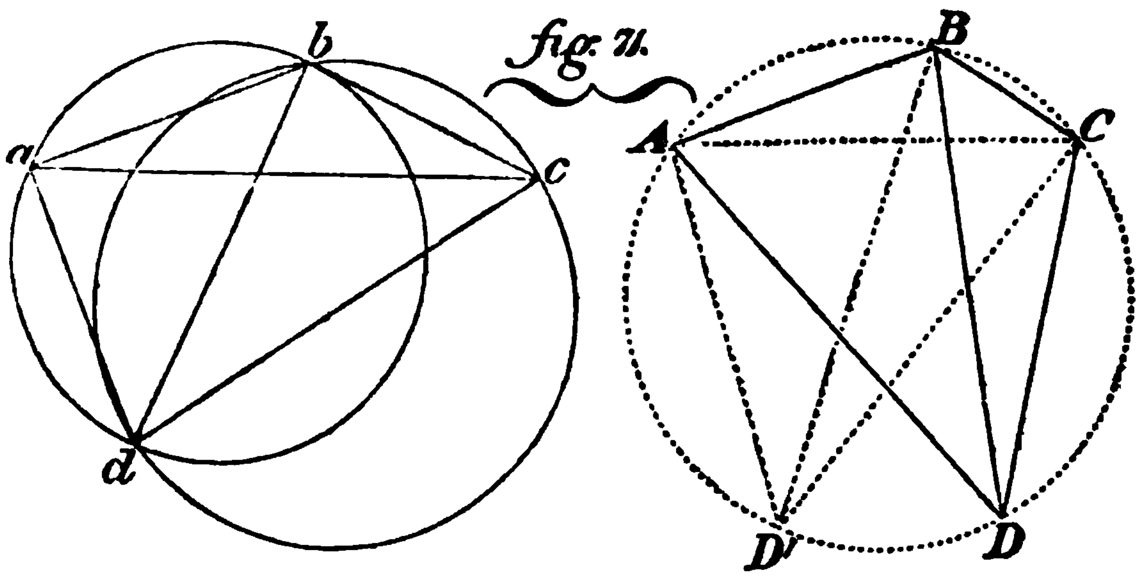
Remark. The visual angles BCA, BDA, BEA, are in fact no other than the angles of elevation of the point A above the horizon, taken from the three stations C, D, E. Now, since it is not necessary to measure towards B, in order to find these angles, we can then, by means of the formulæ already found, determine the altitude of this point A above the horizontal plane, when also from the altitude AB only this point is visible, and in this respect the problem is of great use in practical Geometry.

SECTION LIV.

Prob. *Three places, whose situation is known, are seen from a fourth place, which is in the same plane with*

the others, and there the angle is measured, which the sight-lines make with one another: required to determine the distance of this fourth place from the other three, and also its situation.

SOLUT. Let A, B, C (*fig.* 71) be the three places; the



distances $AB = a$, $BC = b$, and the angle $ABC = \alpha$ are given. D is the fourth place, where the angles $ADB = \beta$, $BDC = \gamma$ are measured: required to determine the distances AD, BD, CD, and the situation of the point D in reference to A, B, C.

1. If the angle BAD be found; then in the quadrilateral figure $ABCD$ the three angles ABC, BAD, ADC are known, consequently also the fourth BCD. If ∴ in each of the two triangles ABC, DBC, there are two angles and one side known; these two triangles are determined, and the sides AD, BD, CD can be calculated. Let ∴ the unknown angle $BAD = \phi$.

2. Since the four angles of every quadrilateral figure are together $= 4\ R$, $BCD = 360^0 - ABC - ADC - BAD$ $= 360^0 - \alpha - \beta - \gamma - \phi$, or when we abbreviate it by putting $360 - (\alpha + \beta + \gamma) = \mu$, $BCD = \mu - \phi$.

3. From the two angles BAD, ADB, and the side AB of the triangle ABD, we obtain

$$BD = \frac{a\ Sin.\ \phi}{Sin.\ \beta},$$

and from the two angles BCD, BDC, and the side BC of the triangle DBC,

$$BD = \frac{b \, Sin. \, (\mu - \phi)}{Sin. \, \gamma}.$$

We .·. have

$$\frac{a \, Sin. \, \phi}{Sin. \, \beta} = \frac{b \, Sin. \, (\mu - \phi)}{Sin. \, \gamma},$$

or $\quad a \, Sin. \, \gamma \, Sin. \, \phi = b \, Sin. \, \beta \, Sin. \, (\mu - \phi),$

or also

$$a \, Sin. \, \gamma \, Sin. \, \phi = b \, Sin. \, \beta \, (Sin. \, \mu \, Cos. \, \phi - Cos. \, \mu \, Sin. \, \phi)$$

If both sides of this equation be divided by $Sin. \, \phi$, we get

$$a \, Sin. \, \gamma = b \, Sin. \, \beta \, (Sin. \, \mu \, Cot. \, \phi - Cos. \, \mu),$$

whence we obtain

$$Cot. \, \phi = Cot. \, \mu + \frac{a \, Sin. \, \gamma}{b \, Sin. \, \beta \, Sin. \, \mu}.$$

4. If the angle ϕ is found; we then have

$$AD = \frac{a \, Sin. \, (\beta + \phi)}{Sin. \, \beta},$$

$$BD = \frac{a \, Sin. \, \phi}{Sin. \, \beta},$$

$$CD = \frac{b \, Sin. \, (\alpha + \beta + \phi - 180^0)}{Sin. \, \gamma}.$$

Cor. 1. Since this problem is of great use in practice, it is .·. worth while to consider a few particular cases, which are contained in the general solution.

1. If $\alpha = 180^0$, or if the point B falls on the line AC, then we have $\mu = 360^0 - 180^0 - \beta - \gamma = 180^0 - (\beta + \gamma)$; consequently $Cot. \, \mu = - Cot. \, (\beta + \gamma)$, $Sin. \, \mu = Sin. \, (\beta + \gamma)$. We have .·. for this case

$$Cot. \, \phi = - Cot. \, (\beta + \gamma) + \frac{a \, Sin. \, \gamma}{b \, Sin. \, \beta \, Sin. \, (\beta + \gamma)};$$

or also, because $Cot.\ (\beta + \gamma) = \dfrac{Cos.\ (\beta + \gamma)}{Sin.\ (\beta + \gamma)}$,

$$Cot.\ \phi = \frac{a\ Sin.\ \gamma - b\ Sin.\ \beta\ Cos.\ (\beta + \gamma)}{b\ Sin.\ \beta\ Sin.\ (\beta + \gamma)}.$$

The first of these expressions is, however, the most convenient for calculation.

2. If the point B be under the line AC, then the convex angle ABC within the quadrilateral figure $ABCD$, and not the concave one ABC, must be taken for α, because the former, and not the latter, together with the three remaining angles of the quadrilateral figure are together $= 4\ R$, as was supposed in the solution.

3. If $ABC + ADC = 2\ R$, or $\alpha + \beta + \gamma = 180^0$: then $\mu = 180^0$, $Cot.\ \mu = -\infty$, $Sin\ \mu = 0$. We ∴ obtain from 3,

$$Cot.\ \phi = -\ \infty + \frac{a\ Sin.\ \gamma}{0}.$$

Consequently the expression for $Cot.\ \phi$ appears here in a form, from which its value cannot be determined.

Describe a circle about the triangle ABC; then, because by the hypothesis $ABC + ADC = 2\ R$, the point D must necessarily fall on the circumference of this circle. This limitation does not obtain, when the angles α, β are assumed to be arbitrary; much more, on account of the given situation of the three points A, B, C, the angles β, γ must be of that magnitude required by the condition, that the point D falls on the circumference of the circle described about the triangle ABC. But under the supposition that the angles β, γ are so assumed that the problem is possible, every point in the circumference $ABCD$ will verify the problem. For let D' be any other in the circumference; then $AD'B = ADB$, $BD'C = BDC$.

As regards the calculation, it is known from Trigonometry, that when ψ denotes any concave or convex angle; for all angles between 90^0 and 180^0, $Cot.\ \psi = -\ Cot.\ (180^0 - \psi)$; for all angles between 180^0 and 270^0, $Cot.\ \psi =$ $+\ Cot.\ (\psi - 180^0)$; and for all angles between 270^0 and 360^0,

$$Cot.\ \psi = -\ Cot.\ (360^0 - \psi).$$

COR. 2. If it is required to find the point D arithmetically only, describe the triangle ABC on paper, then make a triangle abc, which is similar to the former: on ab, as a chord, describe a circular arc adb, which subtends the given angle β; also on bc describe a circular arc bdc, which subtends the given angle γ. The point of intersection d of these two circles, will then give the fourth place on the paper; thus the point d in reference to a, b, c, will have the same situation as the point D has in reference to A, B, C. The reason of this is easily seen.

EXAM. 1. Let $a = 1153'{\cdot}7$, $b = 849'{\cdot}43$, $\alpha = 112^\circ.\ 25'$, $\beta = 27^\circ.\ 31'$, $\gamma = 19^\circ.\ 14'$. Here $\mu = 360^\circ - (\alpha + \beta + \gamma) = 200^\circ.\ 50'$; consequently $Sin.\ \mu = -Sin.\ 20^\circ.\ 50'$, $Cot.\ \mu = Cot.\ 20^\circ.\ 50'$. We have $\therefore$

$$\frac{a\ Sin.\ \gamma}{b\ Sin.\ \beta\ Sin.\ \mu} = -\frac{1153{\cdot}7\ .\ Sin.\ 19^\circ.\ 14'}{849{\cdot}43\ Sin.\ 27^\circ.\ 31'\ Sin.\ 20^\circ.\ 50'}$$

$$= -2{\cdot}7229400$$

$$Cot.\ \mu = Cot.\ 20^\circ.\ 50' = 2{\cdot}6279121$$

$$\therefore\ Cot.\ \phi = -0{\cdot}0950279.$$

Since the cotangent here has been found to be negative, ϕ is an obtuse angle. Find $\therefore$ in the Tables an angle to which the positive cotangent $0{\cdot}0950279$ belongs; we find it to be $84^\circ.\ 34'.\ 18''$. This angle being subtracted from 180°, gives $\phi = 95^\circ.\ 25'.\ 42''$; from which the distances AD, BD, CD may be very easily calculated.

EXAM. 2. Let $a = 1490'$, $b = 768'$, $\alpha = 235^\circ$, $\beta = 37^\circ.\ 10'$. $\gamma = 48^\circ.\ 15'$. Here $\mu = 39^\circ.\ 35'$; $\therefore$

$$\frac{a\ Sin.\ \gamma}{b\ Sin.\ \beta\ Sin.\ \mu} = 3{\cdot}7599939$$

$$Cot.\ \mu = 1{\cdot}2095085;$$

consequently $\quad Cot.\ \phi = 4{\cdot}9695024$

and $\quad \phi = 11^\circ.\ 22'.\ 39''$

In this example it has been assumed, that the point B is on the other side of the line AC, say in B'.

EXAM. 3. When $a = 2514'$, $b = 3796'$, $\alpha = 65^\circ. 7'$, $\beta = 135^\circ. 19'$, $\gamma = 113^\circ. 20'$; we find $\phi = 24^\circ. 53'. 22''$.

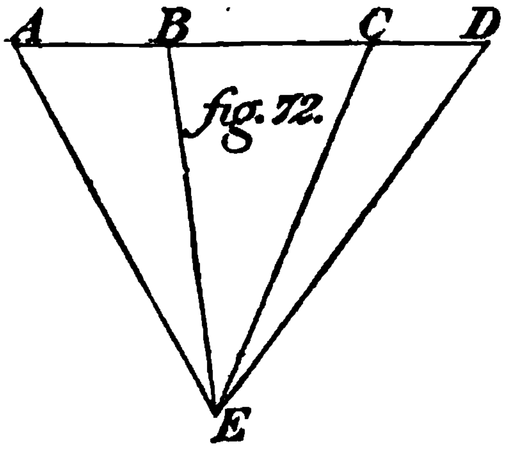
Since in this example the angle $ADC = \beta + \gamma$ is greater than 180°; consequently the point D cannot be below the line AC, because otherwise $ADC < 180^\circ$. Nor can it be in the line AC, for then $ADC = 180^\circ$. Therefore the point D must necessarily be above AC, where, with the points A, C, it forms a convex angle of $135^\circ. 19' + 113^\circ. 20' = 248^\circ. 39'$, or a concave angle of $111^\circ. 21'$; also it must fall within the angle BAC, because the angle $BAD = \phi$ has been found to be positive.

REMARK. Of all the Geometricians who have handled this important problem, I can only, for the sake of brevity, adduce the following: Lambert (Mathematical Contributions, Berlin, 1765, p. 73); Tempelhof (Elements of Analytical Finite Magnitudes, Berlin, 1769, p. 462); Langsdorf (Illustration of Kästner's Principles of Analytical Finite Magnitudes, Manheim, 1777, p. 432); Kästner (Geometrical Treatise, 1st Collection, Göttingen, 1793, 2nd part, p. 289); Pfleiderer (Arch. of pure and practical Mathematics, 2nd Number, p. 318).

SECTION LV.

PROB. *Four objects in the same straight line are seen from a station, and there the angle is measured, which the sight-lines make with one another; the distance of the first object from the second, also that of the third from the fourth: find the distance of the second from the third.*

SOLUT. Let the four objects be A, B, C, D, (*fig.* 72); let E be the station from which they are seen, and the angles $AEB = \alpha$, $BEC = \beta$, $CED = \gamma$, are measured; the distances $AB = a$, $CD = b$ are given: find BC. Let $BC = x$.



1. The triangle AEB gives

$$Sin.\ ABE = \frac{AE\ Sin.\ \alpha}{a},$$

and the triangle BED, in which $BED = \beta + \gamma$, $BD = b + x$

$$Sin.\ DBE = \frac{ED\ Sin.\ (\beta + \gamma)}{b + x}.$$

Now, since $Sin.\ ABE = Sin.\ DBE$,

$$\frac{AE\ Sin.\ \alpha}{a} = \frac{ED\ Sin.\ (\beta + \gamma)}{b + x},$$

and ∴ $$\frac{ED}{AE} = \frac{(b + x)\ Sin.\ \alpha}{a\ Sin.\ (\beta + \gamma)}.$$

2. The triangle ACE, in which $AEC = \alpha + \beta$, $AC = a + x$, gives

$$Sin.\ ACE = \frac{AE\ Sin.\ (\alpha + \beta)}{a + x},$$

and the triangle ECD,

$$Sin.\ DCE = \frac{ED\ Sin.\ \gamma}{b}.$$

Now, since $Sin.\ ACE = Sin.\ DCE$,

$$\frac{AE\ Sin.\ (\alpha + \beta)}{a + x} = \frac{ED\ Sin.\ \gamma}{b},$$

and ∴ $$\frac{ED}{AE} = \frac{b\ Sin.\ (\alpha + \beta)}{(a + x)\ Sin.\ \gamma}.$$

3. If the two expressions found in 1, 2, for $\frac{ED}{AE}$, be put equal to one another, we then obtain

$$(a + x)\ (b + x) = \frac{ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{Sin.\ \alpha\ Sin.\ \gamma}.$$

The solution of this equation gives

$$x = \sqrt{\left[\left(\frac{a-b}{2}\right)^2 + \frac{ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{Sin.\ \alpha\ Sin.\ \gamma}\right]} - \frac{a + b}{2}.$$

4. In order to shorten the calculation, give the first part of this expression the following form:

$$\frac{a - b}{2} \sqrt{\left[1 + \frac{4\ ab\ Sin.\ (\alpha + \beta)\ Sin.\ (\beta + \gamma)}{(a - b)^2\ Sin.\ \alpha\ Sin.\ \gamma}\right]}.$$

Then put

$$\frac{2}{a-b}\sqrt{\frac{ab\ Sin.\ (\alpha+\beta)\ Sin.\ (\beta+\gamma)}{Sin.\ \alpha\ Sin.\ \gamma}} = Tan.\ \phi;$$

and find an angle ϕ, such, that its tangent is equal to the left side of the expression: then we have

$$x = \frac{a-b}{2}\sqrt{(1+Tan.^2\phi)} - \frac{a+b}{2}$$

$$= \frac{a-b}{2}\ Sec.\ \phi - \frac{a+b}{2}.$$

Exam. Let $a=2731'$, $b=1987'$, $\alpha=19^\circ.\ 7'$, $\beta=31^\circ.\ 5'$, $\gamma=14^\circ.\ 57'$. Here

$$log.\ \frac{2}{a-b}\sqrt{\frac{ab\ Sin.\ (\alpha+\beta)\ Sin.\ (\beta+\gamma)}{Sin.\ \alpha\ Sin.\ \gamma}} = 1\cdot2046761,$$

consequently $\qquad log.\ Tan.\ \phi = 11\cdot2046761$

$\qquad\qquad\qquad\phi = 86^\circ.\ 25'.\ 41''\cdot63,$

and $\therefore$. $\qquad x = 3612'\cdot2.$

The angle ϕ must in this case be very accurately calculated, in order to prevent a great mistake. If, for instance, $0\cdot63''$ were left out, we should then get for x only $3611'\cdot9$.

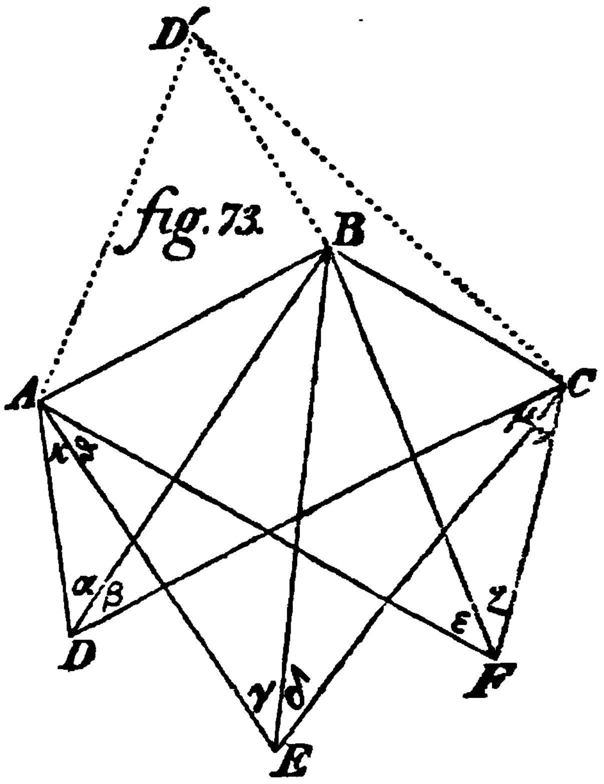
Remark. Another solution of this problem is given by Lambert (Mathematical Contributions, p. 208), which, however, leads to a very difficult formula. My formula agrees essentially with that which Mr. Hauptman Rhode delivered to the Berlin Academy of Science as an Appendix to a Memoir on another subject (Mémoire sur un endroit, &c. Potsdam, chez Horvath, 1804.)

SECTION LVI.

Prob. *The apparent distances of three places from one another, as seen from three different points, are given: likewise the apparent distances of the three points as viewed from one of the three places; required to determine the relative positions of these six points, on the supposition that they are all in the same plane.*

Solut. Let A, B, C, (*fig.* 73) be the three places, which

are seen from the three points D, E, F; the angles $ADB=\alpha$, $BDC=\beta$, $AEB=\gamma$, $BEC=\delta$, $AFB=\epsilon$, $BFC=\zeta$, also the angles $DAE=\kappa$, $EAF=\lambda$, are given or measured; find the positions of the six points, A, B, C, D, E, F.



1. The angles DCE, ECF, may be determined from the given angles. For since in each triangle the sum of all the angles is equal, by merely inspecting the figure, we have $DCE=\kappa+\alpha+\beta-\gamma-\delta$, $ECF=\lambda+\gamma+\delta-\epsilon-\zeta$. Since ∴ these angles are known, for shortness' sake put, $DCE=\mu$, $ECF=\nu$. If, besides, the angles EAB, ECB, are known, we then have all the angles of the figure, and consequently also the positions of the six points. Let, therefore, $EAB=\phi$, $ECB=\psi$.

2. Since $DAB=\phi+\kappa$, $DCB=\psi-\mu$; consequently in the triangle DAB we have

$$BD=\frac{AB\ Sin.\ (\phi+\kappa)}{Sin.\ \alpha}$$

and in the triangle BCD

$$BD=\frac{BC\ Sin.\ (\psi-\mu)}{Sin.\ \beta}.$$

These two expressions for BD, when put equal to one another, give

$$\frac{AB}{BC}=\frac{Sin.\ \alpha\ Sin.\ (\psi-\mu)}{Sin.\ \beta\ Sin.\ (\phi+\kappa)}.$$

3. In like manner, from the two triangles EAB, ECB, we obtain

$$BE=\frac{AB\ Sin.\ \phi}{Sin.\ \gamma},\quad BE=\frac{BC\ Sin.\ \psi}{Sin.\ \delta},$$

and ∴ $$\frac{AB}{BC}=\frac{Sin.\ \gamma\ Sin.\ \psi}{Sin.\ \delta\ Sin.\ \phi}.$$

4. Further, from the two triangles FAB, FCB, in which $FAB = \phi - \lambda$, $FCB = \psi + \nu$, we get

$$BF = \frac{AB\ Sin.\ (\phi - \lambda)}{Sin.\ \varepsilon},\ BF = \frac{BC\ Sin.\ (\psi + \nu)}{Sin.\ \zeta},$$

and $\therefore$ $$\frac{AB}{BC} = \frac{Sin.\ \varepsilon\ Sin.\ (\psi + \nu)}{Sin.\ \zeta\ Sin.\ (\phi - \lambda)}.$$

5. If we put the expressions found for $\frac{AB}{BC}$, in 2, 3, 4, equal to one another, we then obtain

$$\frac{Sin.\ \gamma\ Sin.\ \psi}{Sin.\ \delta\ Sin.\ \phi} = \frac{Sin.\ \alpha\ Sin.\ (\psi - \mu)}{Sin.\ \beta\ Sin.\ (\phi - \kappa)}$$

$$\frac{Sin.\ \gamma\ Sin.\ \psi}{Sin.\ \delta\ Sin.\ \phi} = \frac{Sin.\ \varepsilon\ Sin.\ (\psi + \nu)}{Sin.\ \zeta\ Sin.\ (\phi - \lambda)}.$$

If we expand $Sin.\ (\psi - \mu)$, $Sin.\ (\phi + \kappa)$, $Sin.\ (\psi + \nu)$, $Sin.\ (\phi - \lambda)$ and multiply cross ways, these equations give

$$Sin.\ \beta\ Sin.\ \gamma\ Sin.\ \psi\ (Sin.\ \phi\ Cos\ \kappa + Cos.\ \phi\ Sin.\ \kappa)$$
$$= Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \phi\ (Sin.\ \psi\ Cos.\ \mu - Cos.\ \psi\ Sin.\ \mu)$$

$$Sin.\ \gamma\ Sin.\ \zeta\ Sin.\ \psi\ (Sin.\ \phi\ Cos.\ \lambda - Cos.\ \phi\ Sin.\ \lambda)$$
$$= Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ \phi\ (Sin.\ \psi\ Cos.\ \nu + Cos.\ \psi\ Sin.\ \nu)$$

If each of these equations be divided by $Sin.\ \phi\ Sin.\ \psi$, we have

$$Sin.\ \beta\ Sin.\ \gamma\ (Cos.\ \kappa + Cot.\ \phi\ Sin.\ \kappa)$$
$$= Sin.\ \alpha\ Sin.\ \delta\ (Cos.\ \mu - Cot.\ \psi\ Sin.\ \mu)$$

$$Sin.\ \gamma\ Sin.\ \zeta\ (Cos.\ \lambda - Cot.\ \phi\ Sin.\ \lambda)$$
$$= Sin.\ \delta\ Sin.\ \varepsilon\ (Cos.\ \nu + Cot.\ \psi\ Sin.\ \nu).$$

6. The first of the two equations last found gives

$$Cot.\ \psi = \frac{Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \mu - Sin.\ \beta\ Sin.\ \gamma\ (Cos.\ \kappa + Cot.\ \phi\ Sin.\ \kappa)}{Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \mu}$$

and the second

$$Cot.\ \psi = \frac{Sin.\ \gamma\ Sin.\ \zeta\ (Cos.\ \lambda - Cot.\ \phi\ Sin.\ \lambda) - Sin.\ \delta\ Sin.\ \varepsilon\ Cos.\ \nu}{Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ \nu}$$

If these two expressions be put equal to one another, we obtain

$$Cot.\ \phi = \frac{\left[\begin{array}{l} Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ (Sin.\ \nu\ Cos.\ \mu + Sin.\ \mu\ Cos.\ \nu) - \\ Sin. \beta\ Sin\ \gamma\ Sin. \varepsilon\ Cos. \nu\ Cos. \kappa - Sin\ \alpha\ Sin. \gamma\ Sin\ \zeta\ Sin. \mu\ Cos. \lambda \end{array}\right]}{Sin\ \beta\ Sin. \gamma\ Sin. \varepsilon\ Sin. \nu\ Sin. \kappa - Sin. \alpha\ Sin. \gamma\ Sin. \zeta\ Sin. \lambda\ Sin. \mu}$$

or, if the numerator and denominator be divided by $Sin.\ \gamma$, and $Sin.\ (\mu + \nu)$ be substituted for $Sin.\ \nu\ Cos.\ \mu + Sin.\ \mu$ $Cos.\ \nu$, and $Cosec.\ \gamma$ for $\dfrac{1}{Sin.\ \gamma}$

$$Cot.\ \phi = \frac{\left[\begin{array}{l} Cosec.\ \gamma\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ (\mu + \nu) - \\ Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Cos.\ \kappa - Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \mu\ Cos.\ \lambda \end{array}\right]}{Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Sin.\ \kappa - Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \lambda\ Sin.\ \mu}$$

7. In order to determine $Cot.\ \psi$, it is only requisite to eliminate $Cot.\ \phi$ from the two equations last found in 5; but this object will be much more easily attained by substituting in the expression found for $Cot.\ \phi$ the angles

$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \kappa,\ \lambda,\ \mu,\ \nu,$$

respectively for the angles

$$\zeta,\ \varepsilon,\ \delta,\ \gamma,\ \beta,\ \alpha,\ \nu,\ \mu,\ \lambda,\ \kappa;$$

because the former have the same position with respect to the angle ψ, that the last have with respect to ϕ. By this substitution, we obtain

$$Cot.\ \psi = \frac{\left[\begin{array}{l} Cosec.\ \delta\ Sin.\ \zeta\ Sin.\ \gamma\ Sin.\ \beta\ Sin.\ (\lambda + \kappa) - \\ Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Cos.\ \nu - Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \lambda\ Cos.\ \mu \end{array}\right]}{Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Sin.\ \nu - Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \mu\ Sin.\ \lambda}$$

in which expression the denominator is the same as that in the expression for $Cot.\ \phi$.

In these two expressions for $Cot.\ \phi$, $Cot.\ \psi$,

$$\mu = \kappa + \alpha + \beta - \gamma - \delta,\quad \nu = \lambda + \gamma + \delta - \varepsilon - \zeta;$$

consequently $\quad \mu + \nu = \kappa + \lambda + \alpha + \beta - \varepsilon - \zeta.$

EXAM. Let $\alpha = 40^0.\ 36'$, $\beta = 27^0.\ 9'$, $\gamma = 36^0.\ 49'$, $\delta = 31^0.\ 18'$, $\varepsilon = 22^0$, $\zeta = 13^0.\ 28'$, $\kappa = 25^0.\ 23'$, $\lambda = 20^0.\ 17'$; $\therefore \mu = 25^0.\ 1'$, $\nu = 52^0.\ 56'$. Here

$$\begin{aligned}
&Cosec.\ \gamma\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \varepsilon\ Sin.\ (\mu + \nu) &&= 0{\cdot}20668928\\
&Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Cos.\ \kappa &&= 0{\cdot}12323178\\
&Sin.\ \alpha\ Sin.\ \zeta\ Sin\ \mu\ Cos.\ \lambda &&= 0\ 06011447\\
&Cosec.\ \delta\ Sin.\ \zeta\ Sin.\ \gamma\ Sin.\ \beta\ Sin.\ (\lambda + \kappa) &&= 0{\cdot}08767862\\
&Sin.\ \varepsilon\ Sin.\ \beta\ Sin.\ \kappa\ Cos.\ \nu &&= 0{\cdot}04416768\\
&Sin.\ \zeta\ Sin.\ \alpha\ Sin.\ \lambda\ Cos.\ \mu &&= 0{\cdot}04760865\\
&Sin.\ \beta\ Sin.\ \varepsilon\ Sin.\ \nu\ Sin.\ \kappa &&= 0{\cdot}05847082\\
&Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \lambda\ Sin.\ \mu &&= 0{\cdot}02221714
\end{aligned}$$

Therefore,

$$Cot.\ \phi = \frac{0{\cdot}02334303}{0{\cdot}03625368} = 0{\cdot}6438802$$

$$Cot.\ \psi = \frac{-0{\cdot}00409771}{0{\cdot}03625368} = -\ 0{\cdot}1130288$$

consequently $\phi = 57^0.\ 13'.\ 24''$, $\psi = 96^0.\ 26'.\ 55''$.

When the angles ϕ and ψ are found, it is easy to determine the points A, B, C, D, E, F, and also to calculate the distances of all these points from one another, when only one of these distances (no matter which) has either been measured directly, or is otherwise already known.

COR. When the stations D, E, F, have a different position from that assumed in *fig.* 73, then also the formulæ found for $Cot.\ \phi$, $Cot.\ \psi$ will still obtain, provided the values corresponding to this position are given to the angles α, β, γ, δ, &c. The mode of proceeding in this case will be best elucidated by an example.

Suppose the first station is at D' instead of D, and that it is wished to determine the values of α, β, κ depending upon the position of this point; then suppose that the point D has arrived by slow degrees at D'. Now, as the point D advances to D', and approaches the line AB, the angle $ADB = \alpha$ will always increase, till this point falls in AB itself, and $\alpha = 180^0$. If the point D move towards the upper side of the line AB, then the concave angle will become a convex one, and consequently, if we assume the point D at D', we must under-

stand by α, not the concave, but the convex angle $AD'B$. But if, as is desirable in most cases, we wish to avoid the convex angle, we can then let the point D pass over to D', by producing AB, say towards the side A. Under this supposition, the angle α will constantly decrease, because the lines DB, DA approach each other, till it becomes $= 0$, when the point D is situated in that part of BA which is produced, and the lines DB, DA coincide; and lastly, negative, when the point D moves towards the upper side of AB, and the line BD, which before was on this side of the line DA, is now on the other side of it. If $\therefore$ we make use of the concave angle $AD'B$, we must put $\alpha = -AD'B$.

Proceed in like manner with the angle β. Thus, while the point D is situated under the line BC, or BC produced, and consequently the line DC is on this side of the line DB, the point D is positive; but when it is above the line BC, and $\therefore$ the line DC is on the other side of DB, it is negative. Thus, for the point D', $\beta = -BD'C$.

The angle κ is positive, while it is under AE, but negative as soon as the line AD is on the other side of the line AE. Thus, for the point D', $\kappa = -EAD'$.

With regard to the trigonometrical functions of the negative angles, when θ denotes any angle, $Sin. -\theta = -Sin. \theta$, $Cos. -\theta = Cos. \theta$, $Tan. -\theta = -Tan. \theta$, $Cot. -\theta = -Cot. \theta$, $Sec. -\theta = Sec. \theta$, $Cosec. -\theta = -Cosec. \theta$. Calculations involving angles of this kind occur frequently in the sequel.

REMARK. Lambert's problem which is here solved, is of great practical use; because by its means the positions of six points are obtained at once; it is to be found in his Contributions, I. p. 72. The formula there given (p. 82) for *Cot. x*, or mine for *Cot.* ϕ, is not quite correct, because both in the numerator and denominator a factor has been omitted by mistake. This error was first discovered by Good (Lambert's Scientific Correspondence, 2nd vol. p. 232), and acknowledged (*ibid.* p. 236) by Lambert.

VI. PROBLEMS ON THE CIRCLE.

SECTION LVII.

PROB. *The radius of a circle being given, to calculate from it the circumference and area of the circle.*

SOLUT. Let the radius of a circle $= r$, the circumference $= p$, and the area $= q$; then, as is already known,

$$p = 2\pi r, \; q = \pi r^2;$$

or, when the diameter $= d$,

$$p = \pi d, \; q = \tfrac{1}{4}\pi d^2:$$

in which π is the number which represents the circumference of a circle, whose diameter $= 1$. This number is

$$3{\cdot}14159265358979323846264338327950\ldots$$

Thus far has this number been calculated by Ludolph of Cologne. In Vega's large and small Logarithmic Tables, this number is calculated by the author as far as 143 decimal places. In most practical cases, however, it will be necessary merely to make use of the first five decimal places; and thus $\pi = 3{\cdot}14159$.

By the transformation of this number into a continued fraction, we obtain the following abbreviated values:

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \text{\&c.}$$

Lambert, in his Correspondence, II. pp. 156, 157, calculates 27 values of this kind; the two last, however, are not correct, as Professor Schulz has discovered. (Solution of some of the most important Mathematical Theories, Königsberg, 1803, p. 159.)

In practice, when great accuracy is not requisite, we may put $\pi = \frac{355}{113}$, because this number differs from Ludolph's, beginning from the seventh decimal place: for $\frac{355}{113} =$ 3·1415929 ...

If we make use of logarithms; then

$$log.\ \pi = 0{\cdot}4971498726941338543512 7\ ...$$

or abbreviated

$$log.\ \pi = 0{\cdot}4971499.$$

Cor. From these two equations, for p and q we obtain

$$r = \frac{p}{2\pi},\ r = \sqrt{\frac{q}{\pi}},\ q = \frac{p^2}{4\pi},\ p = 2\sqrt{\pi q}.$$

Exam. 1. The diameter of a circle is 42′. 1″. 2‴: what are its circumference and area? Ans. The circumference = 132′. 3″. 2‴, and the area = 1393 □′. 37 □″. 04 □‴, nearly.

Exam. 2. The area of a circle is 3765 □′. 18 □″: what are the radius and circumference? Ans. The radius = 33′. 6″. 2‴, and the circumference = 217′. 5″. 2‴, nearly.

A few Examples in this Rule.

I. There are three circles given; the diameter of the first is 9′. 7″, the diameter of the second is 13′. 6″, and the diameter of the third is 22′. 9″: find a circle, whose area is equal to the sum of the areas of these three circles. What is the diameter of this circle? Ans. About 28′. 3″. 4‴.

II. From any point let there be two concentric circles described; let the radius of the exterior circle = 1′. 5″. 3‴, and the radius of the interior one = 10″. 9‴: what is the radius of a circle, whose area is equal to the area of the circle described between these two? Ans. About 10″. 7‴. 4ᴵᵛ.

III. The radius of a circle is 39′. 8″: it is required to

describe about it another concentric circle, such that the area of the circle described between these two = 385 □′. What is its area? Ans. About 41′. 3″. 1‴.

IV. The diameter of a circle is 45′. 3″. 7‴: find the diameter of another circle, whose area : the area of the former circle : : 387 : 932. What is the diameter?
Ans. 29′. 2″. 3‴, nearly.

V. There are two circles given; the circumference of one is 69′. 5″, and that of the other 35′. 9″: what is the diameter of a circle, whose area is equal to the sum of the areas of these two circles? Ans. 24′. 8″. 9‴, nearly.

VI. To convert a circle, whose diameter is 9′. 7″, into an equilateral triangle: what is the dimension of a side of this triangle? Ans. 13′. 0″. 6‴ nearly.

SECTION LVIII.

Prob. *From the given radius of a circle, to find the value of an arc expressed in degrees, minutes, and seconds.*

Solut. Let the radius of a circle $= r$, the given number of degrees of an arc $= \phi$; let the length of this arc, expressed in the same terms of unity as the radius, $= l$. Since the circumference of the whole described circle, with the radius r, $= 2\pi r$, then

$$360^0 : \phi = 2\pi r : l,$$

and ∴

$$l = \frac{\pi r \phi}{180^0}.$$

If ϕ, besides degrees, contains also minutes and seconds; the degrees must be converted into minutes or seconds, and the 180 in the denominator multiplied by 60, or by $60 \times 60 = 3600$. If it is considered preferable, the minutes and seconds can be converted into decimal parts of a degree, and the 180 remain unaltered.

Cor. Conversely, from the expression found for l, we obtain

$$\phi = \frac{180^\circ l}{\pi r},\ r = \frac{180^\circ l}{\pi \phi}.$$

Exam. 1. What is the length of an arc of $37^\circ.\ 19'$, in a circle whose radius is $13'.\ 4''$?

Ans. $87'.\ 2''.\ 7'''.\ 4^{\text{iv}}$, nearly.

Exam. 2. What is the length of an arc of $149^\circ.\ 16'.\ 13''$, in a circle whose radius is $19'.\ 7''$?

Ans. $51'.\ 3''.\ 2'''.\ 3^{\text{iv}}$, nearly.

Exam. 3. What is the length of an arc of $253^\circ.\ 9'.\ 3''$, in a circle, whose radius is $23'.\ 8''.\ 6'''$?

Ans. $105'.\ 4''.\ 2'''.\ 1^{\text{iv}}$, nearly.

Exam. 4. How many degrees, minutes, and seconds, does an arc contain, whose chord is $25'.\ 7''$, when the radius of the circle to which it belongs $= 19'.\ 3''.\ 7'''$?

Ans. $76^\circ.\ 1'.\ 11''$, nearly.

Exam. 5. What is the radius of a circle, when an arc of $25^\circ.\ 3'.\ 49''$ has a chord of $247'.\ 8''$?

Ans. $566'.\ 4''.\ 7'''.\ 4^{\text{iv}}$, nearly.

Exam. 6. There are two arcs having equal chords, which belong to two different circles; one is $15^\circ.\ 39'.\ 7''$, the other $56^\circ.\ 9'.\ 43''$; the first belongs to a circle whose radius is $7'.\ 6''.\ 3'''$: what is the radius of the circle to which the other arc belongs? Ans. $2'.\ 1''.\ 2'''.\ 6^{\text{iv}}$ nearly.

SECTION LIX.

Prob. *From the given angle and radius of a segment of a circle, to find its area.*

Solut. Let ϕ be the number of degrees, minutes and seconds, which the angle, and consequently also the arc, of the circle contains, l the chord of the arc, r the radius, and

q the area of the segment; then, by the foregoing §, $l = \frac{\pi r \phi}{180^0}$.

Now, since every segment of a circle is equal to a triangle, whose base is the arc, and whose altitude is the radius of this segment, consequently

$$q = \frac{lr}{2} = \frac{\pi r^2 \phi}{360^0}.$$

Cor. Hence we obtain

$$\phi = \frac{360^0 q}{\pi r^2},\ r = \sqrt{\frac{360^0 q}{\pi \phi}}.$$

Exam. 1. The radius of a circle is $7''.\ 9'''$; the angle of a segment of this circle contains $37^0.\ 5'$: what is the area of this segment? Ans. $20\ \square''.\ 19\ \square'''.\ 67\ \square^{IV}$, nearly.

Exam. 2. What is the angle of a segment, whose radius $= 25783'$, and whose area $= 935\ \square'$?
Ans. $0{\cdot}58023''$, nearly.

Exam. What is the radius of a segment, whose angle $= 46^0.\ 25'.\ 18''$, and whose area $= 367\ \square'.\ 90\ \square''$?
Ans. $301'.\ 3''.\ 5'''$, nearly.

A few practical Examples to this Rule.

I. The area of a segment is equal to the square of its radius: what is its angle? Ans. $114^0.\ 35'.\ 29''$.

II. There is a segment, whose angle $= 69^0.\ 47''$, and is such, that when the radius, arc, and area, in the order in which they are here placed, are expressed in terms of one and the same unity for the chord and area, the three values thus obtained are in continual proportion: what is the area of this segment, the foot calculated at unity?
Ans. $13008\ \square'.\ 28\ \square''$.

III. Find a triangle, whose three sides are in the proportion of the three numbers 11, 13, 20, and whose area is equal

to the area of a segment, whose angle is 19°. 27′. 5″, and whose arc is 27′. 3″. What are the three sides of this triangle? Ans. One side = 44′·8594, another = 53·0157, and the third = 81′·5626.

SECTION LX.

PROB. *To find the area of a segment of a circle, from the arc, and the radius of this arc, expressed in degrees, minutes and seconds.*

SOLUT. Let ADB (*fig*. 74) $= \phi$ be the given arc expressed in degrees, minutes, and seconds; C the center, and r the radius of the arc; then, by the foregoing §,

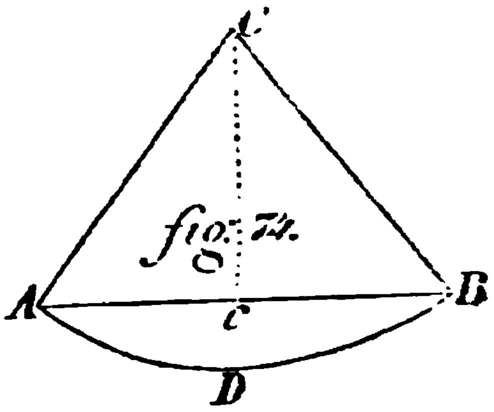


fig. 74.

$$\text{Segment } ACB = \frac{\pi r^2 \phi}{360^0}.$$

Now $\triangle ACB = \frac{1}{2} r^2 Sin. \phi$, (§ XXVI)

$$\text{consequently Segment } ADBA = \frac{\pi r^2 \phi}{360^0} - \tfrac{1}{2} r^2 Sin. \phi$$

$$= \tfrac{1}{2} r^2 \left[\frac{\pi \phi}{180^0} - Sin. \phi\right].$$

EXAM. 1. When $\phi = 29^0. 38'. 15''$, $r = 37'$, then segment $ADBA = 15·5801 \square'$.

EXAM. 2. When $\phi = 73^0. 25'. 11''$, $r = 65'$, then segment $ADBA = 682·3271 \square'$.

SECTION LXI.

PROB. *From the given chord and radius of a segment, to find its area.*

SOLUT. Let ACB (*fig*. 74) be the segment, whose area is sought, let the given radius $= r$, the given chord $AB = a$.

If the angle ACB is found, then the area of the segment is also found (§ LIX).

Put $ACB = \phi$: then, if the perpendicular Cc is drawn, $ACc = \frac{1}{2}\phi$, $Ac = r\,Sin.\,\frac{1}{2}\phi$, $AB = a = 2\,r\,Sin.\,\frac{1}{2}\phi$; consequently

$$Sin.\,\tfrac{1}{2}\phi = \frac{a}{2\,r},$$

from which the angle ϕ may be determined. Having found this angle, then

$$\text{Segment } ACB = \frac{\pi r^2 \phi}{360^{\circ}}.$$

Cor. If the chord AB and the angle ACB be given, then in like manner we obtain the area of the segment from § LIX, by substituting in the expression found in this section for r its value $\dfrac{a}{2\,Sin.\,\frac{1}{2}\phi}$. Thus

$$\text{Segment } ACB = \frac{\pi a^2 \phi}{1440^{\circ}\,Sin.^2\,\frac{1}{2}\phi}:$$

which expression is most readily calculated by means of logarithms.

Exam. 1. The chord of a segment $= 23'$, the radius $= 29' \cdot 7''$; what is its area? Ans. 350 □′. 71 □″.

Exam. 2. The chord of a segment $= 54'$, the angle $= 67^{\circ}.\ 15'.\ 25''$: what is its area? Ans. 1395 □′. 07 □″.

Remark. If the chord be equal to the radius, then the area of the segment $= \frac{1}{6}\pi r^2$, because in this case the chord is a side of a regular hexagon described in the circle of which the segment is a part. If the chord $= r\sqrt{2}$, then the segment is a quadrant; and consequently its area $= \frac{1}{4}\pi r^2$. If the chord $= r\sqrt{3}$, then it is a side of a triangle described in the circle; and ∴ the segment $= \frac{1}{3}\pi r^2$.

SECTION LXII.

Prob. *From the arc and chord of a segment expressed in degrees, minutes, and seconds, to find the area which is included by the chord and arc.*

Solut. By the foregoing section, if we retain the notation

there used (*fig.* 74),

$$\text{Segment } ACB = \frac{\pi a^2 \phi}{1440^0 \, Sin.^2 \frac{1}{2} \phi}.$$

But in the triangle ACB, $Cc = \frac{1}{2} a \, Cot. \frac{1}{2} \phi$, and consequently

$$\triangle \, ACB = \tfrac{1}{4} a^2 \, Cot. \tfrac{1}{2} \phi;$$

we . ·. have

$$\text{Segment } ABDA = \frac{\pi a^2 \phi}{1440^0 \, Sin.^2 \frac{1}{2} \phi} - \tfrac{1}{4} a^2 \, Cot. \tfrac{1}{2} \phi.$$

Exam. 1. When $a = 18'. 9''. 3'''$, $\phi = 113^0. 39'. 25''$, Segment $ABDA = 68{\cdot}2624 \, \square'$.

Exam. 2. When $a = 126'. 5''. 8'''$, $\phi = 269^0. 14'. 7''$, Segment $ABDA = 22527{\cdot}207 \, \square'$.

Remark. In the second example, $Cot. \frac{1}{2} \phi = Cot. \, 134^0. \, 37'. \, 3''{\cdot}5 = - Cot. \, 45^0. \, 22'. \, 56''{\cdot}5$; the section is greater than the segment, because it is greater than a semicircle.

SECTION LXIII.

Prob. *To find a segment of a circle, which is bisected by its chord.*

Solut. Let ACB (*fig.* 74) be the required segment, which is bisected by its chord AB, so that the segment $ADBA = \triangle \, ACB$, or $2 \triangle \, ACB =$ segment ACB. Now the segment $ACB = \frac{\pi r^2 \phi}{360^0}$ (§ LIX), and $\triangle \, ACB = \frac{1}{2} r^2 \, Sin. \, \phi$; we therefore have the equation,

$$\frac{\pi r^2 \phi}{360^0} = r^2 \, Sin. \, \phi,$$

or

$$\frac{\pi \phi}{360^0} = Sin. \, \phi.$$

2. In order to solve an equation of this kind between an arc and its sine, there is scarcely a more convenient method, than that which the rule known by the name of False Position

presents. To apply this mode of calculation with advantage, make first a few rough guesses, in order at least to approximate somewhat near to the value of the angle ϕ. Assume $\phi = 90^0$, this gives, $\frac{\pi\,\phi}{360^0} = \frac{1}{4}\,\pi = 0{\cdot}785 \ldots$ and $Sin.\,\phi = 1$; consequently ϕ must be $> 90^0$. If we assume $\phi = 120^0$; then $\frac{\pi\,\phi}{360} = \frac{1}{3}\,\pi = 1{\cdot}047 \ldots$ and $Sin.\,\phi = 0{\cdot}866 \ldots$; consequently $\phi < 120^0$. Hence it follows, that ϕ must be between 90 and 120^0. Put $\therefore$ successively $\phi = 100^0$, $\phi = 110^0$; then we have,

When $\phi = 100^0$.		When $\phi = 110^0$.	
$log.\,\frac{\pi\phi}{360^0} =$	$0{\cdot}9408474 - 1$	$log.\,\frac{\pi\,\phi}{360^0} =$	$0{\cdot}9822401 - 1$
$log.\,Sin.\,\phi =$	$9{\cdot}9933515 - 10$	$log.\,Sin.\,\phi =$	$9{\cdot}9729858 - 10$
Error	525041	Error	-92543
Subtract	-92543		

617584 Difference of errors.

Now form the following proportion :

$$617584 : 525041 = 10^0 : 8^0\ 30' \text{ nearly.}$$

We have $\therefore$ $\phi = 108^0.\ 30'$, nearly.

3. This value found for ϕ does not differ very much from the real one, as the following calculation shews. In order to render it more accurate, try also further the assumption $\phi = 108^0.\ 35'$: we then have

when $\phi = 108^0.\ 30'$.		When $\phi = 108^0.\ 35'$.	
$log.\,\frac{\pi\phi}{360^0} =$	$0\ 9762771 - 1$	$log.\,\frac{\pi\,\phi}{360^0} =$	$0{\cdot}9766105 - 1$
$log.\,Sin.\,\phi =$	$9{\cdot}9769566 - 10$	$log.\,Sin.\,\phi =$	$9{\cdot}9767447 - 10$
Error	6795	Error	1342
Subtract	1342		

5453 Difference of errors.

Now form the following proportion:

$$5453 : 1342 = 5' : 1'.\ 13'' \text{ nearly.}$$

We $\therefore$ have $\phi = 108^{\circ}.\ 35' + 1'\ 13'' = 108^{\circ}.\ 36'.\ 13''$, nearly.

4. Since the value of ϕ is only a little too small, as the following calculation shews, $\therefore$ assume ϕ only a few seconds larger; we then have

When $\phi = 108^{\circ}.\ 36'.\ 13''$.		When $\phi = 108^{\circ}.\ 36'.\ 14''$.	
$log.\ \frac{\pi\phi}{360^{\circ}}$	$= 0{\cdot}9766916 - 1$	$log.\ \frac{\pi\phi}{360^{\circ}}$	$= 0{\cdot}9766927 - 1$
$log.\ Sin.\ \phi$	$= 9{\cdot}9766930 - 10$	$log.\ Sin.\ \phi$	$= 9{\cdot}9766923 - 10$
Error	14	Error	-4
Subtract	-4		

18 Difference of errors.

Put $\therefore$ again:

$$18 : 4 = 1'' : 13'''.$$

5. Since this last approximation is only represented by $'''$, this denotes that even the seconds in the values of ϕ found in 4 are correct, and we $\therefore$ can put $\phi = 108^{\circ}.\ 36'.\ 13''$. If we wish to determine this value still more accurately, we must use logarithms with more than seven decimal places. Euler (Introduction to the Analysis Infinitorum, translated by Michelsen, 2nd vol. p. 452) performed this; and by these means found

$$\phi = 108^{\circ}.\ 36'.\ 13''.\ 45'''.\ 27^{iv}.\ 6^{v}.$$

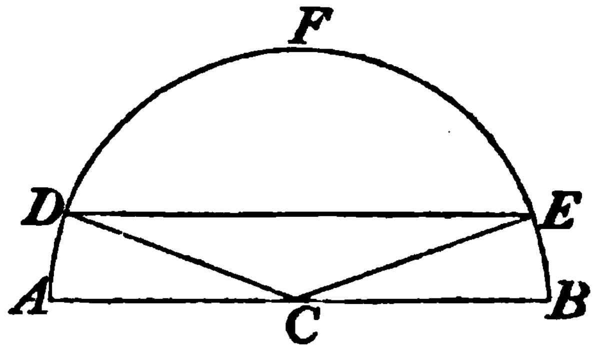
REMARK. The method here adopted is founded on the well-known property of trigonometrical lines and their logarithms, that their differences, in small alterations of the angles, are nearly as the differences of the angles themselves, which supposition is the more correct, the smaller these changes are with respect to the angle.

SECTION LXIV.

Prob. *To bisect a semicircle by a chord which is parallel to the diameter.*

Solut. Let $ADFEB$ (*fig.* 75) be a semicircle, and DE $\parallel$ the chord AB, which bisects it; let the radius $= r$, and the angle $DCE = \phi$.

fig. 75.



1. Since the area of the semicircle $= \frac{1}{4}\pi r^2$, and the area of the segment $DFED = \frac{\pi r^2 \phi}{360^0} - \frac{1}{2} r^2 \, Sin. \, \phi$ (§ LX), we have the equation

$$\tfrac{1}{4}\pi r^2 = \frac{\pi r^2 \phi}{360^0} - \tfrac{1}{2} r^2 \, Sin. \, \phi,$$

or $$90^0 \pi = \pi \phi - 180^0 \, Sin. \, \phi,$$

or also $$\frac{\pi (\phi - 90^0)}{180^0} = Sin. \, \phi.$$

Since ϕ must necessarily be greater than 90^0, put $\phi = 90^0 + \psi$; this gives $\phi = Cos. \, \psi$, and instead of the foregoing equation, we then have the one,

$$\frac{\pi \psi}{180^0} = Cos. \, \psi$$

2. Hence the value of ψ may be determined in the same way as in the foregoing §. Since ψ must be $< 90^0$, because $\phi < 180$; try $\psi = 45^0$. This assumption gives, $\frac{\pi\psi}{180^0} = \frac{1}{4}\pi = 0{\cdot}785 \ldots$, $Cos. \, \psi = 0{\cdot}707 \ldots$; whence we conclude, that ψ is actually less than 45^0, but cannot be widely different from it, because these two magnitudes first differ in the second decimal place. If $\therefore$ we put $\psi = 40^0$; then we have $\frac{\pi\psi}{180^0} = \frac{2}{9}\pi = 0{\cdot}698 \ldots$, and $Cos. \, \psi = 0{\cdot}766 \ldots$,

and consequently $\psi > 40^0$. After having in this way previously convinced ourselves, that the value of ψ lies between 40^0 and 45^0; we can then approximate it more nearly by means of the proportional parts. To effect this, try the two assumptions $\psi = 41^0$, $\psi = 43^0$.

$\psi = 41^0$.	$\psi = 43^0$.
$log. \frac{\pi\psi}{180^0} = 0{\cdot}8546613 - 1$	$log. \frac{\pi\psi}{180^0} = 0{\cdot}8753459 - 1$
$log. Cos. \psi = 9{\cdot}8777799 - 10$	$log. Cos. \psi = 9{\cdot}8641275 - 10$
Error 231186	Error -112184
Subtract -112184	

343370 Difference of errors.

Now put

$$343370 : 231186 = 2^0 : 1^0.\ 20', \text{ nearly.}$$

3. We have $\therefore \psi = 41^0 + 1^0.\ 20'$. Since this value differs but very little from the true one, as the following calculation shews, calculate, in the second place, on the assumption $\psi = 42^0.\ 21'$.

$\psi = 42^0.\ 20'$.	$\psi = 42^0.\ 21'$.
$log. \frac{\pi\psi}{180^0} = 0{\cdot}8685598 - 1$	$log. \frac{\pi\psi}{180^0} = 0{\cdot}8687308 - 1$
$log. Cos. \psi = 9{\cdot}8687851 - 10$	$log. Cos. \psi = 9{\cdot}8686700 - 10$
Error 2253	Error -608
Subtract -608	

2861 Difference of errors.

Again, put

$$2861 : 2253 = 1' : 47'', \text{ nearly.}$$

We obtain $\therefore \psi = 42^0.\ 20'.\ 47''$, and in this value the seconds are perfectly right, of which we can be convinced, by assuming ψ greater or less by $1''$, and making the trial with it. However, the approximation cannot be continued further, for then it would be necessary to make use of logarithms with

more than seven decimal places. Euler, in the above-cited work, by these means finds

$$\psi = 42^\circ.\ 20'.\ 47''.\ 14'''.$$

From this value of ψ we further obtain

$$\phi = \psi + 90^0 = 132^\circ.\ 20'.\ 47''.\ 14'''.$$

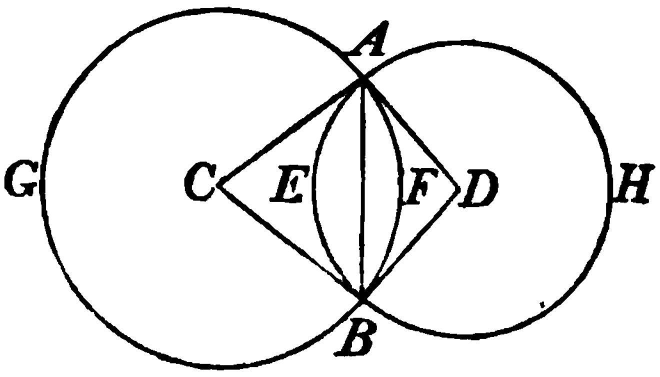
REMARK. Kästner treats this problem in a similar but more general way (Geometrical Treatise, 2nd Collection, p. 129, &c.), for he gives a method how to find a segment having a given proportion to the area of the circle.

SECTION LXV.

PROB. *Two circles intersect each other; the radii of these circles, together with the line which joins the points of section, are given: find the area of the lune which is common to both.*

SOLUT. The two circles $AFBG$, $AEBH$, (*fig.* 76) intersect each other in the points A, B; the radii $AC = r$, $AD = \rho$, together with $AB = a$ are given: find the area of the space included by the two arcs AFB, AEB, or, as it is called, the lune $AFBEA$.

fig. 76.



Put $\angle\ ACB = \phi$, $\angle\ ADB = \psi$; then, by § LXI, $Sin.\ \frac{1}{2}\,\phi = \frac{a}{2\,r}$, $Sin.\ \frac{1}{2}\,\psi = \frac{a}{2\,\rho}$. Having from hence determined the angles ϕ, ψ, then

$$\text{Segment } AFBA = \frac{\pi\, r^2\, Sin.\ \phi}{360^0} - \tfrac{1}{2} r^2\ Sin.\ \phi,$$

$$\text{Segment } AEBA = \frac{\pi\, \rho^2\, Sin.\ \psi}{360^0} - \tfrac{1}{2} \rho^2\ Sin.\ \psi.$$

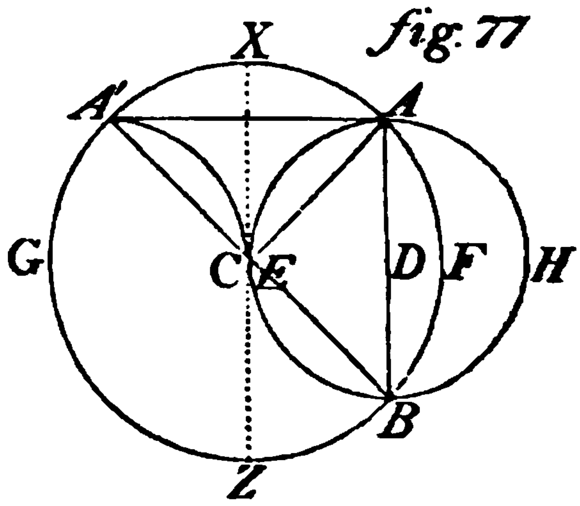
As the lune $AFBEA$ consists of these two segments, consequently this last is found.

EXAM. Let $r = 27'$, $\rho = 14'$, $a = 9'.\ 5''.\ 4'''$. Here

$\phi = 20^\circ.\ 21'.\ 4''.7$, $\psi = 39^\circ.\ 50'.\ 26''\cdot 9$; consequently, Segment $AFBA = 2\cdot705\ \square'$, Segment $AEBA = 5\cdot361\ \square'$, and $\therefore\ AFBEA = 8\cdot066\ \square'$.

Cor. When the lune $AFBEA$ has been found, the areas of the lunes $AFBHA$, $AEBGA$ may also be easily found; for it is only necessary to subtract it from the circles $AFBG$, $AEBH$.

If in any circle $AFBG$ (*fig.* 77), the two radii AC, CB, are perpendicular to one another, and AB be assumed as the diameter of the second circle $AEBH$; then $\phi = 90^\circ$, $\psi = 180^\circ$, $AB = a = r\sqrt{2}$, $\rho = \frac{1}{2}AB = r\sqrt{\frac{1}{2}}$. For this case $\therefore$



$$\text{Segment } AFBA = \frac{\pi r^2 \cdot 90^\circ}{360^\circ} - \tfrac{1}{2} r^2\, Sin. 90^\circ = \tfrac{1}{4}\pi r^2 - \tfrac{1}{2} r^2,$$

$$\text{Segment } AEBA = \frac{\frac{1}{2}\pi r^2 \cdot 180^\circ}{360^\circ} - \tfrac{1}{4} r^2\, Sin. 180^\circ = \tfrac{1}{4}\pi r^2.$$

Consequently the lune $AFBEA = \frac{1}{2}\pi r^2 - \frac{1}{2} r^2$. If we subtract this from the area $AEBH = \frac{1}{2}\pi r^2$, we then have the lune $AFBHA = \frac{1}{2} r^2 = AD^2 = \triangle\ ACB$. The area of this last lune does not depend consequently on the quadrature of the circle: thus it is always equal to the square of the radius of the lesser circle, or also equal to the triangle ACB.

Since ACB is a right angle, then the circle $AEBH$ passes through the center C of the other circle. If now the diameter XZ is drawn parallel to AB, then there are two curvilinear segments ACX, BCZ, the first of which is inclosed by the arcs AC, AX, and the radius CX; the second by the arcs BC, BZ, and the radius CZ. The area of both segments together is found by subtracting the lune $AFBEA = \frac{1}{2}\pi r^2 - \frac{1}{2} r^2$ from the semicircle $XFZ = \frac{1}{2}\pi r^2$. Consequently each of these segments is half of the triangle ACB, or half of the lune $AFBHA$.

Let the curvilinear sector XCA' be equal to the curvilinear sector XCA; then the curvilinear $\triangle ACA'$, inclosed by the circular arcs AA', CA, CA', is equal to the curvilinear $\triangle ACA' = \triangle ACB$. Further, ACB is a right angle, and $\therefore \angle ACX = \angle BCZ = \frac{1}{2}R$; consequently, if the straight lines CA', AA' are drawn, ACA' is also a right angle, and $\triangle ACA' = \triangle ACB$; $\therefore$ also the curvilinear $\triangle ACA' = \triangle ACA'$. Since the arcs AA', AC' are quadrants of their respective circles; then $Arc.\ AA' : Arc.\ AC = r : \rho$, and $\therefore (Arc.\ AA')^2 : (Arc.\ AC)^2 = r^2 : \rho^2 = 2 : 1$. Hence it follows, that $(Arc.\ AA')^2 = (Arc.\ AC)^2 + (Arc.\ A'C)^2$. The area of the rectilinear triangle ACA' is $\therefore$ equal to that of the curvilinear triangle ACA', and they have this common property, that the square of one of their sides is equal to the sum of the squares of the other two.

VII. PROBLEMS, WITH THEIR ANALYTICAL AND GEOMETRICAL SOLUTIONS, CHIEFLY WITH RESPECT TO GEOMETRICAL CONSTRUCTIONS.

SECTION LXVI.

PROB. *The base of a triangle, one of its angles, and the difference of the other two sides are given: find the triangle.*

SOLUT. In the triangle ABC (*fig.* 78, 79), the base BC, the angle ACB, and the difference of the two sides AC, AB are given: find the triangle. Let $BC = a$, $ACB = \alpha$, $AC - AB = \pm d$; the upper sign obtains for *fig.* 78, the lower for *fig.* 79.

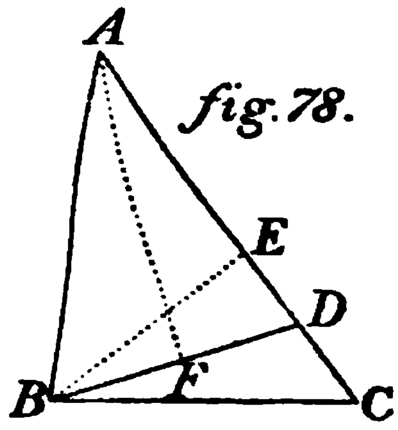


fig. 78.

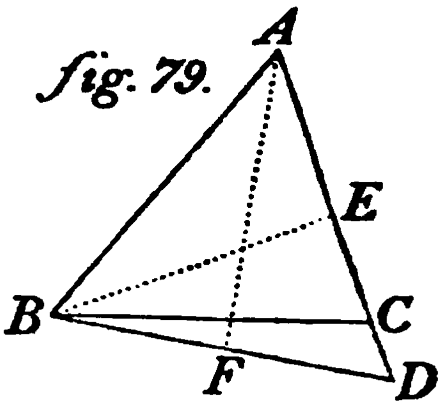


fig. 79.

1. If the angle BAC be known; then in the triangle ABC we have two angles and one side; consequently the triangle itself. Put $\therefore$ $BAC = \phi$.

2. Then

$$AC = \frac{a\ Sin.\ (\alpha+\phi)}{Sin.\ \phi}\ AB = \frac{a\ Sin.\ \alpha}{Sin.\ \phi}:$$

consequently, since $AC - AB = \pm d$,

$$\frac{a\ [Sin.\ (\alpha + \phi) - Sin.\ \alpha]}{Sin.\ \phi} = \pm d.$$

3. But $Sin.\ (\alpha+\phi) - Sin.\ \alpha = 2\ Cos.\ (\alpha + \frac{1}{2}\phi)\ Sin.\ \frac{1}{2}\phi$ and $Sin.\ \phi = 2\ Sin.\ \frac{1}{2}\phi\ Cos.\ \frac{1}{2}\phi$; we have $\therefore$

$$\frac{2\ a\ Cos.\ (\alpha + \frac{1}{2}\phi)\ Sin.\ \frac{1}{2}\phi}{2\ Sin.\ \frac{1}{2}\ \phi\ Cos.\ \frac{1}{2}\ \phi} = \pm\ d,$$

or
$$\frac{a\ Cos.\ (\alpha + \frac{1}{2}\phi)}{Cos.\ \frac{1}{2}\ \phi} = \pm\ d.$$

4. $Cos.\ (\alpha + \frac{1}{2}\phi) = Cos.\ \alpha\ Cos.\ \frac{1}{2}\ \phi - Sin.\ \alpha\ Sin.\ \frac{1}{2}\ \phi$, and $\frac{Sin.\ \frac{1}{2}\ \phi}{Cos.\ \frac{1}{2}\ \phi} = Tan.\ \frac{1}{2}\phi$; consequently

$$a\ (Cos.\ \alpha - Sin.\ \alpha\ Tan.\ \tfrac{1}{2}\ \phi) = \pm\ d,$$

and $\therefore\ Tan.\ \frac{1}{2}\ \phi = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha} = Cot.\ \alpha \mp \frac{d}{a\ Sin.\ \alpha}$.

Exam. When $a = 173'$, $d = -27'$, $\alpha = 56^0.\ 25'.\ 13''$, then $\phi = 80^0.\ 48'.\ 36''$.

Const. To the given base BC, apply the angle $BCA=\alpha$; on CA (*fig.* 78), when $CA > AB$, and consequently d is positive, or on CA produced (*fig.* 79), when $CA < AB$, and consequently d is negative, take $CD=d$, and draw BD; then make the angle $ABD = ADB$, and produce the lines, till they intersect each other in A, then ABC is the triangle sought.

Upon AC, BD, draw the perpendiculars BE, AF: then (because $BC=a$, $BCA=\alpha$), $BE=a\ Sin.\ \alpha$, $CE=a Cos.\ \alpha$; $\therefore\ DE = a\ Cos.\ \alpha \mp d$, (the upper sign for *fig.* 78, and the lower for *fig.* 79) and

$$Tan.\ DBE = \frac{DE}{BE} = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha}.$$

Now, since $\angle\ ABD = \angle\ ADB$, by the construction, consequently ADB is an isosceles triangle, and $\therefore$, because AF is perpendicular to the base BD, $DAF=\frac{1}{2}\ BAD=\frac{1}{2}\ \phi$. But $\angle\ DAF = \angle\ DBE$, because $\triangle\ ADF$ is similar to $\triangle\ DBE$; consequently also,

$$Tan.\ \tfrac{1}{2}\ \phi = \frac{a\ Cos.\ \alpha \mp d}{a\ Sin.\ \alpha}.$$

Synthetic Proof. The triangle ABC has the given base BC, and the given angle BCA (by the construction).

Further, in the isosceles triangle $AB = AD$, and $\therefore$

$$AC - AB = AC - AD = \pm CD = \pm d.$$

Remark. If d is positive, and $= a\ Cos.\ \phi$, the formula gives $Tan. \frac{1}{2} \phi = 0$, and $\therefore \phi = 0$; in the figure the point D will then be in E, and we have $\angle ABD = \angle ADB = 90°$, consequently $AB \parallel AC$. If $d > a\ Cos.\ \phi$; then $Tan. \frac{1}{2} \phi$ is negative, consequently $\frac{1}{2} \phi$ is an obtuse angle, and $\phi > 180°$; then, in the figure, D is above E, $\therefore ACB$, and consequently also ABC is an obtuse angle, and in this case the lines AB, AC, do not meet on this, but on the other side of the line BC. Since $\therefore$ for this case, the angle ACB is no longer in the triangle ACB, but is an exterior angle of this figure; consequently this solution, at least, cannot be made use of here.

SECTION LXVII.

Prob. *The base of a triangle, the sum of its other two sides, and vertical angle are given: find the triangle.*

Solut. In the triangle BAC (*fig.* 80), the base $BC = a$, the sum of the sides $AB + AC = s$, and the vertical angle $BAC = \alpha$, are given: find the triangle.

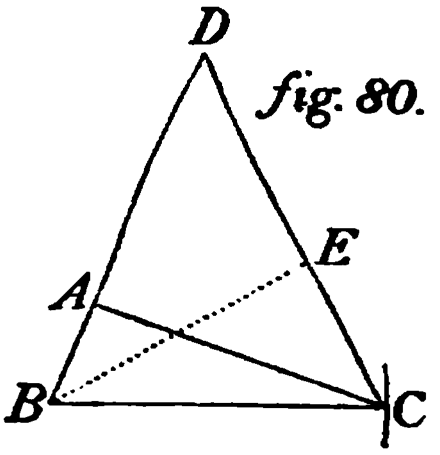


fig. 80.

1. Since $BAC = \alpha$, $ABC + ACB = 180^0 - \alpha$. The sum of the two angles at the base is consequently known, and it is only necessary to know their difference, in order to find the angles themselves. Let $\therefore ABC - ACB = \phi$; then

$$ABC = 90^0 - \tfrac{1}{2}\alpha + \tfrac{1}{2}\phi,\ ACB = 90^0 - \tfrac{1}{2}\alpha - \tfrac{1}{2}\phi.$$

2. We $\therefore$ have

$$AC = \frac{a\ Sin.\ ABC}{Sin.\ \alpha} = \frac{a\ Cos. \frac{1}{2}(\alpha - \phi)}{Sin.\ \alpha}.$$

$$AB = \frac{a\ Sin.\ ACB}{Sin.\ \alpha} = \frac{a\ Cos. \frac{1}{2}(\alpha + \phi)}{Sin.\ \alpha}.$$

Now since $AB + AC = s$,

$$\frac{a\left[Cos. \frac{1}{2}(\alpha + \phi) + Cos. \frac{1}{2}(\alpha - \phi)\right]}{Sin.\ \alpha} = s.$$

3. $Cos. \frac{1}{2}(\alpha + \phi) + Cos. \frac{1}{2}(\alpha - \phi) = 2\ Cos\ \frac{1}{2}\alpha\ Cos. \frac{1}{2}\phi$, and $Sin.\ \alpha = 2\ Sin. \frac{1}{2}\alpha\ Cos. \frac{1}{2}\alpha$, consequently

$$\frac{2\,a\,Cos.\,\frac{1}{2}\,\alpha\,Cos.\,\frac{1}{2}\,\phi}{2\,Sin.\,\frac{1}{2}\,\alpha\,Cos.\,\frac{1}{2}\,\alpha} = s,$$

and $$Cos.\,\tfrac{1}{2}\,\phi = \frac{s\,Sin.\,\frac{1}{2}\,\alpha}{a}.$$

4. Having from hence determined ϕ; then we also have the angles $ABC = 90^0 - \frac{1}{2}(\alpha - \phi)$, $ACB = 90^0 - \frac{1}{2}(\alpha + \phi)$, and consequently also the triangle itself.

Exam. For $a = 125' \cdot 67$, $s = 152' \cdot 39$, $\alpha = 49^0.\ 37'.\ 48''$, then $\phi = 118^0.\ 48'.\ 56''$, then $ABC = 124^0.\ 35'.\ 34''$, $ACB = 5^0.\ 46'.\ 38''$.

Const. Draw a line $BD = s$; to D apply the angle $BDC = \frac{1}{2}\,\alpha$; from B, with a radius $BC = a$, describe a circle, which cuts the line DC in C; make the angle $DCA = ADC$, and draw CB; then ABC is the triangle sought.

Upon DC draw the perpendicular BE; then $BE = BD\,Sin.\,BDC = s\,Sin.\,\frac{1}{2}\,\alpha$, and $Cos.\,CBE = \frac{BE}{BC} = \frac{s\,Sin.\,\frac{1}{2}\,\alpha}{a} = Cos.\,\frac{1}{2}\,\phi$; consequently, $CBE = \frac{1}{2}\,\phi$, $BCE = 90^0 - \frac{1}{2}\,\phi$. We ∴ have $ACB = DCB - DCA = 90^0 - \frac{1}{2}\,\phi - \frac{1}{2}\,\alpha$, $ABC = DBE + EBC = 90^0 - \frac{1}{2}\,\alpha + \frac{1}{2}\,\phi$, which was required.

Synthetic Proof. Since $\angle\ BDC = \angle\ DCA$; $BAC = BDC + DCA = 2\,BDC = \alpha$; also $BA + AC = BA + AD = BD = s$, and $BC = a$; consequently BAC is the required triangle.

Remark. The problem is only positive, when $a < s$, and also $s\,Sin.\,\frac{1}{2}\,\alpha < a$; the first, because the sum of two sides of a triangle is always greater than the third side; the second, because $Cos.\,\frac{1}{2}\,\phi$ cannot be greater than 1. It is also evident from the figure, that $BC = a$ cannot be greater than $BE = s\,Sin.\,\frac{1}{2}\,\alpha$, because otherwise it could not reach the line CD, which is required by the construction.

If in the problem, the difference of the sides, instead of the sum, be given; then it may be solved in a similar way.

SECTION LXVIII.

Prob. *The base of a triangle, the difference of the two angles at the base, and the difference of the sides, are given: find the triangle.*

Solut. In the triangle ABC (*fig.* 81), the base $BC = a$, the difference of the angles at the base, or $ABC - ACB = \alpha$, and the difference of the two sides, or $AC - AB = d$, are given: find the triangle.

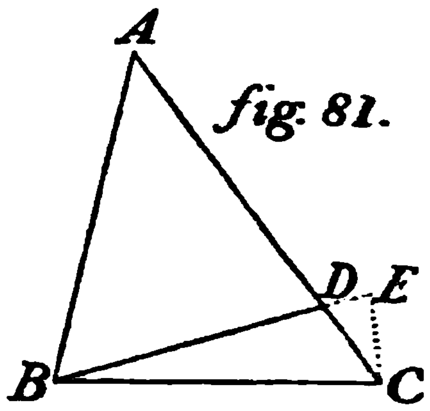


fig. 81.

1. Since the difference of the angles ABC, ACB is given; consequently it is only necessary to know their sum, in order to be able to determine the angles themselves. Put ∴ $ABC + ACB = \phi$; then we have, $ABC = \frac{1}{2}(\phi + \alpha)$, $ACB = \frac{1}{2}(\phi - \alpha)$; further, $BAC = 180^0 - \phi$.

2. Hence it follows, that

$$AC = \frac{a \; Sin. \frac{1}{2}(\phi + \alpha)}{Sin. \; \phi}, \quad AB = \frac{a \; Sin. \frac{1}{2}(\phi - \alpha)}{Sin. \; \phi}.$$

Therefore $AC - AB = \dfrac{a[Sin. \frac{1}{2}(\phi + \alpha) - Sin. \frac{1}{2}(\phi - \alpha)]}{Sin. \; \phi} = d.$

3. $Sin. \frac{1}{2}(\phi + \alpha) - Sin. \frac{1}{2}(\phi - \alpha) = 2 \; Cos. \frac{1}{2}\phi \; Sin. \frac{1}{2}\alpha$, $Sin. \; \phi = 2 \; Sin. \frac{1}{2}\phi \; Cos. \frac{1}{2}\phi$; consequently,

$$\frac{2 \, a \; Cos. \frac{1}{2}\phi \; Sin. \frac{1}{2}\alpha}{2 \; Sin. \frac{1}{2}\phi \; Cos \frac{1}{2}\phi} = d.$$

and ∴ $$Sin. \frac{1}{2}\phi = \frac{a \; Sin. \frac{1}{2}\alpha}{d}.$$

4. Having from hence determined the value of ϕ; we then likewise have the angles $ABC = \frac{1}{2}(\phi + \alpha)$, $ACB = \frac{1}{2}(\phi - \alpha)$, $BAC = 180^0 - \phi$, and ∴, since the line BC is also given, the triangle ABC is known.

Exam. When $a = 234'$, $d = 98'$, $\alpha = 30^0. \; 59'. \; 34''$,

then $\frac{1}{2}\phi = 39^0.\ 38'.\ 22''$, $ABC = \frac{1}{2}\phi + \frac{1}{2}\alpha = 55^0.\ 8'.\ 9''$, $ACB = \frac{1}{2}\phi - \frac{1}{2}\alpha = 24^0.\ 8'.\ 35''$.

CONST. To BC, the given base of the triangle, apply the angle $CBD = \frac{1}{2}\alpha$, and from C, with a distance $CD = d$, cut the line BD in D; draw CD, and produce it; then make the angle $DBA = ADB$, and produce BA, DA, till they meet in A: then ABC is the triangle sought.

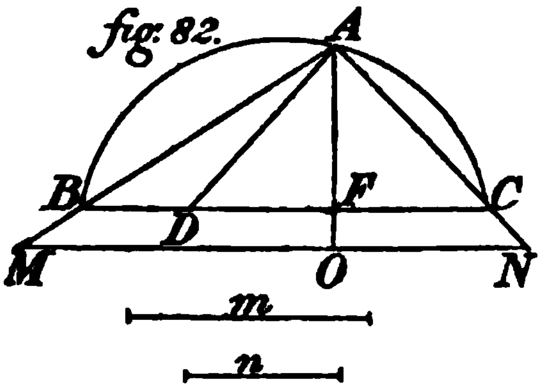
Produce BD, and draw CE perpendicular to it: then $CE = BC\ Sin.\ CBE = a\ Sin.\ \frac{1}{2}\ \alpha$, $Sin.\ CDE = \frac{CE}{CD}$ $= \frac{a\ Sin.\ \frac{1}{2}\alpha}{d} = Sin\ \frac{1}{2}\phi$; consequently $CDE = ADB = ABD$ $= \frac{1}{2}\phi$, $ABC = ABD + CBD = \frac{1}{2}\phi + \frac{1}{2}\alpha$, and $ACB = ADB$ $- CBD = \frac{1}{2}\phi - \frac{1}{2}\alpha$, which was required.

REMARK. The problem is impossible, when CD is less than the perpendicular CE; for then the line CD cannot reach BD. This agrees also with the calculation, since d cannot be less than $a\ Sin.\ \frac{1}{2}\alpha$, because otherwise $Sin.\ \frac{1}{2}\phi > 1$, which is impossible.

SECTION LXIX.

PROB. *The vertical angle of a triangle, and the segments into which the perpendicular drawn from the vertex of this angle divides the base, are given: find the triangle.*

SOLUT. In the triangle ABC (*fig.* 82), the vertical angle $BAC = \alpha$ is given; also the segments $BF = a$, $CF = b$, into which the base is divided by the perpendicular AF, are given: determine the triangle.



1. Since $BF = a =$ $AF\ Tan.\ BAF$, and $CF = b = AF\ Tan.\ CAF$: therefore

$$a : b = Tan.\ BAF : Tan.\ CAF;$$

consequently

$$a+b:a-b = Tan.BAF + Tan.CAF : Tan.BAF - Tan.CAF$$
$$= Sin.\,(BAF+CAF) : Sin.\,(BAF-CAF)$$
$$= Sin.\,\alpha : Sin.\,(BAF-CAF).$$

2. Hence we obtain

$$Sin.\,(BAF - CAF) = \frac{(a-b)\,Sin.\,\alpha}{a+b}$$

or, since $BAF = 90^0 - ABC$, and $CAF = 90^0 - ACB$, and $\therefore BAF - CAF = ACB - ABC$,

$$Sin.\,(ACB - ABC) = \frac{(a-b)\,Sin.\,\alpha}{a+b}$$

3. Having from this equation determined the difference $ACB - ABC$ of the angles at the base; then, because their sum is already known, we have also the angles ACB, ABC, for $ACB + ABC = 180^0 - \alpha$.

Exam. When $a = 247'$, $b = 53'$, $\alpha = 113^0.\ 20'.\ 54''$, we find $ACB - ABC = 36^0.\ 25'.\ 15''{\cdot}25$. Now since $ACB + ABC = 66^0.39'.6''$: therefore $ACB = 51^0.32'.10''{\cdot}62$, $ABC = 15^0.\ 6'.\ 55''{\cdot}37$.

Const. On an indefinite line, make $BF = a$, $CF = b$; upon BC describe a segment of a circle BAC, containing an angle equal to the given angle α. From F draw the perpendicular FA, which meets the segment in A, and draw the lines BA, CA; then BAC is the required triangle.

Make $FD = FC$; then $\angle FAD = \angle FAC$, $\angle ADC = \angle ACD$; further

$$BC : AB = Sin.\,BAC : Sin.\,ACB$$
$$AB : BD = Sin.\,ADB : Sin.\,BAD$$
$$= Sin.\,ACB : Sin.\,BAD,$$

consequently $BC : BD = Sin.\,BAC : Sin.\,BAD$;
or since $BC = a+b$, $BD = a-b$, $BAC = \alpha$, $BAD = BAF - FAD = BAF - FAC = ACB - ABC$,

$$a+b : a-b = Sin.\,\alpha : Sin.\,(ACB - ABC)$$

and $\therefore Sin. (ACB - ABC) = \frac{(a - b) Sin. \alpha}{a + b}$,

which was required.

Cor. If the segments themselves are not given, but merely their proportion to one another, and also the altitude of the required triangle: let $m : n$ be the given proportion between the segments, and p the altitude of the triangle. First construct, as was done before, a triangle BAC, whose segments BF, FC are equal to the lines m, n, and upon the perpendicular AF, produced if necessary, make $AO = p$; through O, parallel to BC, draw the line MN, which cuts AB, AC, or these lines produced, in M, N: the $\triangle\, MAN$ is the required triangle. This method is very easily understood.

If instead of the altitude of the triangle, one of its sides, viz. AM, be given: then in AB, or this line produced, it is only necessary to determine the point M, so that AM may be of the required length, and then from M to draw MN parallel to BC, in order to obtain the required triangle MAN.

SECTION LXX.

Prob. *The vertical angle of a triangle, the sum of its sides, and the difference of the segments into which a perpendicular from the vertex of this angle divides the base, are given: find the triangle.*

Solut. 1. Let BAC (*fig.* 82) be the required triangle, the sum of its sides $AB + AC = a$, the difference of the segments $BF - CF = d$, and the vertical angle $BAC = \alpha$. If the difference of the sides AB, AC be also known; then we can determine each of these. Assume therefore $AB - AC = x$: then $AB = \frac{a + x}{2}$, $AC = \frac{a - x}{2}$. Further, let $BC = y$.

2. Since $AB^2 = AF^2 + BF^2$, $AC^2 = AF^2 + CF^2$: then $AB^2 - AC^2 = BF^2 - CF^2$, or

$$(AB + AC)(AB - AC) = (BF + CF)(BF - CF),$$

or
$$ax = dy.$$

3. In the triangle BAC, we have

$$BC^2 = AB^2 + AC^2 - 2\,AB\,.\,AC\,.\,Cos.\,\alpha,$$

or $y^2 = \left(\frac{a+x}{2}\right)^2 + \left(\frac{a-x}{2}\right)^2 - \frac{a^2-x^2}{2}\,Cos.\,\alpha.$

If in this last equation we substitute for y its value $\frac{ax}{d}$ taken from 2, and solve the equation, we then obtain

$$x = \sqrt{\frac{a^2d^2(1 - Cos.\,\alpha)}{2\,a^2 - d^2(1 + Cos.\,\alpha)}}$$

or, since $1 + Cos.\,\alpha = 2\,Cos.^2\,\frac{1}{2}\alpha$, $1 - Cos.\,\alpha = 2\,Sin.^2\,\frac{1}{2}\alpha$,

$$x = \sqrt{\frac{a^2d^2\,Sin.^2\,\frac{1}{2}\alpha}{a^2 - d^2\,Cos.^2\,\frac{1}{2}\alpha}} = \frac{ad\,Sin.\,\frac{1}{2}\alpha}{\sqrt{(a^2 - d^2\,Cos.^2\,\frac{1}{2}\alpha)}}.$$

Solut. 2. 1 From 2 of the foregoing solution, we have

$$a(AB - AC) = dy,$$

and $\therefore$ $$AB - AC = \frac{dy}{a}.$$

2. Assume the angle $ABC = \phi$; then $ACB = 180^\circ - (\alpha + \phi)$, and

$$AB = \frac{y\,Sin.\,(\alpha + \phi)}{Sin.\,\alpha},\ AC = \frac{y\,Sin.\,\phi}{Sin.\,\alpha};$$

consequently, $$AB - AC = \frac{y\,[Sin.\,(\alpha + \phi) - Sin.\,\phi]}{Sin.\,\alpha}$$

$$= \frac{2\,y\,Cos.\,(\frac{1}{2}\alpha + \phi)\,Sin.\,\frac{1}{2}\alpha}{2\,Sin.\,\frac{1}{2}\alpha\,Cos\,\frac{1}{2}\alpha},$$

$$= \frac{y\,Cos.\,(\frac{1}{2}\alpha + \phi)}{Cos.\,\frac{1}{2}\alpha}$$

3. If the two expressions for $AB - AC$ in 1 and 2 be put equal to one another, we then obtain

$$Cos.\,(\tfrac{1}{2}\alpha + \phi) = \frac{d\,Cos.\,\frac{1}{2}\alpha}{a};$$

whence the angle ϕ may be determined.

4. Further, since $AB+AC=a$; therefore from 2 we have

$$\frac{y\,[Sin.\,(\alpha+\phi)+Sin.\,\phi]}{Sin.\,\alpha}=a,$$

or
$$\frac{2\,y\,Sin.\,(\frac{1}{2}\,\alpha+\phi)\,Cos.\,\frac{1}{2}\,\alpha}{2\,Sin.\,\frac{1}{2}\,\alpha\,Cos.\,\frac{1}{2}\,\alpha}=a,$$

and $\therefore$
$$y=\frac{a\,Sin.\,\frac{1}{2}\,\alpha}{Sin.\,(\frac{1}{2}\,\alpha+\phi)};$$

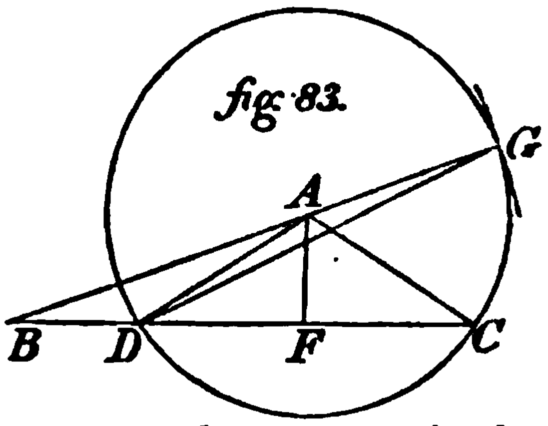
further, $$AB=\frac{y\,Sin.\,(\alpha+\phi)}{Sin.\,\alpha}=\frac{a\,Sin.\,(\alpha+\phi)}{2\,Cos.\,\frac{1}{2}\,\alpha\,Sin.\,(\frac{1}{2}\alpha+\phi)},$$

$$AC=\frac{y\,Sin.\,\phi}{Sin.\,\alpha}=\frac{a\,Sin.\,\phi}{2\,Cos.\,\frac{1}{2}\,\alpha\,Sin.\,(\frac{1}{2}\,\alpha+\phi)}.$$

Since the second solution gives all the parts of the required triangle at once, by means of formulæ, which are simple and easy of calculation, it is consequently preferable to the first.

Exam. When $a=207'$, $d=13'.\,7''$, $\alpha=61^{\circ}.\,23'.\,18''$, we find $\frac{1}{2}\,\alpha+\phi=86^{\circ}.\,44'.\,14''\!\cdot\!8$, $\phi=56^{\circ}.\,2'.\,35''\!\cdot\!8$; and hence $BC=y=105'\!\cdot\!8357$, $AB=107'\!\cdot\!0022$, $AC=99'\!\cdot\!9976$.

Const. Upon an indefinite line (*fig.* 83), take $BD=d$, and make CDG equal to half the adjacent angle of the given vertical angle, consequently $=90^{\circ}-\frac{1}{2}\,\alpha$; from B, with the radius $BG=a$, cut the line DG in G, and join BG; then make the angle $GDA=DGA$, and from A, with the radius AB, describe a circle, which cuts the indefinite line in C, and join AC; then BAC is the required triangle.



For since $CDG=90^{\circ}-\frac{1}{2}\alpha$; therefore $BGD=90^{\circ}-\frac{1}{2}\,\alpha-ABC$; consequently $Sin.BDG=Sin.\,CDG=Cos.\frac{1}{2}\,\alpha$, $Sin.\,BGD=Sin.\,[90^{\circ}-(\frac{1}{2}\alpha+ABC)]=Cos.\,(\frac{1}{2}\,\alpha+ABC)$. But in the triangle BGD

$$BG:BD=Sin.\,BDG:Sin.\,BGD,$$

or $a : d = Cos.\ \frac{1}{2}\alpha : Cos.\ (\frac{1}{2}\alpha + ABC)$;

consequently $Cos.\ (\frac{1}{2}\alpha + ABC) = \frac{d\ Cos.\ \frac{1}{2}\alpha}{a}$.

But analytically

$$Cos.\ (\frac{1}{2}\alpha + \phi) = \frac{d\ Cos.\ \frac{1}{2}\alpha}{a};$$

we have $\therefore \frac{1}{2}\alpha + ABC = \frac{1}{2}\alpha + \phi$, and $ABC = \phi$, which were required.

Synthetic Proof. Upon BC draw the perpendicular AF; then BF, FC, are the segments of the triangle BAC.

Since $\angle GAC = 2 \angle GDC$, (*Euc. III.* 20); therefore GAC is the adjacent angle of the vertical angle in the required triangle; consequently BAC is this vertical angle. Further

$$AB + AC = AB + AG = BG = a,$$
$$BF - CF = BF - DF = BD = d.$$

Q. E. D.

SECTION LXXI.

Prob. *The base of a triangle, its altitude, and the difference of the angles at the base, are given: find the triangle.*

Solut. Let the required triangle be BAC (*fig.* 82); the base $BC = a$, the altitude $AF = h$, and the difference of the angles at the base $ACB - ABC = \alpha$.

1. If from these data the sum of the angles ACB, ABC, can be determined; we then have the angles, and consequently also the triangle. Put $\therefore ACB + ABC = \phi$; then $ACB = \frac{1}{2}(\phi + \alpha)$, $ABC = \frac{1}{2}(\phi - \alpha)$.

2. From the right-angled triangles AFC, AFB, we obtain

$$CF = h\ Cot.\ \tfrac{1}{2}(\phi + \alpha),\quad BF = h\ Cot.\ \tfrac{1}{2}(\phi - \alpha).$$

Now since $CF + BF = a$; we have the equation

$$h\,[Cot.\ \tfrac{1}{2}(\phi + \alpha) + Cot.\ \tfrac{1}{2}(\phi - \alpha)] = a.$$

3. But $Cot. \frac{1}{2}(\phi + \alpha) + Cot. \frac{1}{2}(\phi - \alpha)$

$$= \frac{Sin.\ \phi}{Sin.\ \frac{1}{2}(\phi + \alpha)\ Sin.\ \frac{1}{2}(\phi - \alpha)},$$

$$Sin.\ \frac{1}{2}(\phi + \alpha)\ Sin.\ \frac{1}{2}(\phi - \alpha) = \frac{1}{2}\left[Cos.\ \alpha - Cos.\ \phi\right];$$

instead of the former equation, we consequently have the following one:

$$\frac{h\ Sin.\ \phi}{\frac{1}{2}\left[Cos.\ \alpha - Cos.\ \phi\right]} = a,$$

or $$Sin.\ \phi + \frac{a}{2h}\ Cos.\ \phi = \frac{a}{2h}\ Cos.\ \alpha.$$

4. In order from hence to determine the value of ϕ in the easiest way, we can make use of the following artifice. Put $Tan.\ \mu$ for $\frac{a}{2h}$; consequently determine an angle μ such, that its tangent is equal in numerical value to the expression, $\frac{a}{2h}$; by these means the foregoing equation is transformed into the following one:

$$Sin.\ \phi + Tan.\ \mu\ Cos.\ \phi = Tan.\ \mu\ Cos.\ \alpha\ ;$$

or, when we multiply both sides by $Cos.\ \mu$, and substitute $Sin.\ \mu$ for $Cos.\ \mu\ Tan.\ \mu$,

$$Cos.\ \mu\ Sin.\ \phi + Sin.\ \mu\ Cos.\ \phi = Sin.\ \mu\ Cos.\ \alpha,$$

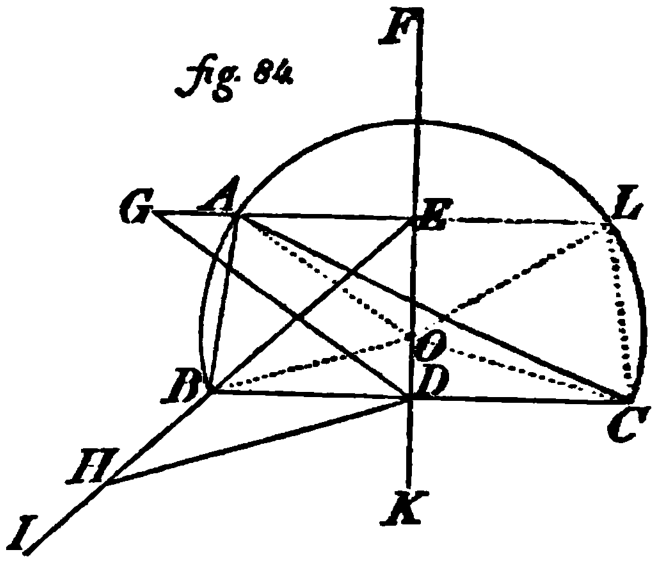
or likewise $$Sin.\ (\phi + \mu) = Sin.\ \mu\ Cos.\ \alpha,$$

whence ϕ may now be determined.

EXAM. Let $a = 1365'$, $h = 789'$, $\alpha = 21^{\circ}.\ 39'.\ 18''$. Here $log.\ Tan.\ \mu = log.\ \frac{a}{2h} = 9{\cdot}9370257$, and $\therefore \mu = 40^{\circ}.\ 51'.\ 37''{\cdot}55$. Hence we obtain further, $log.\ Sin.\ (\phi + \mu)$ $log.\ Sin.\ \mu + log.\ Cos.\ \alpha = 9{\cdot}7839362$, and consequently $\phi + \mu = 37^{\circ}.\ 26'.\ 53''\ 96$, or $\phi + \mu = 142^{\circ}.\ 33'.\ 6''{\cdot}04$. The first of the two values found for $\phi + \mu$ need not to be used here, because $\phi + \mu$ must necessarily be greater than μ. If $\therefore$ we take the second value, we then obtain, $\phi = 101^{\circ}.41'.28''{\cdot}49$, and hence $ACB = \frac{1}{2}(\phi + \alpha) = 61^{\circ}.40'.23''{\cdot}24$, $ABC = \frac{1}{2}(\phi - \alpha) = 40^{\circ}.\ 1'.\ 5''{\cdot}24$.

Const. From D, the centre of the given base BC, (*fig.* 84), draw KF perpendicular to it, and make $DE=h$; from E draw the line EG parallel to BC, and to ED apply the angle $EDG=\alpha$; then by these means the point G is determined. Draw EB, and from D, with the distance DG, cut EB produced in H; then describe upon BC an arc, including the angle HDK; and from the point A, in which this arc cuts the line EG, draw the lines AB, AC; then BAC is the triangle sought.

fig. 84



This construction may be derived from the analytical solution, in the following way:

Since $BD=\frac{1}{2}a$, $DE=h$; therefore $Tan.\ BED=\frac{BD}{DE}=\frac{a}{2h}=Tan.\ \mu$; consequently $BED=\mu$. Further, in the triangle HED,

$$HD : ED = Sin.\ HED : Sin.\ DHE,$$

or since $HD=DG=DE\ Sec.\ GDE=\frac{h}{Cos.\ \alpha}$,

$$\frac{h}{Cos.\ \alpha} : h = Sin.\ \mu : Sin.\ DHE,$$

consequently $Sin.\ DHE = Sin.\ DHI = Sin.\ \mu\ Cos.\ \alpha$.

But by the analytical solution also, $Sin.\ (\phi+\mu)=Sin.\ \mu+Cos.\ \alpha$; we have $\therefore$ either $\phi+\mu=DHE$, or $\phi+\mu=DHI$. The first supposition cannot obtain here; for since $DH=DG>DE$, consequently also $\angle\ DEH>\angle\ DHE$ (*Euc. I.* 18); $\therefore \mu>\phi+\mu$, which is impossible. We $\therefore$ have $DHI=\phi+\mu$, and consequently $EDH=DHI-DEH=\phi$. Now since ϕ denotes the sum of the two angles at the base of the required triangle; consequently HDK must be the vertical angle of this triangle; from which all the rest necessarily follows.

Synthetic Proof. The triangle BAC, as appears immediately

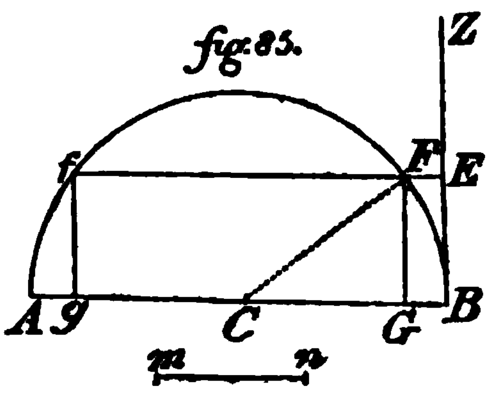
from the construction, has the given base and altitude; $\therefore$ it only remains to be proved, that likewise $ABC - ACB = GDE = a$. Produce $\therefore$ the line GE, till it cuts the circular arc in L, and from the centre O draw the radii OA, OB, OC, OL: then $BOC = 2\,BAC$ (*Euc. III.* 20), and $BOD = \frac{1}{2}BOC = BAC$; but likewise $HDK = BAC$ (construction), consequently $HDK = BOK$, and $BO \parallel HD$. Further, since $DH = DG$, $OB = OA$; consequently $DH : OB = DG : OA$; but likewise $DH : OB = DE : EO$, consequently $DG : OA = DE : EO$, $\triangle AEO$ is similar to $\triangle GED$ (*Euc. VI.* 7), and $\therefore OA \parallel DG$. Now draw CL, then $ACL = BCL - ACB = ABC - ACB$; further, $ACL = \frac{1}{2}AOL = AOE = GDE$; consequently $ABC - ACB = GDE$.

Q. E. D.

SECTION LXXII.

Prob. *To divide a straight line, so that the rectangle contained by the two parts may be equal to a given square.*

Solut. The given line $AB = a$ (*fig.* 85) is required to be divided in G, so that $AG \times GB = mn^2 = b^2$.



Let $AG = x$; then $BG = a - x$; rectangle $AG \times GB = x(a-x)$, and $\therefore$

$$x(a-x) = b^2.$$

The solution of this equation gives

$$x = \tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 - b^2)}.$$

Const. Upon AB describe a semicircle AFB: from B draw the perpendicular BZ, and then make $BE = mn$; from E draw the line EF parallel to AB, which cuts the semicircle in F, f; and from these points draw FG, fg, perpendicular to AB: then G or g is the required point of section.

Draw the radius CF: then $CG^2 = CF^2 - FG^2 = CB^2 - BE^2 = \frac{1}{4}a^2 - b^2$; consequently $CG = Cg = \sqrt{(\frac{1}{4}a^2 - b^2)}$, and $\therefore AG = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - b^2)}$. $AG = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - b^2)}$, which are the two values found for x.

The synthetic proof of this construction is founded on *Euc. VI.* 13, 17.

SECTION LXXIII.

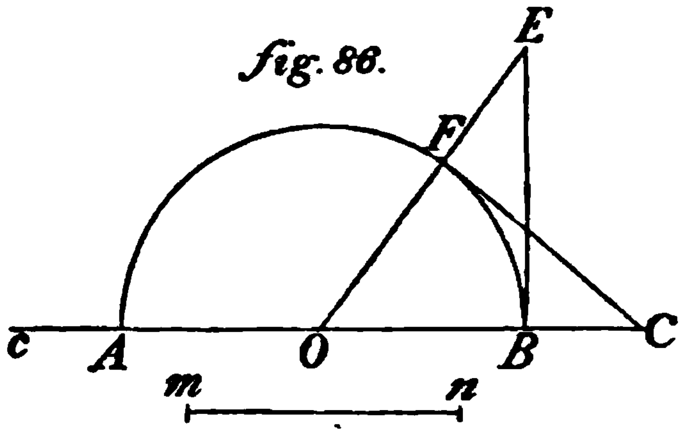
Prob. *To produce a given line, so that the rectangle contained by the whole line thus produced, and the part produced, may be equal to the square of a given line.*

Solut. Let the line AB be given (*fig.* 86): lengthen it by the part BC, so that the rectangle $AC \times CB = mn^2$

Make $AB = a$, $mn = b$, $BC = x$; then $AC = a + x$, and we ∴ have the following equation:

$$x(a + x) = b^2.$$

The solution gives



$$x = -\tfrac{1}{2}a \pm \sqrt{(\tfrac{1}{4}a^2 + b^2.)}$$

Const. Upon AB describe a semicircle; from B draw the perpendicular $BE = mn$; and from the centre O draw the line OE; to the point F, in which this line cuts the semicircle, draw the tangent FC, which meets AB produced in C; then BC is the required part, by which AB must be produced.

For since, as is readily seen, $\triangle\, EOB$ is similar and equal to $\triangle\, FOC$, and $OE^2 = OB^2 + BE^2 = \frac{1}{4}a^2 + b^2$: then $OC = OE = \sqrt{(\frac{1}{4}a^2 + b^2)}$, and ∴
$BC = OC - OB = \sqrt{(\frac{1}{4}a^2 + b^2)} - \frac{1}{2}a.$

If we produce the line AB in the other direction, and make $Oc = OC$, we then have $Bc = OB + OC = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 + b^2)}$, which is the absolute value of this line. But with respect to the position, if Bc be opposite to BC, we ∴ have analytically $Bc = -\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 + b^2)}$, which is the second value of x. Since in the problem nothing is said as to which side the line AB is to be produced on, consequently both values of x verify it, because $Bc \,.\, cA = BC \,.\, CA$. But if it is expressed in the problem, that the

line is to be produced towards B, then the first value of x need only be retained.

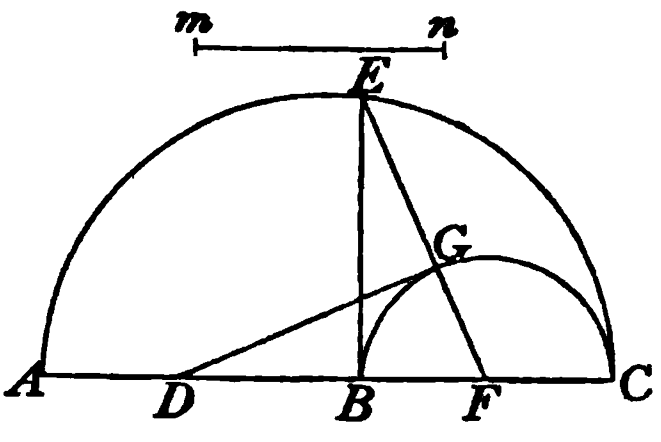
The synthetic proof of the construction is founded on *Euc. III.* 36, and is easily deduced from it.

SECTION LXXIV.

Prob. *To divide a given line into two parts, so that the rectangle contained by one part and another given line, may be equal to the square of the other part.*

Solut. Let (*fig.* 87) two lines AB, mn, be given: divide the first in D, so that rectangle $AD \times mn = BD^2$.

fig. 87.



Let $AB = a$, $mn = b$, $BD = x$: then $AD = a - x$, and $\therefore$, from the conditions of the problem, we have the equation

$$(a - x)\,b = x^2.$$

The solution of this equation gives

$$x = -\tfrac{1}{2}b \pm \sqrt{(ab + \tfrac{1}{4}b^2)}.$$

Const. Upon that part of AB which is produced, take $BC = mn$, and upon AC, BC, describe two semicircles; from B draw the perpendicular BC, and from F, the center of the semicircle BGC, draw the line FE; to the point of intersection G, draw the tangent GD, which meets AB in D: then D is the required point of section.

For since $BE^2 = AB \times BC$ (*Euc. VI.* 13, 17) $= ab$, and $BF = \frac{1}{2}BC = \frac{1}{2}b$: consequently $FE^2 = BE^2 + BF^2 = ab + \frac{1}{4}b^2$, and $FE = \sqrt{(ab + \frac{1}{4}b^2)}$. But $\triangle\, DFG$ is similar and equal to $\triangle\, BFE$, and $\therefore$ $DF = FE$; consequently also $DF = \sqrt{(ab + \frac{1}{4}b^2)}$, and $\therefore$ $DB = DF - BF = \sqrt{(ab + \frac{1}{4}b^2)} - \frac{1}{2}b$, which was required.

The second value of x, because it is negative, does not obtain here, otherwise the point D, which, according to the

problem, is to be an intersection of the line AB, will fall beyond the point B, and consequently not between A and B.

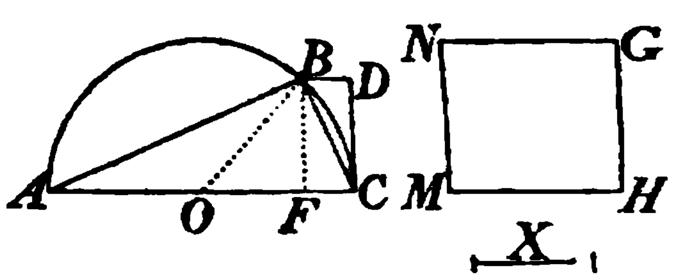
Synthetic Proof. Since $\triangle DFG$ is similar and equal to $\triangle BFE$: then $BE=DG$. Now $BE^2=AB \times BC$ (*Euc. VI.* 13, 17), $DG^2 = BD \times DC$ (*Euc. III.* 36); consequently also $AB \times BC = BD \times DC$. Take the rectangle $BD \times BC$ from both sides; we then have $(AB - BD)$ $BC = (DC - BC) \times BD$, or $AD \times BC = AD \times mn = BD^2$. Q. E. D.

SECTION LXXV.

PROB. *To find two lines such, that the sum of their squares may be equal to a given square, together with the rectangle contained by these lines.*

SOLUT. Let AC (*fig.* 88) the side of the given square $= a$; let the given rectangle be MG, its sides $MN = b$, $NG = c$. Let x, y be the two lines sought. From the conditions of the problem, we obtain the two following equations:

fig. 88.



$$xy = bc, \quad x^2 + y^2 = a^2.$$

The solution of these equations gives:

$$x = \sqrt{\left[\tfrac{1}{2}a^2 + \sqrt{\left(\tfrac{1}{4}a^4 - b^2c^2\right)}\right]}$$
$$y = \sqrt{\left[\tfrac{1}{2}a^2 - \sqrt{\left(\tfrac{1}{4}a^4 - b^2c^2\right)}\right]}$$

Hence we get the construction.

CONST. Find the fourth proportional to the three lines AC, MN, NG (*Euc. VI.* 12); let this be X, so that $AC : MN = NG : X$. Then describe upon AC a semicircle, and from C draw the perpendicular $CD = X$; from D draw DB parallel to AC, which meets the semicircle in B, and from B draw the lines BA, BC; these are the lines sought.

Draw the radius OB, and the perpendicular BF. Since $AC : MN = NG : X$ (Construction), or $a : b = c : X$; therefore $X = CD = BF = \frac{bc}{a}$: consequently $OF = \surd(OB^2 - BF^2)$ $= \surd\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)$. Hence we obtain further: $AF = \frac{1}{2}a + \surd\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)$, $CF = \frac{1}{2}a - \surd\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right)$. Now since $AB^2 = AC \times AF = \frac{1}{2}a^2 + a\surd\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right) = \frac{1}{2}a^2 + \surd(\frac{1}{4}a^4 - b^2c^2)$, $BC^2 = AC \times CF = \frac{1}{2}a^2 - a \times \surd\left(\frac{1}{4}a^2 - \frac{b^2c^2}{a^2}\right) = \frac{1}{2}a^2 - \surd(\frac{1}{4}a^4 - b^2c^2)$: consequently $AB = \surd[\frac{1}{2}a^2 + \surd(\frac{1}{4}a^4 - b^2c^2)]$, $BC = \surd[\frac{1}{2}a^2 - \surd(\frac{1}{4}a^4 - b^2c^2)]$, which was required

Synthetic Proof. Since ABC is a right angle, consequently $AC^2 = AB^2 + BC^2$, which verifies the first condition of the problem. Further, since $AC : MN = NG : X$ $(= BF)$: consequently rectangle $AC \times BF$ = rectangle $MN \times NG$. But rectangle $AC \times BF$ = rectangle $AB \times BC$, because both rectangles are double the triangle ABC; consequently also rectangle $AB \times BC$ = rectangle $MN \times NG$. Q. E. D.

REMARK. We may also, as I presume to be already known, give the analytical expressions for x and y, the following more simple forms:

$$x = \tfrac{1}{2}[\surd(a^2 + 2bc) + \surd(a^2 - 2bc)]$$
$$y = \tfrac{1}{2}[\surd(a^2 + 2bc) - \surd(a^2 - 2bc)].$$

The formulæ given in the solution are, however, better adapted to the construction.

SECTION LXXVI.

PROB. *To find a right-angled triangle such, that its hypothenuse is equal to a given line, and the rectangle contained by its two sides is equal to the square of the difference of these two sides.*

SOLUT. Let ABC (*fig.* 88) be the required triangle,

whose hypothenuse $AC = a$ is given, and in which rectangle $AB \times BC = (AB - BC)^2$.

Let $AB = x$, $BC = y$: then, from the conditions of the problem, we obtain the two following equations:

$$x^2 + y^2 = a^2,\ xy = (x - y)^2;$$

and these give

$$x = \sqrt{\frac{a^2}{6}}(3 + \sqrt{5}),\ y = \sqrt{\frac{a^2}{6}}(3 - \sqrt{5}).$$

In order to adapt these expressions found for x and y to the construction, give them the following forms:

$$\sqrt{a[\tfrac{1}{2}a + \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{9}a^2)}]},\ \sqrt{a[\tfrac{1}{2}a - \sqrt{(\tfrac{1}{4}a^2 - \tfrac{1}{9}a^2)}]}.$$

Const. Upon a given hypothenuse $AC = a$, describe a semicircle; from C draw the perpendicular $CD = \frac{1}{3}AC = \frac{1}{3}a$, and from D draw DB parallel to AC, which meets the semicircle in B; from B draw the lines BA, BC: ABC is the triangle sought.

Draw the radius OB, and the perpendicular BF; then $BF = CD = \frac{1}{3}a$, $OB = \frac{1}{2}a$, consequently $OF = \sqrt{(OB^2 - BF^2)} = \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$, and $\therefore AF = \frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$, $CF = \frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}$. Now since $AB^2 = AC \times AF$, $BC^2 = AC \times CF$; consequently $AB = \sqrt{a[\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}]}$, $BC = \sqrt{a[\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - \frac{1}{9}a^2)}]}$, which was required.

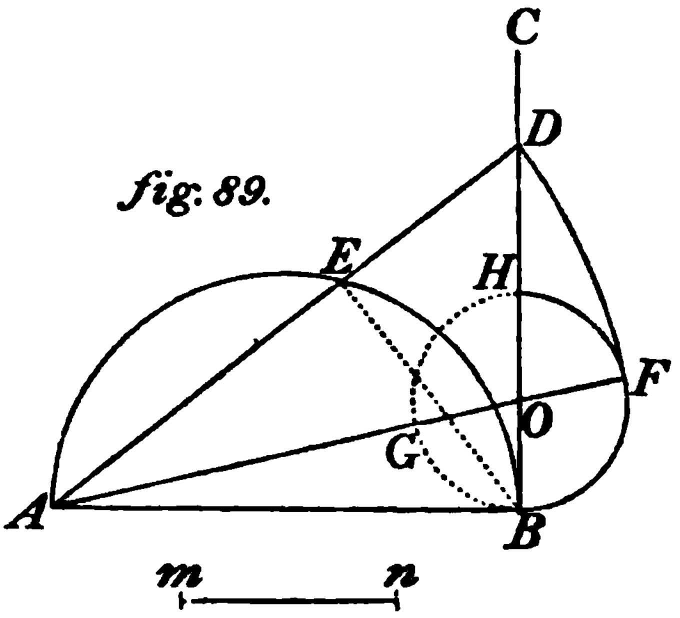
Synthetic Proof. Since $\triangle ABC$ is similar to $\triangle BFC$: then $AB : AC = BF : BC$; consequently also $3\,AB : AC = 3\,BF : BC = AC : BC$, and $\therefore 3\,AB \times BC = AC^2$. But because ABC is a right angle, $AC^2 = AB^2 + BC^2$; we consequently have $3\,AB \times BC = AB^2 + BC^2$, and $AB \times BC = AB^2 - 2\,AB \times BC + BC^2 = (AB - BC)^2$. Q. E. D.

SECTION LXXVII.

Prob. *At the extremity of the given diameter of a semicircle, a perpendicular is drawn: find a point in this*

perpendicular such, that when a line is drawn from this point to the other extremity of the diameter, that part of it which is without the circle, may be equal to a given line.

Solut. Let AB (*fig.* 89) be the given diameter of a semicircle, BC a perpendicular upon it: find a point D such, that when DA is drawn, DE may be equal to a given line mn.



Let $AB = a$, $mn = b$, $AD = x$, and draw BE. Since AEB is a right angle, consequently $\triangle\, ADB$ is similar to $\triangle\, ABE$, and $\therefore$

$$AD : AB = AB : AE$$

$$x : a = a : x - b.$$

Hence we obtain

$$x^2 - bx = a^2,$$

and $$x = \tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + a^2)}.$$

Const. Upon BC, make $BH = mn = b$, and upon BH describe a semicircle BFH; from A through its center O draw the line AF, and with a radius AF, describe an arc, which cuts the line BC in D: D is the required point.

For since $AB = a$, $BO = \tfrac{1}{2}BH = \tfrac{1}{2}b$; consequently $AO = \sqrt{(BO^2 + AB^2)} = \sqrt{(\tfrac{1}{4}b^2 + a^2)}$; then $AD = AF = AO + OF = \tfrac{1}{2}b + \sqrt{(\tfrac{1}{4}b^2 + a^2)}$, which was required.

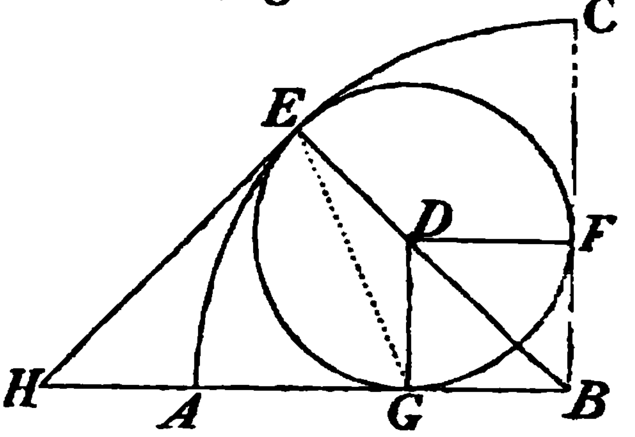
Synthetic Proof. Complete the circle $BFHG$. Since BD touches the circle AEB, and AB the circle $BFHG$; $\therefore BD^2 = AD \times DE$, $AB^2 = FA \times AG$; consequently $AD \times DE + FA \times AG = BD^2 + AB^2 = AD^2$, and $\therefore FA \times AG = AD^2 - AD \times DE = AD \times AE$. Now $FA = AD$; consequently also $AG = AE$, and $\therefore DE = GF$, or, since $GF = BH = mn$, $DE = mn$. Q. E. D.

SECTION LXXVIII.

PROB. *In a given quadrant to describe a circle which touches both the circumference and the two radii.*

SOLUT. Let ACB (*fig.* 90) be the given quadrant, and its radius $= a$. Further, let EFG be the required circle, D its centre, and E, F, G, the points in which it touches the circumference, and the two radii of the quadrant.

fig. 90.



1. Draw DF, DG: then DFB, DGB, are right angles (*Euc. III.* 18); likewise ABC is a right angle; consequently $BFDG$ is a parallelogram, and since $DG = DF$, it is also a square; $\therefore$ DBC is half a right angle. Produce BD; consequently this line passes through the point E. (*Euc. III.* 2).

2. Let $DE = DF = DG = x$; then $BD = \surd(BG^2 + DG^2) = \surd\, 2\, x^2 = x\,\surd\, 2$; consequently $BE = BD + DE = x\surd 2 + x$. Now, since also $BE = a$: therefore

$$x\,\surd\, 2 + x = a,$$

and $\therefore$ $$x = \frac{a}{\surd\, 2 + 1} = a\,(\surd\, 2 - 1),$$

or likewise $$x = \surd\, 2\, a^2 - a.$$

Hence we obtain the following construction.

CONST. Bisect the angle ABC by the line BE; to the point E, where this line meets the circumference AC, draw the tangent EH, which meets BA produced in H; then make $BG = AH$, and from G draw the perpendicular GD; from D, where this perpendicular meets BE, with a radius DG, describe the circle EFG: this is the required circle.

For since BEH is a right angle, and EBH half a right angle; consequently $BE = EH = a$, and $\therefore$ $BH =$

$BH = \sqrt{(BE^2 + EH^2)} = \sqrt{2a^2}$; consequently $BG = AH = BH - BA = \sqrt{2a^2} - a$, which was required.

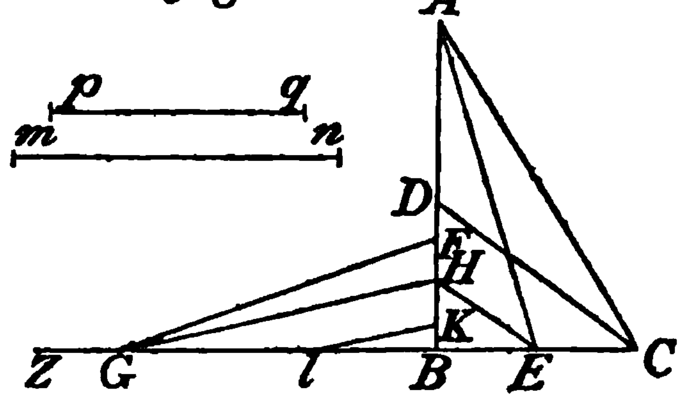
Synthetic Proof. Draw DF perpendicular to BC; then $DFB = DGB$, $DBG = DBF$, $BD = BD$; consequently $\triangle DGB$ is similar and equal to $\triangle DFB$, and $DF = DG$; hence the circle described with the radius DG passes through F, and touches the lines BC, BA, in F, G (*Euc. III.* 16). Further, since $HBE = \frac{1}{2}R$: then $BE = EH$; but likewise $AH = BG$ (Construction), consequently $HG = AB = BE = EH$, and $\therefore \triangle HGE$ is isosceles, and $HGE = HEG$. But also $DEH = DGH$ ($= R$); consequently $DEG = DGE$, and $\therefore DG = DE$. Therefore the circle described with the radius DG, also passes through E, and it meets the circumference AB in no other point, otherwise a line drawn from B to this point would be equal to BE, which is impossible. (*Euc. III.* 8).

SECTION LXXIX.

Prob. *The lines drawn from the vertical angle of a right-angled triangle to the centre of the opposite sides, are given: find the triangle.*

Solut. Find a right-angled triangle ABC (*fig.* 91) such, that the sides AB, BC, containing the right angle, when bisected in D, E, and the lines AE, CD are drawn, $AE = mn = a$, and $CD = pq = b$.

fig.91.



Put $BD = x$: then $BC^2 = CD^2 - BD^2 = b^2 - x^2$; consequently $BE^2 = \frac{1}{4} BC^2 = \frac{b^2 - x^2}{4}$, and $AE^2 = AB^2 + BE^2 = 4x^2 + \frac{b^2 - x^2}{4} = \frac{b^2 + 15x^2}{4}$. Now since $AE = a$; we consequently have the equation

$$\frac{b^2 + 15\,x^2}{4} = a^2,$$

and hence $x = \sqrt{\dfrac{4\,a^2 - b^2}{15}}$.

CONST. Upon any line AB, to the point B draw a perpendicular CZ, and make $BF = \frac{1}{2}pq = \frac{1}{2}b$; from F with a distance $FG = mn = a$ cut BZ in G; in BA take any part BK, from K, with a distance $Kl = 4\,BK$, cut BZ in l; and from G draw GH parallel to Kl, which meets the line BA in H; then from H make $HE = \frac{1}{2}pq = \frac{1}{2}b$ cut the line BC in E, and make $BC = 2\,BE$, $BA = 4\,BH$. Draw AC: then ABC is the triangle sought.

For since $FG = a$, $BF = \frac{1}{2}b$ (Construction): then $BG^2 = FG^2 - BF^2 = a^2 - \frac{1}{4}b^2$: Further, because $\triangle\ BHG$ is similar to $\triangle\ BKI$, $GH : BH = Kl : KB = 4 : 1$ (Construction), and $\therefore$ $GH = 4\,BH$, $GH^2 = 16\,BH^2$, $BG^2 = GH^2 - BH^2 = 15\,BH^2$; consequently $15\,BH^2 = a^2 - \frac{1}{4}b^2$, $BH^2 = \dfrac{a^2 - \frac{1}{4}b^2}{15}$; $\therefore$ (because $x = \frac{1}{2}AB = 2\,BH$),

$x^2 = \dfrac{4\,a^2 - b^2}{15}$, and $x = \sqrt{\dfrac{4\,a^2 - b^2}{15}}$, which was required.

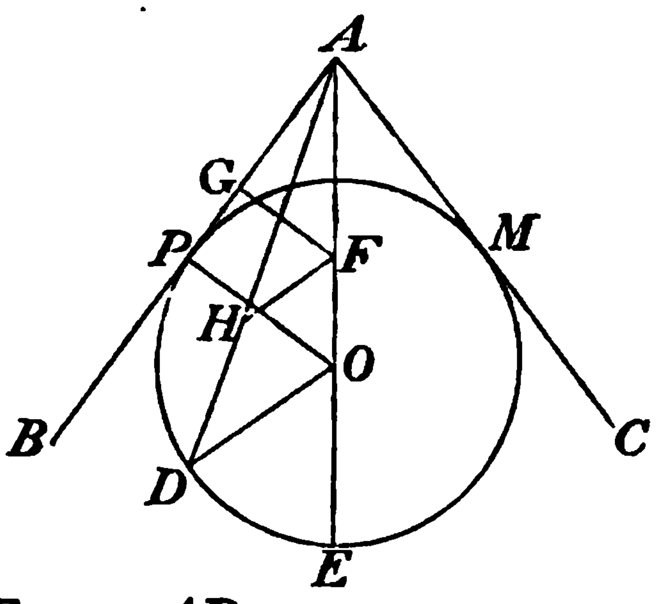
Synthetic Proof. Assume $BD = 2\,BH = \frac{1}{2}AB$, and draw CD: then, because also $BC = 2\,BE$, $CD \parallel HE$, and $\therefore$ $CD = 2\,HE = pq$ (Construction); wherefore the line CD, which is drawn from C to the centre of AB, is of the given length. Further, because the triangles BKI, BGH, are similar, as was proved before, $GH = 4\,BH$, and also $AB = 4\,BH$ (Construction): then $AB = GH$, and $\therefore$ $GB^2 = GH^2 - BH^2 = AB^2 - BH^2$. But $AE^2 - HE^2 = AB^2 - BH^2$, (because $AE^2 = AB^2 + BE^2$, $HE^2 = BH^2 + BE^2$), consequently $GB^2 = AE^2 - HE^2$. But likewise $GB^2 = GF^2 - BF^2 = GF^2 - HE^2$, (because by the Construction, $BF = \frac{1}{2}pq = HE$); consequently we have $AE^2 - HE^2 = GF^2 - HE^2$, and $\therefore$ $AE = GF$; hence, since $GF = mn$ (Construction), also $AE = mn$. Q. E. D.

SECTION LXXX.

Prob. *An angle and a point in it are given: describe a circle which touches both the lines containing the angle, and passes through the given point.*

Solut. Let the given angle be *BAC* (*fig.* 92), the given point *D*, *PMD* the required circle, which touches the lines *AB*, *AC*, in *P*, *M*, and passes through the point *D*.

fig. 92.



1. Through *O*, the centre of the circle, draw the line *AE*; consequently this bisects the angle *BAC*. Further, since the point *D* is given, we likewise have the angle *DAE*, and the line *AD*. Therefore let $BAE = \alpha$, $DAE = \beta$, $AD = a$.

2. If the angle *ADO* be known; then in the triangle *ADO* we have one side and two angles, consequently also the lines *AO*, *OD*, and at the same time the centre and radius of the circle. However, this angle can be very easily found; for since $AO : OP = AO : OD$, and $AO : OP = 1 : \text{Sin.}\ \alpha$, $AO : OD = \text{Sin.}\ ADO : \text{Sin.}\ \beta$: then

$$1 : \text{Sin.}\ \alpha = \text{Sin.}\ ADO : \text{Sin.}\ \beta,$$

$$\text{and} \therefore \quad \text{Sin.}\ ADO = \frac{\text{Sin.}\ \beta}{\text{Sin.}\ \alpha}.$$

Hence we get the following very simple Construction.

Const. Bisect the angle *BAC* by the line *AE*, and then take any point *F*; from this point draw *FG* perpendicular to *AC*, and with the distance $FH = FG$, cut the line *AD* in *H*: then from *D* draw *DO* parallel to *HF*: *O* is the centre of the circle.

For since $FH = FG$ (Construction); therefore $AF : FG =$

$AF : FH$. But $AF : FG = 1 : Sin.\ \alpha$, $AF : FH = Sin.\ AHF : Sin.\ \beta$; consequently $1 : Sin.\ \alpha = Sin.\ AHF : Sin.\ \beta$, and $\therefore\ Sin.\ AHF = Sin.\ ADO = \frac{Sin.\ \beta}{Sin.\ \alpha}$, which was required.

Synthetic Proof. Draw OP perpendicular to AB: then $OP \parallel FG$, $DO \parallel HF$; consequently $AF : FG = AO : OP$, and $AF : FH = AO : OD$. But $AF : FG = AF : FH$, (because $FG = FH$, by the construction); consequently $AO : OP = AO : OD$, and $\therefore\ OP = OD$. A circle described with a radius OD, will consequently pass through P, and at this point touch the line AB, consequently also AC.

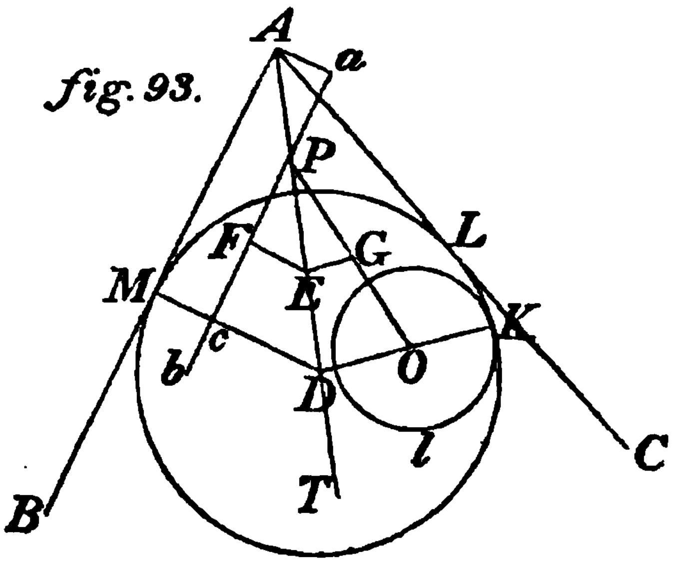
Remark. Since there is another point besides H in the line AD, from which $FH = FG$: consequently strictly there are two circles, which verify the problem.

SECTION LXXXI.

Prob. *An angle, and a circle within it, are given: describe another circle, which at the same time touches the two lines containing the given angle and the given circle.*

Solut. Let BAC (*fig.* 93) be the given angle, Kl the given circle, whose centre is O, and MLK the required circle, which touches the lines AB, AC in M, L, and the circle in K. The centre D of the required circle must, as in the preceding section, be in the line AD, which bisects the angle BAC; further, the centres of the two circles, and the point of contact, must be in a straight line (*Euc.* III. 11.)

fig. 93.



1. In AD take any point P, and draw PO: then, besides

the angle BAC, and the radius OK of the given circle, the lines AP, PO, and the angle OPD may be considered as given. Therefore let $DAC = \frac{1}{2} BAC = \alpha$, $OPD = \beta$, $AP = a$, $PO = b$, $OK = r$.

2. If the angle POD be known; then we can draw the line OD, and then the centre D is determined. Assume ∴ $POD = \phi$: then

$$DO = \frac{b \ Sin. \beta}{Sin. (\beta + \phi)}, \quad PD = \frac{b \ Sin. \phi}{Sin. (\beta + \phi)},$$

consequently $DK = DO + OK = \dfrac{b \ Sin. \beta}{Sin. (\beta + \phi)} + r$,

$$AD = PD + AP = \frac{b \ Sin. \phi}{Sin. (\beta + \phi)} + a.$$

3. Now $DK = DM = AD \ Sin. \alpha$; we consequently have the equation

$$\frac{b \ Sin. \beta}{Sin. (\beta + \phi)} + r = \left[\frac{b \ Sin. \phi}{Sin. (\beta + \phi)} + a\right] Sin. \alpha,$$

or

$$b \ Sin. \beta + r \ Sin. (\beta + \phi) = b \ Sin. \alpha \ Sin. \phi + a \ Sin. \alpha \ Sin. (\beta + \phi):$$

from which equation the value of ϕ may be determined.

4. Since the point P may be assumed arbitrarily: determine it ∴, so that the obtained equation may be more simple. This is the case, when we put $AP = a = \dfrac{r}{Sin. \alpha}$; for by these means the foregoing equation is transformed into the following one:

$$Sin. \beta = Sin. \alpha \ Sin. \phi,$$

and this gives $Sin. \phi = \dfrac{Sin. \beta}{Sin. \alpha}$;

from which we obtain a very easy construction.

CONST. Bisect the given angle BAC by the line AT; from A draw the perpendicular $Aa = r$, make ab parallel to

AB, and from the point *P*, where it cuts the line *AT*, draw *PO* to the centre of the given circle. In *AT* take any point *E*, draw *EF* perpendicular to *ab*, and with the distance $EG = EF$, cut the line *PO* in *G*. Through *O* draw the line *DK* parallel to *EG*; then the point *D*, in which this line meets *AT*, is the centre, and *DK* the radius of the required circle.

For we have $AP = \dfrac{Aa}{Sin.\ APa} = \dfrac{r}{Sin.\ \alpha}$, as was required. Further, because $EF = EG$ (Construction), $EF : EP = EG : EP$; but $EF : EP = Sin.\ \alpha : 1$, $EG : EP = Sin.\ \beta : Sin.PGE$; consequently $Sin.\alpha : 1 = Sin.\beta : Sin.PGE$, and $\therefore Sin.\ PGE = Sin.\ POD = \dfrac{Sin.\ \beta}{Sin.\ \alpha}$.

Synthetic Proof. Draw *DM* perpendicular to *AB*: then $Dca = DMA = R$; consequently, since also $EFP = R$ (Construction) $Dc \parallel EF$. Now likewise $DO \parallel EG$, hence $PE : EF = PD : Dc$, and $PE : EG = PD : DO$. But $PE : EF = PE : EG$ (because $EF = EG$), consequently $PD : Dc = PD : DO$, and $\therefore Dc = DO$; and since also $cM = Aa = OK$; then $DM = DK$. A circle described with the radius *DK*, passes $\therefore$ through *M*, and touches the line *AB*, consequently also *AC*. But since the two circles touch in *K*, it appears, from hence, that they can have no other common point, because otherwise a line drawn from *D* to this point must be equal to *DK*, which is impossible (*Euc. III.* 8.)

Remark. Since in *PO* there are always two points, such as *G*, from which $EG = EF$; consequently there are also always, as in the foregoing section, two circles, which verify the problem.

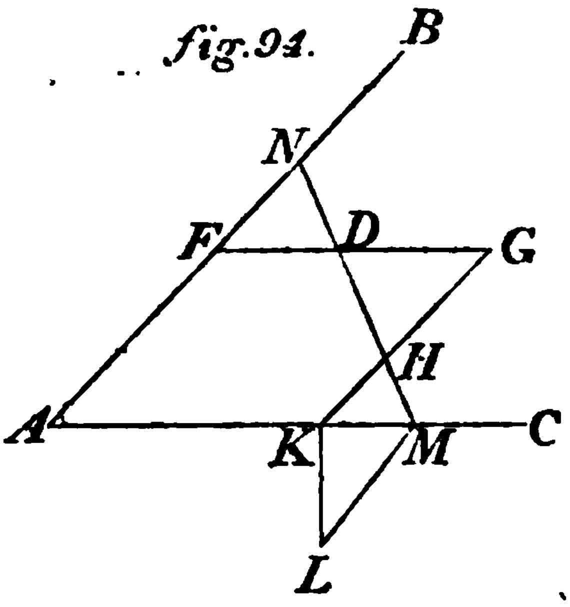
SECTION LXXXII.

Prob. *An angle and a point within it are given: through this point draw a line, which meets the two lines including the angle, and with them forms a triangle of a given area.*

Solut. Let *BAC* (*fig.* 94) be the given angle, and *D*

the given point within it: through this point draw a line MN such, that the $\triangle NAM$ has a given area.

fig. 94.



1. Through D draw a line FG parallel to AC: then the lines AF, FD may be considered as known. To AF apply a parallelogram $AFGK$, which has the given area (*Euc. I.* 45): consequently the solution of the problem depends merely upon this, to draw the line MN in such a way, that $\triangle NAM$ = parallelogram $AFGK$.

2. If this be done: then $\triangle DGH = \triangle DNF + \triangle HKM$. But $\triangle DGH : \triangle DNF = DG^2 : DF^2$, and $\triangle DGH : \triangle HKM = DG^2 : KM^2$ (*Euc. VI.* 19); consequently also $\triangle DGH : \triangle DNF + \triangle HKM = DG^2 : DF^2 + KM^2$, and since $\triangle DGH = \triangle DNF + \triangle HKM$, then likewise $DG^2 = DF^2 + KM^2$, and $\therefore$

$$KM = \sqrt{(DG^2 - DF^2)}.$$

Hence, since the lines DG, DF are known, the line KM, and at the same time also the point M, may be very easily determined, both by calculation and by notation.

CONST. After having applied a parallelogram $AFGK$ to AF, as required in the solution, upon AC draw from K the perpendicular KL, and with the distance $LM = DG$, cut the line KC in M: then from M through D draw the line MN: what was required is now done.

The proof readily follows from the solution itself.

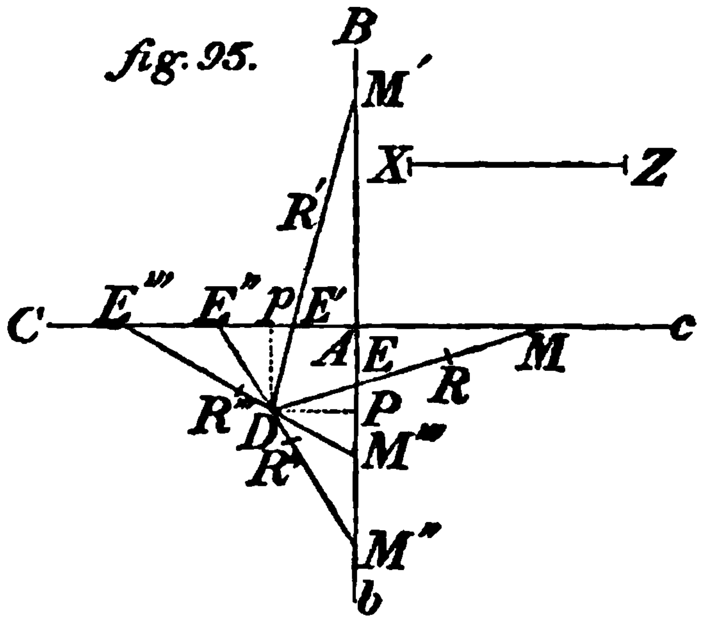
SECTION LXXXIII.

PROB. *Two lines intersecting each other at right angles, and a point which is equally distant from these two lines are given: through this point draw a line, so*

that the part of it contained between the two lines containing one of the right angles may be equal to a given line.

SOLUT. Let the two lines Bb, Cc (*fig.* 95) intersect each other at right angles in the point A; let the point D be so situated, that the two perpendiculars DP, Dp are equal to one another; through D draw a line DM, so that its part EM, which lies between the lines Ab, Ac, may be equal to the given line XZ.

fig. 95.



1. If the line DE or DM be determined: then the problem is solved. Instead, however, of finding these lines immediately, it will be better, to assume their sum for the unknown magnitude in the calculation, because their difference is already given. Let $\therefore DP = Dp = a$, $DM - DE = XZ = 2b$, $DM + DE = 2x$: then $DM = x+b$, $DE = x - b$.

2. Since $\triangle DPE$ is similar to $\triangle EAM$: therefore

$$DE : DP = EM : AM$$

or

$$x - b : a = 2b : AM;$$

and $\therefore$

$$AM = \frac{2ab}{x-b},$$

$$Mp = a + \frac{2ab}{x-b} = \frac{a(x+b)}{x-b}.$$

3. Now $DM^2 = Mp^2 + Dp^2$; we $\therefore$ have the equation

$$(x+b)^2 = a^2 + \frac{a^2(x+b)^2}{(x-b)^2},$$

or

$$x^4 - 2(a^2 + b^2)x^2 + b^4 - 2a^2b^2 = 0.$$

This bi-quadratic equation is a quadratic one for x^2; its

solution gives

$$x = \pm \sqrt{[a^2 + b^2 \pm a \sqrt{(a^2 + 4b^2)}]}.$$

4. Consequently x has four values, all of which verify the equation. For since it was not expressed in the problem, in which of the four right angles the part EM between the lines containing the angles was situated; consequently this line may have four different situations; viz. either the position EM, within the angle bAc, as was assumed in the solution, or the position $E'M'$, within the angle BAC, or even the two positions $E''M''$, $E'''M'''$, within the angle bAC. If the line EM be bisected in R: then $DR = \frac{1}{2}(DM + DE) = x$; likewise if the lines $E'M'$, $E''M''$, $E'''M'''$, be bisected in R', R'', R''': then DR', DR'', DR''', are the three other values of x. The lines DR, DR', are equal with respect to their absolute magnitudes, and only differ in their positions; these are expressed by the two values $\pm \sqrt{[a^2 + b^2 + a \sqrt{(a^2 + 4b^2)}]}$. In like manner, the lines DR'', DR''', only differ in their position, and are expressed by the two values $\pm \sqrt{[a^2 + b^2 - a \sqrt{(a^2 + 4b^2)}]}$. The two first values of x, which correspond to the lines DR, DR', are always positive; the two last are only so, when $a^2 + b^2 > a \sqrt{(a^2 + 4b^2)}$, or $a^4 + 2a^2b^2 + b^4 > a^4 + 4a^2b^2$, that is $b^4 > 2a^2b^2$, and $b^2 > 2a^2$.

CONST. Let Bb, Cc, (*fig.* 96) be the two lines intersecting each other, D the given point, and $DP = Dp = a$, be two perpendiculars to the former of these lines. In Dp take the part $DS = XZ$ (preceding figure) $= 2b$, and draw PS; then in DP produced, take $PT = PS$; upon DT describe a semicircle, which cuts the line Cc in M, and draw DM: then the part EM, which lies between the two lines including the angle bAc, is equal to the given line XZ.

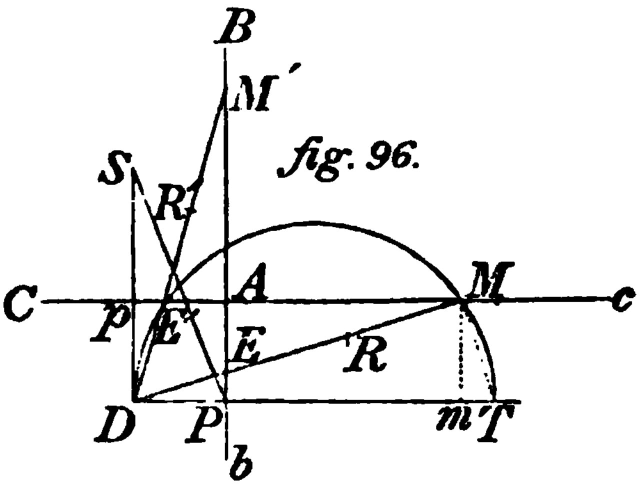


fig. 96.

For since $DP = a$, $DS = 2b$: then $PS = \sqrt{(a^2 + 4b^2)}$, and $\therefore DT = DP + PT = DP + PS = a + \sqrt{(a^2 + 4b^2)}$. Now draw MT, and the perpendicular Mm. Since $\angle TMm$

$= \angle\, MDT$, and $DP = Mm$: consequently the right-angled triangles TMm, DPE, are equal, and $\therefore\ MT = DE$. If EM be bisected in R: then $DR = \frac{1}{2}(DM + DE) = \frac{1}{2}(DM + MT)$. Now $DM^2 + MT^2 = DT^2$, and $DM \times MT = Mm \times DT = a . DT$, hence $DM^2 + 2DM . MT + MT^2 = DT^2 + 2a . DT = DT(2a + DT) = [a + \sqrt{(a^2 + 4b^2)}] \times [3a + \sqrt{(a^2 + 4b^2)}] = 4a^2 + 4b^2 + 4a\sqrt{(a^2 + 4b^2)}$, and $\therefore\ DM + MT = \sqrt{[4a^2 + 4b^2 + 4a\sqrt{(a^2 + 4b^2)}]}$, $DR = \sqrt{[a^2 + b^2 + a\sqrt{(a^2 + 4b^2)}]}$, as was required.

Synthetic Proof. Since $EM^2 = AM^2 + AE^2$, and $AP^2 = (AE + EP)^2 = AE^2 + 2AE \times EP + EP^2$: therefore $AM^2 + 2AE^2 + 2AE \times EP + EP^2 = EM^2 + AP^2$, or $AM^2 + 2PA \times AE + EP^2 = EM^2 + DP^2$. In the similar triangles AEM, DEP, $DP : AM = PE : AE$; and $DP \times AE\ (= PA \times AE) = AM \times EP$. We therefore have $AM^2 + 2AM \times EP + EP^2 = EM^2 + DP^2$, or $(AM + EP)^2 = EM^2 + DP^2$; consequently since $AM = Pm$, and $EP = mT$, $(Pm + mT)^2 = PT^2 = EM^2 + DP^2$. But likewise $PT^2 = PS^2 = DS^2 + DP^2$; hence $EM^2 + DP^2 = DS^2 + DP^2$, $\therefore\ EM^2 = DS^2$, and $EM = DS$. Q. E. D.

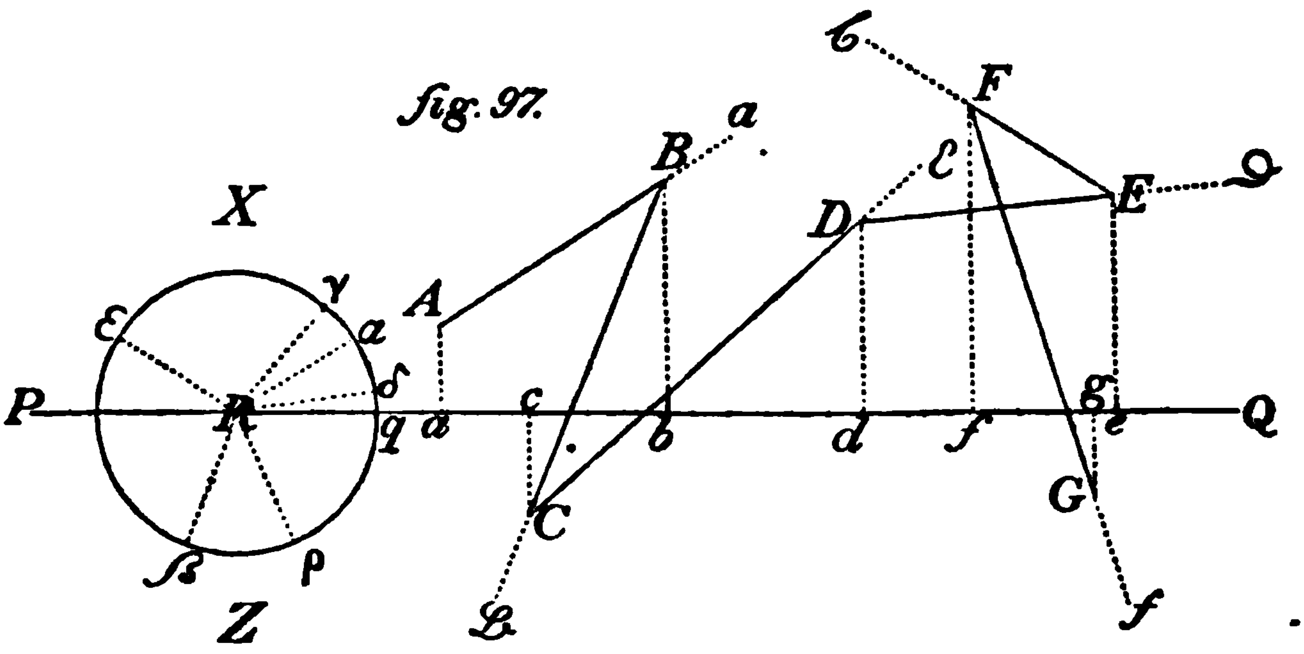
REMARK. The circle cuts the line Cc twice in E'. If the line DM' be drawn through this point, which meets Bb in M': then, if EM' be bisected in R', DR' is the second value of x denoted by the same letters as in *fig.* 95, and consequently $= DR$. This is very easily proved. For the other two values of x, a similar construction may be found, which, however, I shall not give here, in order that I may have room for other matter.

VIII. POLYGONOMETRICAL PROBLEMS.

SECTION LXXXIV.

DEFINITIONS.

1. If from all the angular points and extremities of a crooked line *ABCDEF* (*fig.* 97), the perpendiculars *Aa*,



Bb, *Cc*, *Dd*, *Ee*, *Ff*, *Gg*, be let fall upon any given straight line *PQ*: these are called Ordinates. Further, if in the line *PQ* we assume any point *R*; then the parts of this line which lie between this point and the ordinates, are called Abscissæ; thus *Ra* is the abscissa of the point *A*, *Rb* the abscissa of the point *B*, and so on: further, *PQ* is the axis of abscissæ, *R* the origin of abscissæ; the abscissa and ordinate of a point form together the co-ordinates of this point.

If the co-ordinates of a point are given; consequently also the point itself is. But the ordinate of a point may fall on both sides of the line *PQ*: consequently the determination

of the absolute magnitude of the ordinates is not sufficient; we must also know their position. In order to distinguish the above two positions from one another, the ordinates above the line PQ are expressed by $+$; on the other hand, those which are below it, by $-$. Therefore the points A, B, C, D, E, F, G, have the ordinates $+Aa$, $+Bb$, $-Cc$, $+Dd$, $+Ee$, $+Ff$, $-Gg$. There is a similar relation between the abscissæ; thus, the abscissæ which are from R, in the direction of Q, are denoted by $+$, and those which are in the direction of P, are expressed by $-$.

2. The lines AB, BC, CD, DE, EF, FG, of which the crooked line $ABCDEFG$ is composed, are called its lines of division. But if the crooked line inclose a space; consequently these lines, as we already know, are sides of the figure thus formed. If a line of division be produced, in order to avoid mistakes, it is denoted according to the order of the alphabet. Produce ∴ (when the contrary is not expressly required) the line AB not towards A, but towards B; in like manner BC, not towards B, but towards C, and so on. In this case it is advisable to denote the crooked line, from its first to its last point, according to the letters of the alphabet.

3. When from any point R of the axis of abscissæ, which need not be exactly the beginning of the abscissa, a circle is described, and the radii $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\varepsilon$, $R\zeta$, are respectively drawn parallel to the lines of division AB, BC, CD, DE, EF, FG, according to the order in which they are produced: then these radii are called the corresponding radii of these lines; viz. $R\alpha$ is the corresponding radius of AB, $R\beta$ the corresponding radius of BC, and so on.

4. By the exterior angles of a crooked line are here meant those angles which the continuation of a line of division makes with the other at their point of junction; thus CBa, DCb, EDc, FEd, GFe.

It is known, that when two lines, say M, N, are parallel to two others, say P, Q, viz. $M \parallel P$, $N \parallel Q$, the angle, which the lines M, N form at their junction, is equal to the

angle included by the lines P, Q. Consequently also the exterior angle of two lines of division, is equal to the angle formed by their corresponding radii; consequently $aBb = = aR\beta$, $bCc = \beta R\gamma$, $cDd = \gamma R\delta$, $dEe = \delta R\epsilon$, $eFf = \epsilon R\zeta$.

The absolute magnitude of the exterior angles is not sufficient, however, to determine the situation of two lines of division, because these lines may form a convex as well as a concave angle, although the exterior angle continues to be of the same magnitude: this $\therefore$ must be considered. With this view, imagine a moving line turning round the point R, and advancing successively in the directions $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\epsilon$, $R\zeta$, which, by these means, describe the angles $\alpha R\beta = aBb$, $\beta R\gamma = bCc$, $\gamma R\delta = cDd$: then it is evident, that this radius in advancing from $R\beta$ towards $R\gamma$, must have a motion, which, as to its direction, is exactly contrary to that which it describes in moving from $R\alpha$ towards $R\beta$. If $\therefore$ we consider the angles, which are generated by the motion of a radius from the object Q towards the objects X, P, Z, as positive; those angles which require a contrary movement may be considered as negative. Therefore the angles bCc, dEe, eFf, require the sign $+$, and the angles aBb, cDd, the sign $-$.

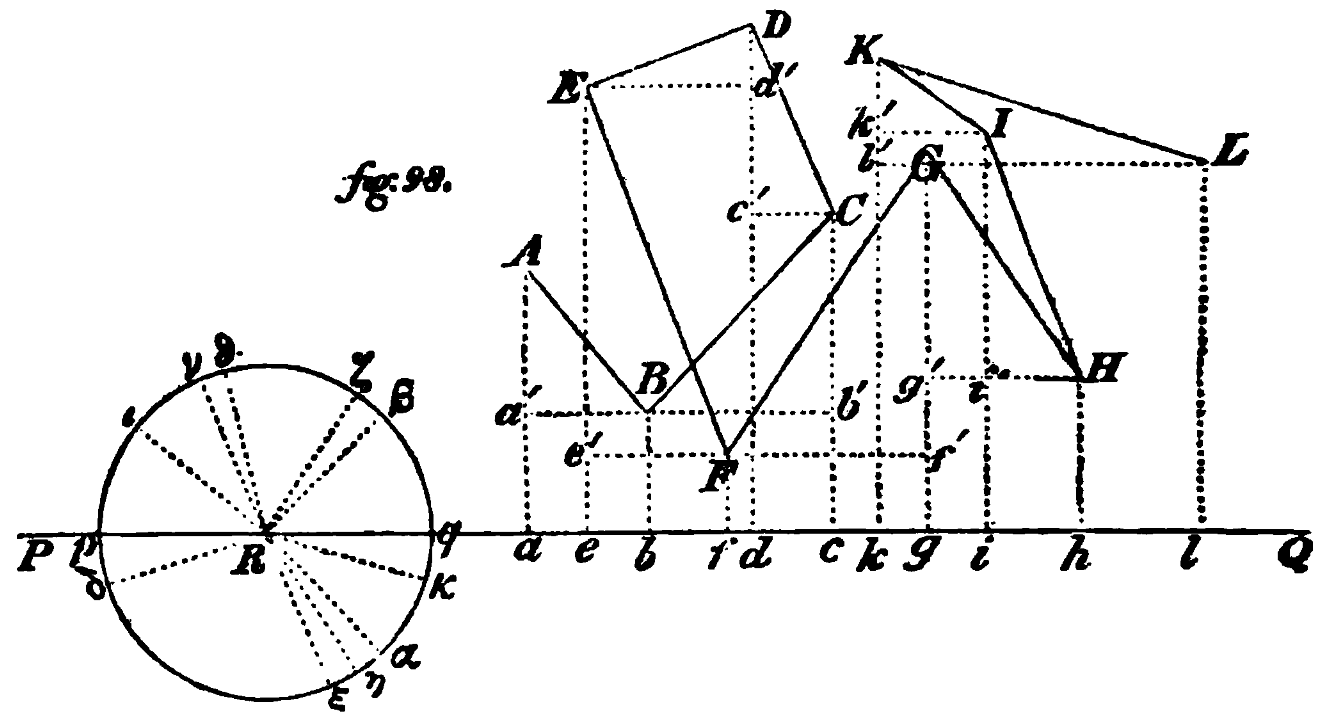
Let the exterior angles of a crooked line aBb, bCc, cDd, dEe, eFf, with their proper signs, $+$ or $-$, be constantly expressed according to the order, by the single letters B, C, D, E, F; $\therefore$ in the foregoing figure $B = - aBb$, $C = + bCc$, $D = - cDd$, $E = dEe$, $F = eFf$. Further, let the angle which the first corresponding radius, and consequently also the first line of division, makes with the line of abscissæ, here $qR\alpha$, be denoted by A. Let the line $R\alpha$ be above the line rq, then $A = - qR\alpha$.

SECTION LXXXV.

PROB. *The lines of division of a crooked line, its exterior angles, and the co-ordinates of the first point, together with the angle, which the first corresponding radius*

makes with the line of abscissæ, are given: determine the co-ordinates of each angular point.

Solut. Let $ABCDEFGHIKL$ (*fig.* 98) be the crooked



line, its lines of division $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$, $FG = f$, $GH = g$, $HI = h$, $IK = i$, $KL = k$; let the corresponding radii of these lines, according to their order, be $R\alpha$, $R\beta$, $R\gamma$, $R\delta$, $R\epsilon$. Further, let R be the origin of the abscissæ, the abscissa of the first point $Ra = p$, its ordinate $Aa = q$.

1. From the points B, C, D, E, &c. to the line of abscissæ PQ, draw the perpendiculars Bb, Cc, Dd, Ee, &c. We then have

$$\begin{aligned}
Aa &= q \\
Bb &= Aa - Aa' = q - a\,Sin.\,ABa' = q - a\,Sin.\,q\,R\,\alpha \\
Cc &= Bb + Cb' = Bb + b\,Sin.\,CBb' = Bb + b\,Sin.\,q\,R\,\beta \\
Dd &= Cc + Dc' = Cc + c\,Sin.\,DCc' = Cc + c\,Sin.\,p\,R\,\gamma \\
Ee &= Dd - Dd' = Dd - d\,Sin.\,DEd' = Dd - d\,Sin.\,p\,R\,\delta \\
Ff &= Ee - Ee' = Ee - e\,Sin.\,EFe' = Ee - e\,Sin.\,q\,R\,\epsilon \\
Gg &= Ff + Gf' = Ff + f\,Sin.\,GFf' = Ff + f\,Sin.\,q\,R\,\zeta \\
Hh &= Gg - Gg' = Gg - g\,Sin.\,GHg' = Gg - g\,Sin.\,q\,R\,\eta \\
Ii &= Hh + Ii' = Hh + h\,Sin.\,IHi' = Hh + h\,Sin.\,p\,R\,\theta \\
Kk &= Ii + Kk' = Ii + i\,Sin.\,KIk' = Ii + i\,Sin.\,p\,R\,\iota \\
Ll &= Kk - Kl' = Kk - k\,Sin.\,KLl' = Kk - k\,Sin.\,q\,R\,\kappa,
\end{aligned}$$

further :

$Ra = p$
$Rb = Ra + ab = p + a\,Cos.\,ABa' = p + a\,Cos.\,q\,R\,\alpha$
$Rc = Rb + bc = Rb + b\,Cos.\,CBb' = Rb + b\,Cos.\,q\,R\,\beta$
$Rd = Rc - cd = Rc - c\,Cos.\,DCc' = Rc - c\,Cos.\,p\,R\,\gamma$
$Re = Rd - de = Rd - d\,Cos.\,DEd' = Rd - d\,Cos.\,p\,R\,\delta$
$Rf = Re + ef = Re + e\,Cos.\,EFe' = Re + e\,Cos.\,q\,R\,\varepsilon$
$Rg = Rf + fg = Rf + f\,Cos.\,GFf' = Rf + f\,Cos.\,q\,R\,\zeta$
$Rh = Rg + gh = Rg + g\,Cos.\,GHg' = Rg + g\,Cos.\,q\,R\,\eta$
$Ri = Rh - hi = Rh - h\,Cos.\,IHi' = Rh - h\,Cos.\,p\,R\,\theta$
$Rk = Ri - ik = Ri - i\,Cos.\,KIk' = Ri - i\,Cos.\,p\,R\,\iota$
$Rl = Rk + kl = Rk + k\,Cos.\,KLl' = Rk + k\,Cos.\,q\,R\,\kappa.$

2. But

$Sin.\,A = Sin. - q\,R\,\alpha = -\,Sin.\,q\,R\,\alpha$

$Sin.\,(A + B) = Sin.\,(-\,q\,R\,\alpha + \alpha\,R\,\beta) = Sin.\,q\,R\,\beta$

$Sin.\,(A + B + C) = Sin.\,(-\,q\,R\,\alpha + \alpha\,R\,\beta + \beta\,R\,\gamma) = Sin.\,q\,R\,\gamma = Sin.\,p\,R\,\gamma$

$Sin.\,(A+B+C+D) = Sin.\,(-q\,R\alpha+\alpha R\beta+\beta R\gamma+\gamma R\delta) = Sin.\,q\,\dot{R}\,\delta$ *) $= -\,Sin.\,p\,R\,\delta$

$$Sin.\,(A + B + C + D + E) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon\end{array}\right\}$$
$$= Sin.\,q\,\dot{R}\,\varepsilon = -\,Sin.\,q\,R\,\varepsilon$$

$$Sin.\,(A+B+C+D+E+F) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\end{array}\right\}$$
$$= Sin.\,(360^\circ + q\,R\,\zeta) = Sin.\,q\,R\,\zeta$$

$$Sin.\,(A+B+C+\ldots+G) = Sin.\left\{\begin{array}{c}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta B\eta\end{array}\right\}$$
$$= Sin.\,q\,\dot{R}\,\eta = -\,Sin.\,q\,R\,\eta$$

* The asterisk above $Sin.\,q\,R\,\delta$ denotes, that by $q\,R\,\delta$ is to be understood not the concave, but the convex angle.

$$Sin.\,(A+B+C+\,..\,+H) = Sin.\left\{\begin{matrix}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\eta+\eta R\theta\end{matrix}\right\}$$

$$= Sin.(360^0 + qR\theta) = Sin.\,qR\theta = Sin.\,pR\theta$$

$$Sin.\,(A+B+C+\,...+I) = Sin.\left\{\begin{matrix}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\theta+\eta R\theta+\theta R\iota\end{matrix}\right\}$$

$$= Sin.\,(360^0 + qR\iota) = Sin.\,pR\iota$$

$$Sin.\,(A+B+C+\,...+K) = Sin.\left\{\begin{matrix}-qR\alpha+\alpha R\beta+\beta R\gamma\\ +\gamma R\delta+\delta R\varepsilon+\varepsilon R\zeta\\ -\zeta R\eta+\eta R\theta+\theta R\iota\\ -\iota R\kappa\end{matrix}\right\}$$

$$= Sin.\,q\dot{R}\kappa^* = -Sin.\,qR\kappa;$$

and in like manner

$Cos.\ A = Cos. - qR\alpha = Cos.\ qR\alpha$
$Cos.\ (A+B) = Cos.\,qR\beta$
$Cos.\ (A+B+C) = Cos.\,qR\gamma = -Cos.\,p\,R\gamma$
$Cos.\ (A+B+C+D) = Cos.\,q\dot{R}\delta = -Cos.\,pR\delta$
$Cos.\ (A+B+C+D+E) = Cos.\,q\dot{R}\varepsilon = Cos.\,qR\varepsilon$
$Cos.\ (A+B+B+\ldots+F) = Cos.(360^0+qR\zeta) = Cos.\,qR\zeta$
$Cos.\ (A+B+C+\ldots+G) = Cos.\,q\dot{R}\eta = Cos.\,qR\eta$
$Cos.\ (A+B+C+\,..+H) = Cos.(360^0+qR\theta) = -Cos.\,pR\theta$
$Cos.\ (A+B+C+\ldots+I) = Cos.(360^0+qR\iota) = -Cos.\,pR\iota$
$Cos.\ (A+B+C+\ldots+K) = Cos.\,q\dot{R}\kappa = Cos.\,qR\kappa.$

3. Now, if we substitute the values of $Sin.\,qR\alpha$, $Sin.\,qR\beta$, $Sin.\,pR\gamma$, $Sin.\,pR\delta$, &c. $Cos.\,qR\alpha$, $Cos.\,qR\beta$, $Cos.\,pR\gamma$, $Cos.\,pR\delta$, &c., found in 2, in the expression in 1, for the lines Aa, Bb, Cc, &c., Ra, Rb, Rc, &c., we obtain

$Aa = q$
$Bb = q + a\,Sin.\,A$
$Cc = Bb + b\,Sin.\,(A+B) = q + a\,Sin.\,A + b\,Sin\,(A+B)$
$Dd = Cc + c\,Sin.(A+B+C) = q + a\,Sin.\,A + b\,Sin.\,(A+B) + c\,Sin.\,(A+B+C)$
$Ee = Dd + d\,Sin.\,(A+B+C+D) = q + a\,Sin.\,A + b\,Sin.\,(A+B) + c\,Sin.\,(A+B+C) + d\,Sin.\,(A+B+C+D)$

* *Vide* Note in preceding page.

$$Ff = Ec + e\,Sin.\,(A + B + C + \ldots + E) = q + a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C) + d\,Sin.\,(A + B + C + D) + e\,Sin.\,(A + B + C + \ldots E)$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$Ll = Kk + k\,Sin.\,(A + B + C + \ldots + K) = q + a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C) + d\,Sin.\,(A + B + C + D) + e\,Sin.\,(A + B + \ldots + E) + f\,Sin.\,(A + B + \ldots + F) + g\,Sin.\,(A + B + \ldots + G) + h\,Sin.\,(A + B + \ldots + H) + i\,Sin.\,(A + B + \ldots + I) + k\,Sin.\,(A + B + \ldots + K).$$

Likewise,

$$Ra = p$$
$$Rb = p + a\,Cos.\,A$$
$$Rc = p + a\,Cos.\,A + b\,Cos.\,(A + B)$$
$$Rd = p + a\,Cos.\,A + b\,Cos.\,(A + B) + c\,Cos.\,(A + B + C)$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$Rl = p + a\,Cos.\,A + b\,Cos.\,(A + B) + c\,Cos.\,(A + B + C) + \ldots\ldots\ldots + k\,Cos.\,(A + B + C + \ldots\ldots + K).$$

Cor. These formulæ also obtain, when, as in *fig.* 97, the crooked line is intersected by the line of abscissæ, or when the beginning of the abscissæ is within the crooked line. For it is only necessary to suppose, that the line of abscissæ and the beginning of the abscissæ, are first without the crooked line, and that this last moves more to the right towards Q, but the former parallel with itself vertically: then, because the angles A, B, C, D, &c. and the lines of division a, b, c, d, &c. remain the same, the expressions obtained undergo no further change, than that the co-ordinates of the first point differ. If ∴ these are properly determined

according to the alteration in the position, then all remains the same.

If the line of abscissæ passes through A, and we also assume this point as the origin of the abscissæ; then $p = 0$, $q = 0$, and $\therefore$, when y denotes any ordinate, and x any abscissa,

$$y = a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C) + \&c.$$
$$x = a\,Cos.\,A + b\,Cos.\,(A + B) + c\,Cos.\,(A + B + C) + \&c.$$

EXAM. Let (*fig.* 98) $a = 542$, $b = 698$, $c = 511$, $d = 469$, $e = 970$, $f = 902$, $g = 689$, $h = 660$, $i = 299$, $k = 783$, $p = 817$, $q = 711$; $A = -47^\circ.\ 45'$, $B = +94^\circ.\ 31'$, $C = +66^\circ.\ 19'$, $D = +84^\circ.\ 15'$, $E = +92^\circ.\ 34'$, $F = +126^\circ.\ 32'$, $G = -115^\circ.\ 12'$, $H = +168^\circ.\ 49'$, $I = +33^\circ.\ 5'$, $K = -161^\circ.\ 50'$.

Here

$Sin.\,A = Sin. - 47^\circ.\ 45', = - Sin.\,47^\circ.\ 45'$,
$Cos.\,A = Cos. - 47^\circ.\ 45' = Cos.\,47^\circ.\ 45'$;
$Sin.\,(A + B) = Sin.\,46^\circ.\ 46'$,
$Cos.\,(A + B) = Cos.\,46^\circ.\ 46'$;
$Sin.\,(A + B + C) = Sin.\,113^\circ.\ 5' = Sin\ 66^\circ.\ 55'$,
$Cos.\,(A + B + C) = Cos.\,113^\circ.\ 5' = - Cos.\,66^\circ.\ 55'$;
$Sin.\,(A + B + C + D) = Sin.\,197^\circ.\ 20' = - Sin.\,17^\circ.\ 20'$,
$Cos.\,(A + B + C + D) = Cos.\,197^\circ.\ 20' = - Cos.\,17^\circ.\ 20'$;
$Sin.\,(A + B + \ldots + E) = Sin.\,289^\circ.\ 54' = - Sin.\,70^\circ.\ 6'$,
$Cos.\,(A + B + \ldots + E) = Cos.\,289^\circ.\ 54' = Cos.\,70^\circ.\ 6'$;
$Sin.\,(A + B + \ldots + F) = Sin.\,416^\circ.\ 26' = Sin.\,56^\circ.\ 26'$,
$Cos.\,(A + B + \ldots + F) = Cos.\,416^\circ.\ 26' = Cos.\,56^\circ.\ 26'$;
$Sin.\,(A + B + \ldots + G) = Sin.\,301^\circ.\ 14' = - Sin.\,58^\circ.\ 46'$,
$Cos.\,(A + B + \ldots + G) = Cos.\,301^\circ.\ 14' = Cos.\,58^\circ.\ 46'$;
$Sin.\,(A + B + \ldots + H) = Sin.\,470^\circ.\ 3' = Sin.\,69^\circ.\ 57'$,
$Cos.\,(A + B + \ldots + H) = Cos.\,470^\circ.\ 3' = - Cos.\,69^\circ.\ 57'$;
$Sin.\,(A + B + \ldots + I) = Sin.\,503^\circ.\ 8' = Sin.\,36^\circ.\ 52'$,
$Cos.\,(A + B + \ldots + I) = Cos.\,503^\circ.\ 8' = - Cos.\,36^\circ.\ 52'$;

$Sin. (A+B+...+K) = Sin. 341°. 18' = - Sin. 18°. 42'$,
$Cos. (A+B+...+K) = Cos. 341°. 18' = Cos. 18°. 42'$.

Therefore we have

$$
\begin{aligned}
a\ Sin.\ A &= -542\ Sin.\ 47°.45' = -401·1981\\
b\ Sin.\ (A+B) &= +698\ Sin.\ 46°.46' = +508·5420\\
c\ Sin.\ (A+B+C) &= +511\ Sin.\ 66°.55' = +470·0870\\
d\ Sin.\ (A+B+C+D) &= -469\ Sin.\ 17°.20' = -139·7292\\
e\ Sin.\ (A+B+...+E) &= -970\ Sin.\ 70°.\ 6' = -912·0794\\
f\ Sin.\ (A+B+...+F) &= +902\ Sin.\ 56°.26' = +751·5852\\
g\ Sin.\ (A+B+...+G) &= +689\ Sin.\ 58°.46' = -589·1382\\
h\ Sin.\ (A+B+...+H) &= +660\ Sin.\ 69°.57' = +619·9998\\
i\ Sin.\ (A+B+...+I) &= +299\ Sin.\ 36°.52' = +179·3865\\
k\ Sin.\ (A+B+...+K) &= -783\ Sin.\ 18°.42' = -251·0400
\end{aligned}
$$

$$
\begin{aligned}
a\ Cos.\ A &= +542\ Cos.\ 47°.45' = +364·4228\\
b\ Cos.\ (A+B) &= +698\ Cos.\ 46°.46' = +478·1098\\
c\ Cos.\ (A+B+C) &= -511\ Cos.\ 66°.55' = -200·3475\\
d\ Cos.\ (A+B+C+D) &= -469\ Cos.\ 17°.20' = -447·7015\\
e\ Cos.\ (A+B+...+E) &= +970\ Cos.\ 70°.\ 6' = +330·1681\\
f\ Cos.\ (A+B+...+F) &= +902\ Cos.\ 56°.26' = +498·7219\\
g\ Cos.\ (A+B+...+G) &= +689\ Cos.\ 58°.46' = +357·2633\\
h\ Cos.\ (A+B+...+H) &= -660\ Cos.\ 69°.57' = -226·2744\\
i\ Cos.\ (A+B+...+I) &= -299\ Cos.\ 36°.52' = -239·2101\\
k\ Cos.\ (A+B+...+K) &= +783\ Cos.\ 18°.42' = +741·6656
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
Bb &= 711 - 401·1981 = 309·8019\\
Cc &= Bb + 508·5420 = 818·3439\\
Dd &= Cc + 470·0870 = 1288·4309\\
Ee &= Dd - 139·7292 = 1148·7017\\
Ff &= Ee - 912·0794 = 236·6223\\
Gg &= Ff + 751·5852 = 988·2075\\
Hh &= Gg - 589·1382 = 399·0693\\
Ii &= Hh + 619·9998 = 1019·0691\\
Kk &= Ii + 179·3865 = 1198·4556\\
Ll &= Kk - 251·0400 = 947·4156
\end{aligned}
$$

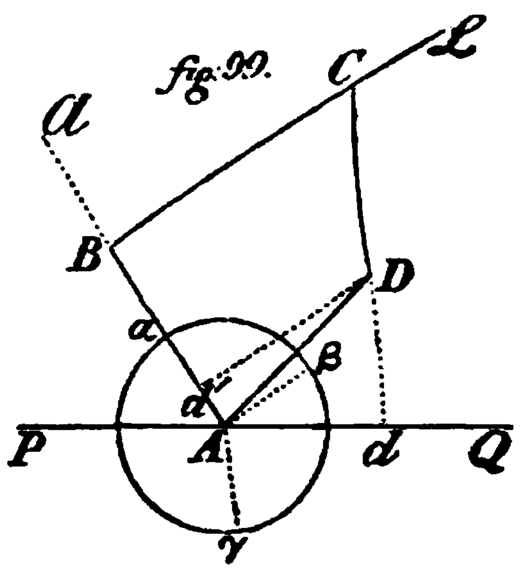
and in like manner, $Rb = 1181{\cdot}4228$, $Rc = 1659{\cdot}5326$, $Rd = 1459{\cdot}1851$, $Re = 1011{\cdot}4836$, $Rf = 1341{\cdot}6517$, $Rg = 1840{\cdot}3736$, $Rh = 2197{\cdot}6369$, $Ri = 1971{\cdot}3625$, $Rk = 1732{\cdot}1524$, $Rl = 2473{\cdot}8180$.

REMARK. If the co-ordinates of the crooked line are found, it is easy to express this line. The formulæ here found form, besides, the basis of all polygonometry, as we shall now show.

SECTION LXXXVI.

PROB. *In a quadrilateral, three sides, and the two angles between these sides, are given: find the fourth side, and the remaining angles.*

SOLUT. Let $ABCD$ (*fig.* 99) be the quadrilateral; AB, BC, CD, the given sides, and ABC, BCD, the given angles. Let $AB = a$, $BC = b$, $CD = c$.



1. Consider $ABCD$ as a crooked line, which both begins and ends in A, and through this point draw any line PQ, which may be its line of abscissæ. Let A be the beginning of the abscissæ, and about this point describe a circle; let $A\alpha$, $A\beta$, $A\gamma$, be the corresponding radii of the sides AB, BC, CD. By comparing *figs.* 97, 98, we get $A = +QA\alpha$, $B = -\alpha A\beta = -\alpha BC$, $C = -\beta A\gamma = -bCD$.

2. Draw Dd perpendicular to PQ; then by § LXXXV when, for shortness sake, we put $A = \alpha$, $A + B = \beta$, $A + B + C = \gamma$, we obtain,

$$Dd = a\ Sin.\ \alpha + b\ Sin.\ \beta + c\ Sin.\ \gamma;$$
$$Ad = a\ Cos.\ \alpha + b\ Cos.\ \beta + c\ Cos.\ \gamma;$$

consequently

$$Dd^2 = a^2 \; Sin.^2 \, \alpha + b^2 \; Sin.^2 \, \beta + c^2 \; Sin.^2 \, \gamma + 2 \, ab \; Sin. \, \alpha \; Sin. \, \beta$$
$$+ \; 2 \; ac \; Sin. \; \alpha \; Sin. \; \gamma \; + \; 2 \; bc \; Sin. \; \beta \; Sin. \; \gamma$$
$$AD^2 = a^2 \; Cos.^2 \, \alpha + b^2 \; Cos.^2 \, \beta + c^2 \; Cos.^2 \, \gamma + 2 \, ab \; Cos. \, \alpha \; Cos. \, \beta$$
$$+ \; 2 \; ac \; Cos. \; \alpha \; Cos. \; \gamma + 2 \; bc \; Cos. \; \beta \; Cos. \; \gamma.$$

Hence, because $AD = \sqrt{\,}(Dd^2 + Ad^2)$, we obtain

$$AD = \sqrt{\left[\begin{array}{c} a^2 + b^2 + c^2 + 2 \, ab \; Cos. \; (\beta - \alpha) \\ + \; 2 \, ac \; Cos. \; (\gamma - \alpha) + 2 \; bc \; Cos. \; (\gamma - \beta) \end{array}\right]}$$

(*fig.* 1 *and* 10.)

or, since $\beta - \alpha = B$, $\gamma - \alpha = B + C$, $\gamma - \beta = C$,

$$AD = \sqrt{\left[\begin{array}{c} a^2 + b^2 + c^2 + 2 \, ab \; Cos. \; B \\ + \; 2 \, ac \; Cos. \; (B + C) + 2 \; bc \; Cos. \; C \end{array}\right]}.$$

an expression, which, as must be the case, does not depend on the angle A.

3. We must further determine the two angles BAD, ADC, or even one of them only, because we then know three angles, and consequently the fourth. With this view, suppose the line PQ turns round the point A, and moves towards AD: then we have

$$Dd = a \; Sin. \; A + b \; Sin. \; (A + B) + c \; Sin. \; (A + B + C) = 0,$$

where $A = DAB$.

4. By solving this equation, we obtain (*fig.* 9)

$$\left[\begin{array}{l} a \; Sin. \; A + b \; [Sin. \; A \; Cos. \; B + Cos. \; A \; Sin. \; B] \\ + c[Sin. \; A \; Cos. \; (B + C) + Cos. \; A \; Sin. \; (B + C)] \end{array}\right] = 0,$$

and by dividing by $Sin. \; A$,

$$\left[\begin{array}{l} a + b \; [Cos. \; B + Cot. \; A \; Sin. \; B] \\ + c[Cos. \; (B + C) + Cot. \; A \; Sin. \; (B + C)] \end{array}\right] = 0;$$

hence further,

$$Cot. \; A = Cot. \; DAB = - \frac{a + b \; Cos. \; B + c \; Cos. \; (B + C)}{b \; Sin. \; B + c \; Sin. \; (B + C)}.$$

5. If the line PQ be so turned that it falls on AB: then we have $A = QAa = 180^\circ$. On this supposition the line Dd is converted into Dd', and the line Ad into $- Ad'$. The two equations

$$Dd = a \; Sin. \; A + b \; Sin. \; (A + B) + c \; Sin. \; (A + B + C)$$
$$Ad = a \; Cos. \; A + b \; Cos. \; (A + B) + c \; Cos. \; (A + B + C)$$

are transformed into the following ones:

$$Dd' = a\,Sin.\,180^0 + b\,Sin.(180^0 + B) + c\,Sin.(180^0 + B + C)$$
$$-Ad' = a\,Cos.\,180^0 + b\,Cos.(180^0 + B) + c\,Cos.(180^0 + B + C)$$

or

$$Dd' = -\ b\ Sin.\ B - c\ Sin.\ (B + C)$$
$$Ad' = a + b\ Cos.\ B + c\ Cos.\ (B + C).$$

Hence we obtain, as in 4,

$$Cot.\ DAB = \frac{Ad'}{Dd'} = -\frac{a + b\ Cos.\ B + c\ Cos.\ (B + C)}{b\ Sin.\ B + c\ Sin.\ (B + C)}.$$

6. If the side AD be calculated; then also the following formula may be used to determine the angle DAB:

$$Sin.\ DAB = \frac{Dd'}{AD} = -\frac{b\ Sin.\ B + c\ Sin.\ (B + C)}{AD}.$$

Exam. Let $AB = a = 452$, $BC = b = 610$, $CD = c = 411$, $ABC = 92^0.\ 5'$, $BCD = 68^0.\ 53'$. Here $B = -aBC = -87^0.\ 55'$, $C = -bCD = -111^0.\ 7'$, and $\therefore B + C = -199^0.\ 2'$; we then have, $Cos.\ B\ Cos. - 87^0.\ 55' = Cos.\ 87^0.\ 55'$, $Cos.\ C = Cos. - 111^0.\ 7' = Cos.\ 111^0.\ 7' = -Cos.\ 68^0.\ 53'$, $Cos.\ (B + C) = Cos. - 199^0.\ 2' = Cos.\ 199^0.\ 2' = -Cos.\ 19^0.\ 2'$, $Sin.\ B = Sin. - 87^0.\ 55' = -Sin.\ 87^0.\ 55'$, $Sin.\ (B + C) = Sin. - 199^0.\ 2' = -Sin.\ 199^0.\ 2' = Sin.\ 19^0.\ 2'$;

$$a^2 + b^2 + c^2 = 745325$$
$$2\ ab\ Cos.\ B = 20046\ 5$$
$$2\ ac\ Cos.\ (B + C) = -351231{\cdot}3$$
$$2\ bc\ Cos.\ C = -180645{\cdot}6;$$

consequently by the formula in 2,

$$AD = \sqrt{(745325 + 20046\ 5 - 351231{\cdot}3 - 180645{\cdot}6)}$$
$$= 483{\cdot}2127.$$

Further we have,

$$b\ Cos.\ B = 22{\cdot}1753,\ b\ Sin.\ B = -609{\cdot}5966$$
$$c\ Cos.\ (B + C) = -388{\cdot}5302,\ c\ Sin.\ (B + C) = 134{\cdot}0346;$$

consequently

$$a + b\,Cos.\,B + c\,Cos.\,(B + C) = \quad 85{\cdot}6451$$
$$b\,Sin.\,B + c\,Sin.\,(B + C) = -475{\cdot}5620;$$

$\therefore$ by the formula in 4, 5,

$$Cot.\,DAB = \frac{85{\cdot}6451}{475{\cdot}5620} = 0{\cdot}1800924$$

and $\qquad DAB = 79^{\circ}.\ 47'.\ 26''.$

By the form in 6, we have

$$Sin.\,DAB = \frac{475{\cdot}5620}{483{\cdot}2127} = 0{\cdot}9841670$$

and $\qquad DAB = 79^{\circ}.\ 47'.\ 26''$

as before.

When the angle DAB is found, we have also ADC: thus it is

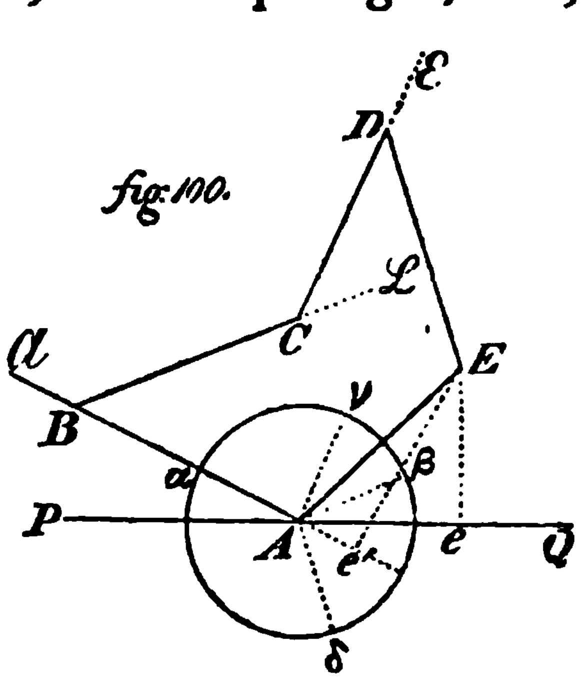
$$ADC = 360^{\circ} - ABC - BCD - BAD = 119^{\circ}.\ 14'.\ 34''.$$

SECTION LXXXVII.

Prob. *Four sides of a pentagon, and the three angles included by them, are given: find its fifth side and the two remaining angles.*

Solut. Let $ABCDE$ (*fig.* 100) be the pentagon, AB, BC, CD, DE, the given sides and ABC, BCD, CDE, the given angles. Let $AB = a$, $BC = b$, $CD = c$, $DE = d$.

fig. 100.



1. Through A, one of the extremities of the required line AE, let any line PQ be drawn, and from the other extremity E the perpendicular Ee. Further, let a circle be described about A, and let $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, according to their order be the corresponding radii of AB, BC, CD, DE

From the opposite situation of these radii we obtain for the present figure, $A = + QA\alpha$, $B = - \alpha AB = - aBC$, $C = + \beta A\gamma = + bCD$, $D = - \gamma A\delta = - cDE$.

2. By § LXXXIV, in the crooked line $ABCDEA$, when, for shortness sake, we put $A = \alpha$, $A + B = \beta$, $A + B + C = \gamma$, $A + B + C + D = \delta$, we get

$$Ee = a \operatorname{Sin.} \alpha + b \operatorname{Sin.} \beta + c \operatorname{Sin.} \gamma + d \operatorname{Sin.} \delta$$
$$Ae = a \operatorname{Cos.} \alpha + b \operatorname{Cos.} \beta + c \operatorname{Cos.} \gamma + d \operatorname{Cos.} \delta,$$

and $\therefore$

$$\begin{aligned} Ee^2 = {} & a^2 \operatorname{Sin.}^2 \alpha + b^2 \operatorname{Sin.}^2 \beta + c^2 \operatorname{Sin.}^2 \gamma + d^2 \operatorname{Sin.}^2 \delta \\ & + 2ab \operatorname{Sin.} \alpha \operatorname{Sin.} \beta + 2ac \operatorname{Sin.} \alpha \operatorname{Sin.} \gamma + 2ad \operatorname{Sin.} \alpha \operatorname{Sin.} \delta \\ & + 2bc \operatorname{Sin.} \beta \operatorname{Sin.} \gamma + 2bd \operatorname{Sin.} \beta \operatorname{Sin.} \delta + 2cd \operatorname{Sin.} \gamma \operatorname{Sin.} \delta \end{aligned}$$

$$\begin{aligned} Ae^2 = {} & a^2 \operatorname{Cos.}^2 \alpha + b^2 \operatorname{Cos.}^2 \beta + c^2 \operatorname{Cos.}^2 \gamma + d^2 \operatorname{Cos.}^2 \delta \\ & + 2ab \operatorname{Cos.} \alpha \operatorname{Cos.} \beta + 2ac \operatorname{Cos.} \alpha \operatorname{Cos.} \gamma + 2ad \operatorname{Cos.} \alpha \operatorname{Cos.} \delta \\ & + 2bc \operatorname{Cos.} \beta \operatorname{Cos.} \gamma + 2bd \operatorname{Cos.} \beta \operatorname{Cos.} \delta + 2cd \operatorname{Cos.} \gamma \operatorname{Cos.} \delta. \end{aligned}$$

3. Now since $AE = \sqrt{(Ee^2 + Ae^2)}$: $\therefore$ (*fig.* 1 *and* 10),

$$AE = \sqrt{\left\{ \begin{array}{c} a^2 + b^2 + c^2 + d^2 + 2\,ab \operatorname{Cos.} (\beta - \alpha) + \\ 2ac \operatorname{Cos.} (\gamma - \alpha) + 2ad \operatorname{Cos.} (\delta - \alpha) + 2bc \operatorname{Cos.} (\gamma - \beta) \\ + 2bd \operatorname{Cos.} (\delta - \beta) + 2cd \operatorname{Cos.} (\delta - \gamma) \end{array} \right\}}$$

or, since $\beta - \alpha = B$, $\gamma - \alpha = B + C$, $\delta - \alpha = B + C + D$, $\gamma - \beta = C$, $\delta - \beta = C + D$, $\delta - \gamma = D$,

$$AE = \sqrt{\left\{ \begin{array}{c} a^2 + b^2 + c^2 + 2\,ab \operatorname{Cos.} B + 2\,ac \operatorname{Cos.} (B + C) \\ + 2\,ad \operatorname{Cos.} (B + C + D) + 2\,bc \operatorname{Cos.} C + \\ 2\,bd \operatorname{Cos.} (C + D) + 2\,cd \operatorname{Cos.} D \end{array} \right\}}$$

4. Let the line PQ so move towards P, that it coincides with AB: then $A = 180^\circ$, and Ee is converted into Ee'. We consequently have

$$Ee' = a \operatorname{Sin.} 180^\circ + b \operatorname{Sin.} (180 + B) + c \operatorname{Sin.} (180^\circ + B + C) + d \operatorname{Sin.} (180^\circ + B + C + D),$$

or

$$Ee' = - b \operatorname{Sin.} B - c \operatorname{Sin.} (B + C) - d \operatorname{Sin.} (B + C + D).$$

Now since $\operatorname{Sin.} BAE = \operatorname{Sin.} EAe' = \frac{Ee'}{AE}$: therefore

$$Sin.\, BAE + \frac{b\, Sin.\, B + c\, Sin.\, (B+C) + d\, Sin.\, (B+C+D)}{AE}.$$

If ∴ the side AE is calculated by means of the formula in 3, we then obtain from this formula the triangle BAE.

5. But if the line PQ move towards Q, in the direction AE; then $A = BAE$, and $Ee = 0$, and we have the equation,

$$\left\{\begin{matrix} a\, Sin.\, A + b\, Sin.\, (A+B) + c\, Sin.\, (A+B+C) \\ + d\, Sin.\, (A+B+C+D) \end{matrix}\right\} = 0,$$

or

$$\left\{\begin{matrix} a\, Sin.\, A + b\, [Sin.\, A\, Cos.\, B + Cos.\, A\, Sin.\, B] + \\ c\, [Sin.\, A\, Cos.\, (B+C) + Cos.\, A\, Sin.\, (B+C)] + \\ d\, [Sin.\, A\, Cos.\, (B+C+D) + Cos.\, A\, Sin.\, (B+C+D)] \end{matrix}\right\} = 0.$$

Divide this equation by $Sin.\, A$, substitute $Cot.\, A$ for $\frac{Cos.\, A}{Sin.\, A}$, and subtract $Cot.\, A = Cot.\, BAE$: this gives

$$Cot.\, BAE = -\frac{a + b\, Cos.\, B + c\, Cos.\, (B+C) + d\, Cos.\, (B+C+D)}{b\, Sin.\, B + c\, Sin.\, (B+C) + d\, Sin.\, (B+C+D)}$$

or also

$$Tan.\, BAE = -\frac{b\, Sin.\, B + c\, Sin.\, (B+C) + d\, Sin.\, (B+C+D)}{a + b\, Cos.\, B + c\, Cos.\, (B+C) + d\, Cos.\, (A+C+D)}$$

We have ∴ two different formulæ for the angle BAE, which mutually serve to prove the calculation.

EXAM. Let $a = 540$, $b = 519$, $c = 438$, $d = 586$, $ABC = 46^{\circ}.\ 38'$, $BCD = 136^{\circ}.\ 5'$, $CDE = 38^{\circ}.\ 51'$; consequently $B = -133^{\circ}.\ 22'$, $C = 43^{\circ}.\ 55'$, $D = -141^{\circ}.\ 9'$. Here $Sin.\, B = Sin. - 133^{\circ}.\ 22' = - Sin.\ 133^{\circ}.\ 22' = - Sin.\ 46^{\circ}.\ 38'$, $Cos.\, B = Cos. - 133^{\circ}.\ 22' = Cos.\ 133^{\circ}.\ 22' = - Cos.\ 46^{\circ}.\ 38'$, $Cos.\, C = Cos.\ 43^{\circ}.\ 55'$, $Cos.\, D = Cos. - 141^{\circ}.\ 9' = Cos.\ 141^{\circ}.\ 9'$, $= - Cos.\ 38^{\circ}.\ 51$, $Sin.\, (B+C) = Sin. - 89^{\circ}.\ 27' = - Sin.\ 89^{\circ}.\ 27'$, $Sin.\, (B+C+D) = Cos. - 89^{\circ}.\ 27' = - Cos.\ 89^{\circ}.\ 27'$, $Sin.\, (B+C+D) =$

$Sin. - 230°. 36' = - Sin. 230°. 36' = Sin. 50°. 36'$, $Cos. (B+C+D) = Cos. - 230°. 36' = Cos. 230°. 36' = - Cos. 50°. 36'$, $Cos. (C+D) = Cos. - 97°. 14' = Cos. 97°. 14' = - Cos. 82°. 46'$; consequently

$$
\begin{array}{lll}
a^2 + b^2 + c^2 + d^2 & = & 1040101 \\
2\,ab\ Cos.\ B & =- & 384889{\cdot}4 \\
2\,ac\ Cos.\ (B + C) & = & 4540{\cdot}8 \\
2\,ad\ Cos.\ (B+C+D) & =- & 367432{\cdot}8 \\
2\,bc\ Cos.\ C & = & 327502{\cdot}5 \\
2\,bd\ Cos.\ (C+D) & =- & 70052{\cdot}5 \\
2\,cd\ Cos.\ D & == & 365670{\cdot}3;
\end{array}
$$

and ∴ $AE = \sqrt{184099{\cdot}3} = 429{\cdot}0679.$

Further

$$
\begin{array}{lll}
b\ Sin.\ B & =- & 377{\cdot}2996 \\
b\ Cos.\ B & =- & 356{\cdot}3720 \\
c\ Sin.\ (B + C) & =- & 437{\cdot}9797 \\
c\ Cos.\ (B + C) & = & 4{\cdot}2044 \\
d\ Sin.\ (B+C+D) & = & 414{\cdot}1852 \\
d\ Cos.\ (B+C+D) & =- & 340{\cdot}2155,
\end{array}
$$

∴ by the formula in 4,

$$Sin.\ BAE = \frac{401{\cdot}0941}{429{\cdot}0679} = 0{\cdot}9348033$$

and by the formula in 5,

$$Cot.\ BAE = -\frac{152{\cdot}3901}{401{\cdot}0941} = -0{\cdot}3799360.$$

From both we obtain

$$BAE = 110°.\ 48'.\ 13''.$$

The negative co-tangent which is obtained from the second formula, shows that the angle is obtuse, which leaves the first indeterminate, because two angles belong to each sine.

When BAE is found, we have also the angle AED.

SECTION LXXXVIII.

Prob. *All the sides of a polygon except one are given, also all the angles included by these sides: find the unknown sides and the two remaining angles.*

Solut. The treatment of the quadrilateral and pentagon in §§ LXXXVI, LXXXVII, clearly shows the method to be adopted for every other polygon. The law by which the formulæ are governed is simple, and easily discovered: it may be expressed by words in the following way:

1. In order to find the unknown sides of the polygon, take the squares of all the given sides; take, further, twice the products of every two of these sides combined in every possible way, and multiply each of them by the cosine of the algebraical sum of the exterior angles between the respective sides; then add all together, and extract the square root from the product.

2. In order to find one of the two required angles of the polygon, assume the given line which is adjacent to the required angle for the first, and multiply each of the remaining given sides both by the sine and the cosine of the algebraical sum of all the exterior angles between it and the above first side; then add all the products arising from the cosines to the first side, and divide the sum by the sum of all the products arising from the sines; the quotient, with a different sign, gives the co-tangent of the required angle. Or, divide the sum of the products arising from the sines, by the side found by the first rule: then the quotient, with a different sign, gives the sine of the required angle.

Thus, if a, b, c, d, e, in the order in which they are here placed, be the five given sides of a hexagon, and B, C, D, E the four given exterior angles formed by these sides: then the unknown side =

$$\sqrt{\left\{\begin{array}{c} a^2+b^2+c^2+d^2+e^2+2ab\,\mathrm{Cos}.\,B+2ac\,\mathrm{Cos}.(B+C) \\ +2ad\,\mathrm{Cos}.(B+C+D)+2ae\,\mathrm{Cos}.(B+C+D+E) \\ +2bc\,\mathrm{Cos}.\,C+2bd\,\mathrm{Cos}.(C+D)+2be\,\mathrm{Cos}.(C+D+E) \\ +2cd\,\mathrm{Cos}.\,D+2ce\,\mathrm{Cos}.(D+E)+2de\,\mathrm{Cos}.\,E \end{array}\right\}}$$

and the cotangent of the required angle adjacent to the side $a =$

$$\frac{\left[\begin{array}{c} a + b\,Cos.\,B + c\,Cos.\,(B + C) + d\,Cos.\,(B + C + D) + \\ e\,Cos.\,(B + C + D + E) \end{array}\right]}{\left[\begin{array}{c} b\,Sin.\,B + c\,Sin.\,(B + C) + d\,Sin.\,(B + C + D) + \\ e\,Sin.\,(B + C + D + E) \end{array}\right]}$$

or the sine of this angle $=$

$$\frac{\left[\begin{array}{c} b\,Sin.\,B + c\,Sin.\,(B + C) + d\,Sin.\,(B + C + D) + \\ e\,Sin.\,(B + C + D + E) \end{array}\right]}{x}$$

where x denotes the required side.

That these formulæ are also applicable to the triangle, and that hence the known trigonometrical rules may be derived, are necessary consequences: this subject is treated in Däzel's Elements of Goniometry; Munich, 1800.

SECTION LXXXIX.

PROB. *In a polygon, all the sides except two are given, likewise all the angles: find the two unknown sides.*

SOLUT. Thus, let $ABCDEF$ (*fig.* 101) be a hexagon, in which all the angles, and all the sides, except the two AF, CD, are known.

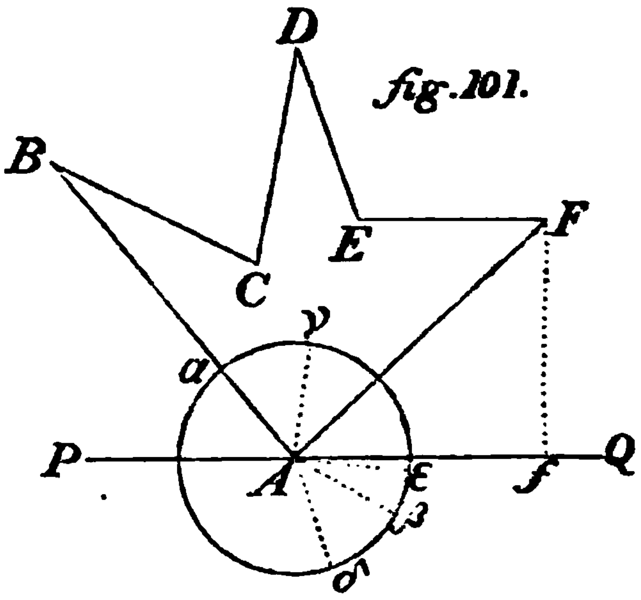


fig. 101.

1. Through A draw any line of abscissæ PQ, and Ff perpendicular to it: then, when $ABCDEF$ is considered as a crooked line, by § LXXXV,

$$Ff = AB.Sin.A + BC.Sin.(A + B) + CD.Sin.(A + B + C) + DE.Sin.(A + B + C + D) + EF.Sin.(A + B + C + D + E)$$

$$Af = AB.Cos.A + BC.Cos.(A + B) + CD.Cos.(A + B + C) + DE.Cos.(A + B + C + D) + EF.Cos.(A + B + C + D + E)$$

in which $A = QAB$.

2. Let the line PQ move towards Q, so that it may fall on AF; then $Ff = 0$, and $Af = AF$, and instead of the two foregoing equations, we have the following one:

$$0 = AB.\ Sin.\ A + BC.\ Sin.\ (A+B) + CD.\ Sin.\ (A+B+C) + DE.Sin.(A+B+C+D) + EF.Sin.(A+B+C+D+E)$$

$$AB = AB.\ Cos.\ A + BC.\ Cos.(A+B) + CD.\ Cos.(A+B+C) + DE.Cos.(A+B+C+D) + EF.Cos.(A+B+C+D+E),$$

in which $A = FAB$.

3. From the first of these two equations, we obtain

$$CD = \frac{-\left[\begin{array}{c} AB.Sin.A + BC.Sin.(A+B) + DE.Sin.(A+B+C+D) \\ + EF.\ Sin.\ (A+B+C+D+E) \end{array}\right]}{Sin.\ (A+B+C)}$$

Having from hence determined CD; then the second equation gives the line AF.

Exam. Let $AB = 1040$, $BC = 624$, $DE = 533$, $EF = 481$; $ABC = 23^{\circ}.\ 52'$, $BCD = 69^{\circ}.\ 14'$, $CDE = 30^{\circ}.\ 24'$, $DEF = 115^{\circ}.\ 30'$, $AFE = 46^{\circ}.\ 44'$, and consequently $BAF = 83^{\circ}.\ 44'$. Let $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, $A\epsilon$, according to their order, be the corresponding radii of the sides AB, BC, CD, DE, EF: then it follows from the situation of these lines with respect to each other, and to AF, into which PQ moves, that $A = FA\alpha = 83^{\circ}.\ 44'$, $B = -\alpha A\beta = -156^{\circ}.\ 8'$, $C = \beta A\gamma = 110^{\circ}.\ 46'$, $D = -\gamma A\delta = -149^{\circ}.\ 36'$, $E = \delta A\epsilon = 64^{\circ}.\ 30'$. We consequently have $Sin.\ (A+B) = Sin. -72^{\circ}.\ 24' = -Sin.\ 72^{\circ}.\ 24'$, $Cos.\ (A+B) = Cos. -72^{\circ}.\ 24' = Cos.\ 72^{\circ}.\ 24'$, $Sin.\ (A+B+C) = Sin.\ 38^{\circ}.\ 22'$, $Cos.(A+B+C) = Cos.\ 38^{\circ}.\ 22'$, $Sin.\ (A+B+C+D) = Sin. -111^{\circ}.\ 14' = -Sin.\ 68^{\circ}.\ 46'$, $Cos.\ (A+B+C+D) = Cos. -111^{\circ}.\ 14' = -Cos.\ 68^{\circ}.\ 46'$, $Sin.\ (A+B+\ldots+E) = Sin. -46^{\circ}.\ 44' = -Sin.\ 46^{\circ}.\ 44'$, $Cos.\ (A+B+\ldots+E) = Cos. -46^{\circ}.\ 44' = Cos.\ 46^{\circ}.\ 44'$; consequently,

$$
\begin{aligned}
AB.\ Sin.\ A &= \quad 1033\cdot7853 \\
AB.\ Cos.\ A &= \quad 113\cdot5222 \\
BC.\ Sin.\ (A+B) &= -\ 594\cdot7909 \\
BC.\ Cos.\ (A+B) &= \quad 188\ 6788 \\
DE.\ Sin.\ (A+B+C+D) &= -\ 496\cdot8163 \\
DE.\ Cos.\ (A+B+C+D) &= -\ 193\cdot0349 \\
EF.\ Sin.\ (A+B+C+D+E) &= -\ 350\cdot2507 \\
EF.\ Cos.\ (A+B+C+D+E) &= \quad 329\cdot6749
\end{aligned}
$$

Hence we find,

$$CD = \frac{-(1033\cdot7853 - 594\cdot7909 - 496\ 8163 - 350\cdot2507)}{Sin.\ 38^{0}.\ 22'}$$

$$= \frac{408\cdot0726}{Sin.\ 38^{0}.\ 22'} = 657\cdot4481,$$

$$\therefore \quad CD.\ Cos.\ (A+B+C) = 515\cdot4753,$$

$$AF = 113\cdot5222 + 188\cdot6788 + 515\cdot4753 - 193\cdot0349 + 329\cdot6749 = 954\cdot3163.$$

In the values found for CD, AF, the error which occurs on account of the incompleteness of the logarithmic tables, cannot amount to 0·001.

REMARK. Although in the solution, the calculation, for the sake of perspicuity, has only been performed for one figure; yet it is sufficiently evident from hence, how we are to proceed in the case of every other figure.

COR. If CD be parallel to AF: then $FA\gamma = FA\alpha - \alpha A\beta + \beta A\gamma = A + B + C = 0$; consequently $Sin.\ (A + B + C) = 0$; the expression for CD, and consequently also that for AF, is determined. Of the accuracy of this result we can easily convince ourselves merely by inspecting the figure. For let the points A, B, C, retain their places, but move the angle DEF between the two parallel lines AF, CD, forwards or backwards, in such a way, that the lines DE, EF, continue to be parallel to themselves, consequently by these means neither the lines DE, EF, nor the angles CDE, DEF, AFE, undergo any change with respect to their magnitude or position; whence follows the indeterminateness of the lines CD, AF.

SECTION XC.

PROB. *In a polygon all the sides but one, and all the angles but two, are given, likewise the two unknown angles are at one of the given sides: find these angles and the unknown side.*

SOLUT. For the sake of perspicuity, take the pentagon $ABCDE$ (*fig.* 100); let CD be the unknown side, and BAE, DEA the unknown angles. Let $AB = a$, $BC = b$, $CD = x$, $DE = d$, $EA = e$.

1. By §§ LXXXVII, LXXXVIII, we have

$$AE^2\ (= e^2) = a^2 + b^2 + x^2 + b^2 + 2\,ab\ \text{Cos.}\ B + 2ax\ \text{Cos.}\ (B + C) + 2ad\,\text{Cos}(B + C + D) + 2bx\ \text{Cos.}\ C + 2bd\ \text{Cos.}\ (C + D) + 2dx\ \text{Cos.}\ D.$$

This equation, when solved, gives

$$x = -a\ \text{Cos.}\ (B + C) - b\ \text{Cos.}\ C - d\ \text{Cos.}\ D \pm \sqrt{\begin{bmatrix} e^2 - a^2 - b^2 - d^2 - 2ab\,\text{Cos.}\ B - 2ad\ \text{Cos.}\ (B + C + D) \\ -2bd\text{Cos.}(C + D) + [a\text{Cos.}(B + C) + b\text{Cos.}C + d\text{Cos.}D]^2 \end{bmatrix}}$$

Having from this determined the line CD; then the angles BAE, DEA, may be found by the formulæ given above.

EXAM. Let $a = 540$, $b = 519$, $d = 536$, $e = 429{\cdot}0679$, $\beta = -133^0.\ 22'$, $C = 43^0.\ 55'$, $D = -141^0.\ 9'$; which values are taken from the example in § LXXXVII, where $AE = e$ was sought, whereas here, on the contrary, e is assumed to be known, while in the above example c was assumed to be unknown. Here $e^2 - a^2 - b^2 - d^2 = -664157{\cdot}73718959$; further, we have

$$\begin{aligned} a\ \text{Cos.}\ (B + C) &= -\ 540\ \text{Cos.}\ 89^0.\ 27' = 5{\cdot}1835 \\ b\ \text{Cos.}\ C &= \quad 519\ \text{Cos.}\ 43^0.\ 55' = 373{\cdot}8614 \\ d\ \text{Cos.}\ D &= -\ 536\ \text{Cos.}\ 38^0.\ 51' = -\ 417{\cdot}4317. \end{aligned}$$

The other members, of which the expression found for x is composed, are already calculated above. We consequently have

$$x = -5\cdot1835 - 373\cdot8614 + 417\cdot4317 \pm$$
$$\sqrt{\begin{bmatrix} -664157\cdot73718959 + 384889\cdot4 + 367432\cdot8 \\ +70052\cdot5 + [5\cdot1835 + 373\cdot8614 - 417\cdot4317]^2 \end{bmatrix}}$$

or $\quad x = 38\cdot3868 \pm 399\cdot6129 \ldots = 437\cdot9997 \ldots$

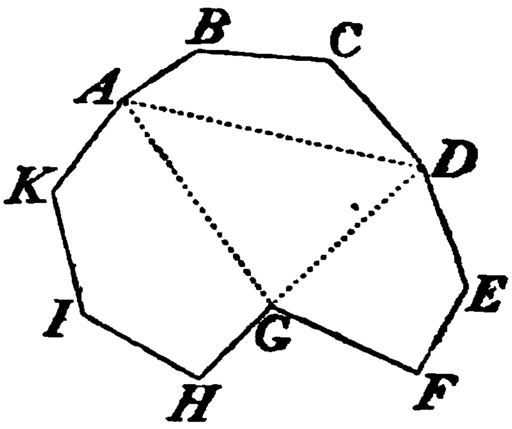
The negative value cannot be used here. In § LXXXVII, we assumed $c = 438$; the difference does not quite amount to 0·0003.

SECTION XCI.

PROB. *In a polygon all the sides but one are given, also all the angles except two, but these last are not, as in the foregoing problem, assumed to be at one side: find the unknown side, and the two unknown angles.*

SOLUT. In the polygon *ABCDEFGHIK* (*fig.* 102) all the angles, except *KAB*, *HGF*, and all the sides except *CD*, are given.

fig. 102.



Draw the diagonal *AG*, and by these means divide the polygon into two others. In the polygon *AKIHG*, all the sides, except *AG*, and the angles *K*, *I*, *H*, included by the given sides, are given; consequently by § LXXXVIII, the two unknown angles *KAG*, *AGH*, and the side *AG* may be found. If *AG* be found, then in the polygon *ABCDEFG*, we have all the sides except *CD*, and all the angles except *BAG*, *AGF*; these may ∴ be found by § XC. Having by these means determined the angles *KAG*, *AGH*, *BAG*, *AGF*; we consequently have $KAB = KAG + BAG$, $HGF = 360° - AGH - AGF$.

SECTION XCII.

Prob. *In a polygon, all the sides, and all the angles except three, are given: find the unknown angles.*

Solut. Let $ABC \ldots K$ (*fig.* 102) be a polygon, all of whose sides, and all its angles except the three A, D, G, are given.

Draw the diagonals AD, AG, DG. Then in the polygon $ABCD$ all the sides except AD, and the angles included by these sides, are known; consequently (§ LXXXVIII) the angles BAD, CDA, together with the side AD, may be found. In like manner, in the polygon $DEFG$, we find the side DG, and the angles EDG, FGD, and in the polygon $AKIHG$ the side AG, and the angles KAG, HGA. Since ∴ the three sides of the triangle ADG are known; consequently we also have its angles, from which, and the above-named, the angles A, D, G, may be determined. Thus we have $KAB = KAG + GAD + DAB$, $CDE = CDA + ADG + GDE$, $HGF = 360^0 - (HGA + AGD + DGF)$.

SECTION XCIII.

Prob. *In a quadrilateral, three sides, and the angles included by them, are given: find its area.*

Solut. Let $ABCD$ (*fig.* 103) be the quadrilateral whose area is sought; $AB = a$, $BC = b$, $CD = c$, the given sides, and ABC, BCD, the given angles. Let PQ be any line of abscissa, Dd perpendicular to it, and $A\alpha$, $A\beta$, $A\gamma$, $A\delta$, the corresponding radii of the sides AB, BC, CD, DA.

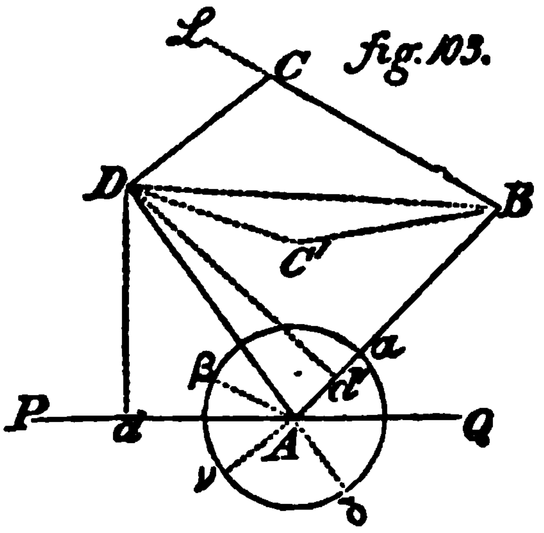


fig. 103.

1. By § XXVI, $\Delta\ BCD = \frac{1}{2} bc\ Sin.\ BCD$; or since $BCD = 180^0 - bCD = 180^0 - \beta A\gamma = 180^0 - C$,

$$\Delta\ BCD = \tfrac{1}{2} bc\ Sin.\ C$$

2. By § LXXXV we have
$Dd = a\,Sin.\,A + b\,Sin.\,(A + B) + c\,Sin.\,(A + B + C)$,
where $A = QA\kappa$.

3. Let the line PQ so move, that it coincides with AB: then $A = 0$, and Dd is transformed into Dd'; we consequently have

$$Dd' = b\,Sin.\,B + c\,Sin.\,(B + C),$$

and $\therefore$

$$\triangle BAD = \tfrac{1}{2}\,AB \times Dd' = \tfrac{1}{2}\,ab\,Sin.\,B + \tfrac{1}{2}\,ac\,Sin.\,(B + C).$$

4. Now from 1 and 3 it follows, that $ABCD = \triangle\,BAD + \triangle BCD = \frac{1}{2}\left[ab Sin. B + ac Sin. (B + C) + bc Sin. C\right]$. This is the same expression, which, by a suitable alteration of the exterior angles into interior, was found in § XXXVII, by another method.

Cor. The expression found is still correct, when the quadrilateral has an angle tending inwards. For let the point C be in C': we must then take the triangle BCD away from the triangle BAD, instead of adding it as before.

But for this case also, the angle C is negative, which follows from the change which takes place in the position of the corresponding radii, and we obtain for the triangle $BCD = \frac{1}{2}\,bc\,Sin.\,C$ a negative value.

Exam. Let $A = 682$, $b = 616$, $c = 407$, $B = 113^{\circ}.46'$, $C = 63^{\circ}.49'$. Here $ab\,Sin.\,B = 384484{\cdot}1$; $ac\,Sin.\,(B + C) = 11704{\cdot}3$; $bc\,Sin.\,C = 224985{\cdot}6$; consequently the quadrilateral $ABCD = 310587$.

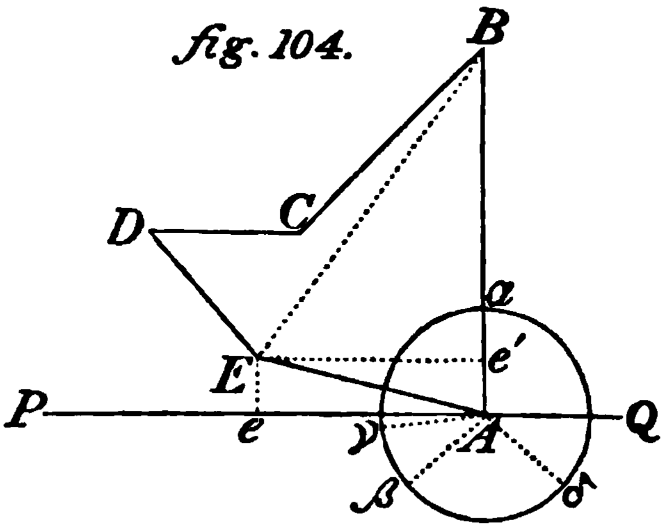
SECTION XCIV.

Prob. *In a pentagon, four sides, and the angles included by them, are given: find its area.*

Solut. Let $ABCDE$ (*fig.* 104) be the pentagon;

$AB = a$, $BC = b$, $CD = c$, $DE = d$, the given sides, and B, C, D, the given angles.

fig. 104.



1. Divide the pentagon by the diagonal BE, into the quadrilateral $BCDE$, and the triangle BAE; through A draw any line of abscissæ PQ, and the perpendicular Ee.

2. Then by § LXXXV,

$$Ee = a \text{ Sin. } A + b \text{ Sin. } (A + B) + c \text{ Sin. } (A + B + C) + d \text{ Sin. } (A + B + C + D),$$

in which $A = QAa$. Now let the line PQ move towards Q, so that it coincides with AB: then $A = 0$, and Ee is transformed into Ee'; we consequently have

$$Ee' = b \text{ Sin. } B + c \text{ Sin. } (B + C) + d \text{ Sin. } (B + C + D)$$

and $\therefore \triangle BEA = \frac{1}{2} AB \times Ee' =$

$$\tfrac{1}{2}\,[ab \text{ Sin. } B + ac \text{ Sin. } (B + C) + ad \text{ Sin. } (B + C + D)].$$

3. But by the foregoing section,

quadrilateral $BCDE = \frac{1}{2}\,[bc \text{ Sin. } C + bd \text{ Sin. } (C + D) + cd \text{ Sin. } D]$; consequently pentagon $ABCDE =$

$$\frac{1}{2}\begin{bmatrix} ab \text{ Sin. } B + ac \text{ Sin. } (B + C) + ad \text{ Sin. } (B + C + D) \\ + bc \text{ Sin. } C + bd \text{ Sin. } (C + D) + cd \text{ Sin. } D \end{bmatrix}$$

Exam. Let $a = 332$, $b = 248$, $c = 128$, $d = 152$, $B = 132^\circ.\ 14'$, $C = -38^\circ.\ 29'$, $D = 140^\circ.\ 47'$. Here $\text{Sin. } B = \text{Sin. } 47^\circ.\ 46'$, $\text{Sin. } (B + C) = \text{Sin. } 93^\circ.\ 45' = \text{Sin. } 86^\circ.\ 15'$, $\text{Sin. } (B + C + D) = \text{Sin. } 234^\circ.\ 32' = -\text{Sin. } 54^\circ.\ 32'$, $\text{Sin. } C = -\text{Sin. } 38^\circ.\ 29'$, $\text{Sin. } (C + D) = \text{Sin. } 102^\circ.\ 18' = \text{Sin. } 77^\circ.\ 42'$, $\text{Sin. } D = \text{Sin. } 39^\circ.\ 13'$; we consequently have

$$ab \text{ Sin. } B \quad = \quad 60962{\cdot}69$$
$$ac \text{ Sin. } (B + C) \quad = \quad 42405{\cdot}01$$

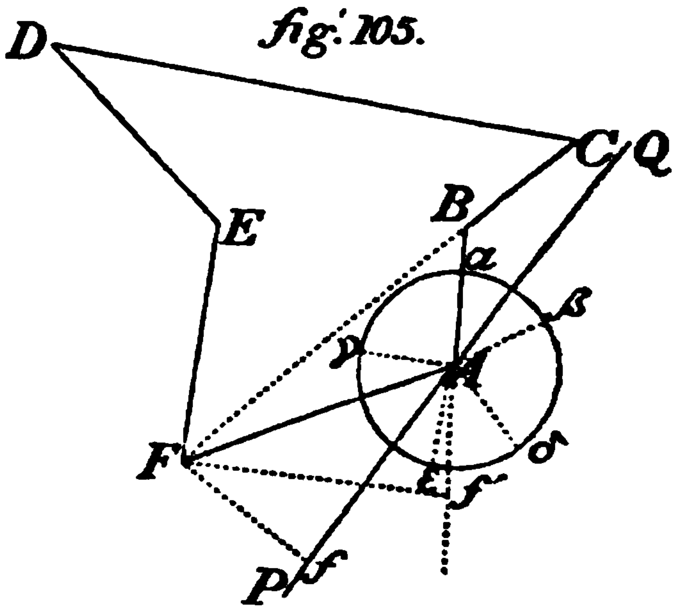
$$\begin{array}{lcr} ad\ Sin.\ (B + C + D) & = & -\ 41100{\cdot}56 \\ bc\ Sin.\ C & = & -\ 19753{\cdot}88 \\ bd\ Sin.\ (C + D) & = & 36830{\cdot}71 \\ cd\ Sin.\ D & = & 12301{\cdot}15\,; \end{array}$$

and $\therefore$ pentagon $ABCDE = 45822{\cdot}56$.

SECTION XCV.

PROB. *In a hexagon, five sides, and the four angles included by them, are given : find its area.*

SOLUT. Let $ABCDEF$ (*fig.* 105) be the hexagon; $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EF = e$, the given sides, and B, C, D, E, the given angles.



1. Draw the diagonal BF, and by it divide the hexagon into the pentagon $BCDEF$, and the triangle BAF; further, let PQ be the arbitrary line of abscissæ, and Ff, perpendicular to it : then (§ LXXXV),

$$Ff = a\ Sin.\ A + b\ Sin.\ (A + B) + c\ Sin.\ (A + B + C) + d\ Sin.\ (A + B + C + D) + e\ Sin.\ (A + B + C + D + E),$$

in which $A = QAa$. Let the line PQ move into AB: then $A = 0$, and Ff is transformed into Ff'; we consequently have

$$Ff' = b\ Sin.\ B + c\ Sin.\ (B + C) + d\ Sin.\ (B + C + D) + e\ Sin.\ (B + C + D + E),$$

and $\therefore$ $\triangle\, BAF = \frac{1}{2} AB \times Ff' =$

$$\tfrac{1}{2}\left\{\begin{array}{c} ab\ Sin.\ B + ac\ Sin.\ (B \parallel C) + ad\ Sin.\ (B + C + D) \\ + ae\ Sin.\ (B + C + D + E) \end{array}\right\}$$

2. By the preceding section, pentagon $BCDEF =$

$$\tfrac{1}{2}\left\{\begin{array}{c} bc\ Sin.\ C + bd\ Sin.\ (C + D) + be\ Sin.\ (C + D + E) \\ + cd\ Sin.\ D + ce\ Sin.\ (D + E) + de\ Sin.\ E \end{array}\right\}$$

we .·. have hexagon $ABCDEF =$

$$\frac{1}{2}\left\{\begin{array}{c} ab\ Sin.\ B + ac\ Sin.\ (B+C) + ad\ Sin.\ (B+C+D) \\ + ae Sin.(B+C+D+E) + bc Sin. C + bd Sin. (C+D) \\ + be\ Sin.\ (C+D+E) + cd\ Sin.\ D + ce\ Sin.\ (D+E) \\ + de\ Sin.\ E \end{array}\right\}$$

Exam. Let $a = 324$, $b = 288$, $c = 1102$, $d = 528$, $e = 504$, $B = -\alpha A\beta = -60^0.\ 52'$, $C = \beta A\gamma = 129^0.\ 20'$, $D = \gamma A\delta = 139^0.\ 43'$, $E = \delta A\epsilon = -55^0.\ 50'$. Here

$ab\ Sin.\ B$	$= -$	81507·02
$ac\ Sin.\ (B + C)$	$=$	332127·56
$ad\ Sin.\ (B + C + D)$	$= -$	80796·38
$ae\ Sin.\ (B + C + D + E)$	$=$	85780·64
$bc\ Sin.\ C$	$=$	245481·35
$bd\ Sin.\ (C + D)$	$= -$	152043·07
$be\ Sin.\ (C + D + E)$	$= -$	79515·22
$cd\ Sin.\ D$	$=$	376209·31
$ce\ Sin.\ (D + E)$	$=$	552245·95
$de\ Sin.\ E$	$= -$	229182·98;
and .·. hexagon $ABCDEF$	$=$	483900·09.

SECTION XCVI.

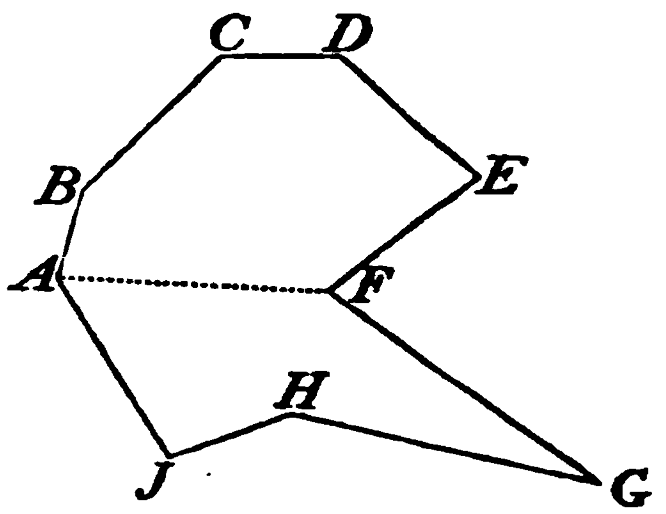
Prob. *All the sides of a polygon but one, and all the angles included by these sides, are given: find its area.*

Solut. From §§ XCIII, XCIV, XCV, the law may be easily perceived, by which the area of every polygon may be calculated from the parts given; this may be expressed in words as follow:

To calculate the area of a polygon from the parts given, assume half the sum of the product of every two of the given sides, combined in every possible way, each multiplied by the sine of the algebraical sum of the exterior angles lying between them.

Cor. If a polygon has a great number of sides, as $ABCDEFGHI$ (*fig.* 106), and if all its sides and angles are known: then the best plan would be, to divide it by a diagonal AE into two other polygons $ABCDEF$, $FGHIA$, so that one may have the same number of sides as the other, or one more, and then to calculate each separately, while we consider AF as the side of both polygons which is not given. By these means we obtain this advantage, that it is not necessary to calculate so many terms as we otherwise should have: likewise the sums of the angles are also less, by which the calculation is essentially shortened.

fig. 106.



SECTION XCVII.

Prob. *All the angles and sides of a polygon are given: find any arbitrary diagonal of this figure.*

Solut. Let $ABCDEFGHI$ (*fig.* 106) be any polygon, and AF the diagonal sought.

In the polygon $ABCDEF$, which is divided by the diagonal AF, the sides AB, BC, CD, DE, EF, and the angles included by them, viz. B, C, D, E, are given: it is required ∴ to find AF (§ LXXXVIII).

Cor. But the line AF may also be found from the other polygon $FGHIA$, which is cut off by this diagonal. We consequently have two different expressions for these lines; and since this likewise obtains for every other diagonal; hence ∴ there are a great number of equations, which express the relations between the sides and angles of a polygon. Thus, for the diagonal AF, when the sides AB, BC, CD, DE, EF, FG, GH, HI, IA, are respectively denoted by a, b, c, d, e, f, g, h, i, we obtain the following equation:

$$a^2 + b^2 + c^2 + d^2 + e^2 + 2\,ab\,Cos.\,B + 2\,ac\,Cos.\,(B + C) + 2\,ad\,Cos.(B + C + D) + 2\,ae\,Cos.(B + C + D + E) + 2\,bc\,Cos.\,C + 2\,bd\,Cos.(C + D) + 2\,be\,Cos.(C + D + E) + 2\,cd\,Cos.\,D + 2\,ce\,Cos.\,(D + E) + 2\,de\,Cos.\,E = f^2 + g^2 + h^2 + i^2 + 2\,fg\,Cos.\,G + 2\,fh\,Cos.\,(G + H) + fi\,Cos.(G + H + I) + 2\,gh\,Cos.\,H + 2\,gi\,Cos.(H + I) + 2\,hi\,Cos.\,I.$$

SECTION XCVIII.

Prob. *To determine the distance between two places, when it is rather considerable.*

Solut. Let A, F, (*fig.* 107) be the two places, whose distance is required.

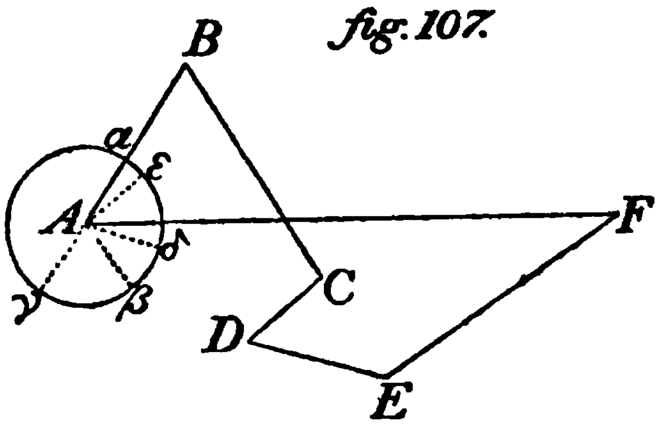


fig. 107.

Connect A, F, with proper distances AB, BC, CD, DE, EF: measure these lines in a direct or indirect way, also the angles included by them, viz. ABC, BCD, CDE, DEF: hence AF may be determined by § LXXXVIII. Thus, if we denote, as before, the sides AB, BC, CD, DE, EF, by a, b, c, d, e, respectively, and the exterior angles by B, C, D, E: then

$$AF = \sqrt{\begin{bmatrix} a^2 + b^2 + c^2 + d^2 + e^2 + 2\,ab\,Cos.\,B + \\ 2\,ac\,Cos.(B + C) + 2\,ad\,Cos.(B + C + D) + \\ 2\,ae\,Cos.(B + C + D + E) + 2\,bc\,Cos.\,C + \\ 2\,bd\,Cos.(C + D) + 2\,be\,Cos.(C + D + E) + \\ 2\,cd\,Cos.\,D + 2\,ce\,Cos.\,(D + E) + 2\,de\,Cos.\,E \end{bmatrix}}$$

where $B = -\,aA\beta = -\,(180^0 - ABC)$, $C = -\,\beta A\gamma = -\,(180^0 - BCD)$, $D = \gamma A\delta = 180^0 - CDE$, $E = \delta A\varepsilon = 180^0 - DEF$.

Remark. The extension which has been here given to the problem in § LXXXVIII, is correct, if in the proofs on which its solution is founded, nothing is assumed which is not applicable to every divided line included by it.

SECTION XCIX.

Prob. *To find the area of a polygon from one of its sides, the two angles adjacent to it, and the angles, which the diagonals drawn from the angular points of this side make with it.*

Solut. Let *ABCDEFG* (*figs.* 108, 109) be a polygon,

fig. 108.

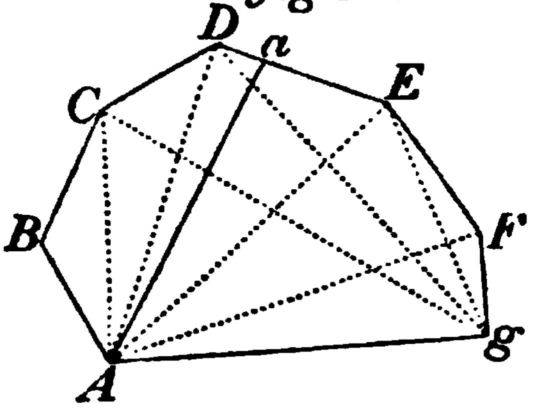
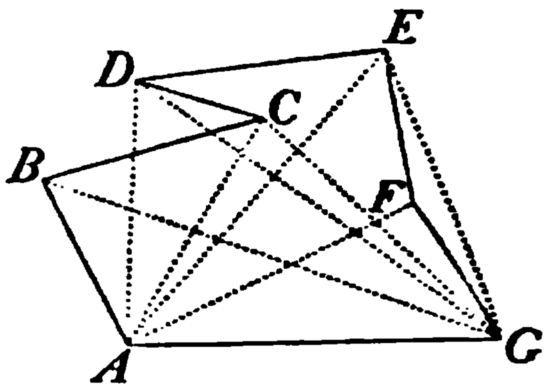


fig. 109.



in which the side *AG*, together with the angles *BAG*, *CAG*, *DAG*, *EAG*, *FAG*, *BGA*, *CGA*, *DGA*, *EGA*, *FGA*, are given; from them find its area. The polygon, *fig.* 108, has all its angles outwards, the figure 109 has some of its angles inwards.

1. From the given angles we may find the angles *BAC*, *CAD*, *DAE*, *EAF*, *FAG*, likewise *ABG*, *ACG*, *ADG*, *AEG*, *AFG*, merely by subtraction. If these are found; then from the triangles *BAG*, *CAG*, *DAG*, *EAG*, *FAG*, we obtain the following expressions:

$$AB = \frac{AG \text{ Sin. } BGA}{\text{Sin. } ABG}, \quad AC = \frac{AG \text{ Sin. } CGA}{\text{Sin. } ACG},$$

$$AD = \frac{AG \text{ Sin. } DGA}{\text{Sin. } ADG}, \quad AE = \frac{AG \text{ Sin. } EGA}{\text{Sin. } AEG},$$

$$AF = \frac{AG \text{ Sin. } FGA}{\text{Sin. } AFG}.$$

2. Hence, and from § XXVI, we obtain

$$\triangle BAC = \tfrac{1}{2} AB \,.\, AC \text{ Sin. } BAC = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } BGA \text{ Sin. } CGA \text{ Sin. } BAC}{\text{Sin. } ABG \text{ Sin. } ACG}$$

$$\triangle CAD = \tfrac{1}{2} AC \,.\, AD \text{ Sin. } CAD = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } CGA \text{ Sin. } DGA \text{ Sin. } CAD}{\text{Sin. } ACG \text{ Sin. } ADG}$$

$$\triangle DAE = \tfrac{1}{2} AD \,.\, AE \text{ Sin. } DAE = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } DGA \text{ Sin. } EGA \text{ Sin. } DAE}{\text{Sin. } ADG \text{ Sin. } AEG}$$

$$\triangle EAF = \tfrac{1}{2} AE \,.\, AF \text{ Sin. } EAF = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } EGA \text{ Sin. } FGA \text{ Sin. } EAF}{\text{Sin. } AEG \text{ Sin. } AFG}$$

$$\triangle FAG = \tfrac{1}{2} AF \,.\, AG \text{ Sin. } FAG = \frac{\tfrac{1}{2} AG^2 \text{ Sin. } FGA \text{ Sin. } FAG}{\text{Sin. } AFG}$$

3. Now since the heptagon $ABCDEFG = \triangle BAC \pm \triangle CAD + \triangle DAE + \triangle EAF + \triangle FAG$, (the upper of the two signs $\pm$ obtains for *fig.* 108); then it is =

$$\tfrac{1}{2} AG^2 \left\{ \begin{array}{l} \dfrac{\text{Sin. } BGA \text{ Sin. } CGA \text{ Sin. } BAC}{\text{Sin. } ABG \text{ Sin. } ACG} \\ \pm \dfrac{\text{Sin. } CGA \text{ Sin. } DGA \text{ Sin. } CAD}{\text{Sin. } ACG \text{ Sin. } ADG} \\ + \dfrac{\text{Sin. } DGA \text{ Sin. } EGA \text{ Sin. } DAE}{\text{Sin. } ADG \text{ Sin. } AEG} \\ + \dfrac{\text{Sin. } EGA \text{ Sin. } FGA \text{ Sin. } EAF}{\text{Sin. } AEG \text{ Sin. } AFG} \\ + \dfrac{\text{Sin. } FGA \text{ Sin. } FAG}{\text{Sin. } AFG} \end{array} \right\}$$

4. If we take G, instead of A, for the common vertex of the triangles, we then obtain besides, because $ABCDEFG = \pm \triangle FGE + \triangle EGD \pm \triangle DGC + \triangle CGB + \triangle BGA$, the following expression for the area:

$$\tfrac{1}{2} AG^2 \left\{ \begin{array}{l} \pm \dfrac{Sin.\ FAG\ Sin.\ EAG\ Sin.\ FGE}{Sin.\ AFG\ Sin.\ AEG} \\ + \dfrac{Sin.\ EAG\ Sin.\ DAG\ Sin.\ EGD}{Sin.\ AEG\ Sin.\ ADG} \\ \pm \dfrac{Sin.\ DAG\ Sin.\ CAG\ Sin.\ DGC}{Sin.\ ADG\ Sin.\ ACG} \\ + \dfrac{Sin.\ CAG\ Sin.\ BAG\ Sin.\ CGB}{Sin.\ ACG\ Sin.\ ABG} \\ + \dfrac{Sin.\ BGA\ Sin.\ BAG}{Sin.\ ABG} \end{array} \right\}$$

Cor. Although the calculation has been effected here for a heptagon, we can readily perceive from hence, how to proceed in the case of every other polygon. The law, which the terms of the expressions in 3 and 4 observe, may be more easily understood by inspection than by words. Moreover, these expressions mutually serve to prove the calculation.

Exam. Let (*fig.* 108) $BAG = 120°$, $CAG = 101°.\ 37'$, $DAG = 81°.\ 20'$, $EAG = 48°.\ 49'$, $FAG = 23°.\ 48'$; $FGA = 88°.\ 14'$, $EGA = 70°$, $DGA = 51°.\ 27'$, $CGA = 34°.\ 7'$, $BGA = 22°.\ 10'$. Here $ABG = 37°.\ 50'$, $ACG = 44°.\ 16'$, $ADG = 47°.\ 13'$, $AEG = 61°.\ 41'$, $AFG = 67°.\ 58'$; $BAC = 18°.\ 23'$, $CAD = 20°.\ 17'$, $DAE = 33°.\ 1'$, $EAF = 24°.\ 31'$; $FGE = 18°.\ 14'$, $EGD = 18°.\ 33'$, $DGC = 17°.\ 20'$, $CGB = 11°.\ 57'$. We consequently have

$$\frac{Sin.\ BGA\ Sin.\ CGA\ Sin.\ BAC}{Sin.\ ABG\ Sin.\ ACG} = 0{\cdot}1558865$$

$$\frac{Sin.\ CGA\ Sin.\ DGA\ Sin.\ CAD}{Sin.\ ACG\ Sin.\ ADG} = 0{\cdot}2968328$$

$$\frac{Sin.\ DGA\ Sin.\ EGA\ Sin.\ DAE}{Sin.\ ADG\ Sin.\ AEG} = 0{\cdot}6197669$$

$$\frac{Sin.\ EGA\ Sin.\ FGA\ Sin.\ EAF}{Sin.\ AEG\ Sin.\ AFG} = 0{\cdot}4776058$$

$$\frac{Sin.\ FGA\ Sin.\ FAG}{Sin.\ AFG} = 0{\cdot}4351331\ ;$$

and $\therefore$, by 3, heptagon $ABCDEFG = 0{\cdot}9926125 \,.\, AG^2$.

Further, we have

$$\frac{Sin.\ FAG\ Sin.\ EAG\ Sin.\ FGE}{Sin.\ AFG\ Sin.\ AEG} = 0{\cdot}1155552$$

$$\frac{Sin.\ EAG\ Sin.\ DAG\ Sin.\ EGD}{Sin.\ AEG\ Sin.\ ADG} = 0{\cdot}3635294$$

$$\frac{Sin.\ DAG\ Sin.\ CAG\ Sin.\ DGC}{Sin.\ ADG\ Sin.\ ACG} = 0{\cdot}5631592$$

$$\frac{Sin.\ CAG\ Sin.\ BAG\ Sin.\ CGB}{Sin.\ ACG\ Sin.\ ABG} = 0{\cdot}4102605$$

$$\frac{Sin.\ BGA\ Sin.\ BAG}{Sin.\ ABG} = 0{\cdot}5327207\ ;$$

and $\therefore$ by 4, heptagon $ABCDEFG = 0{\cdot}9926125 \,.\, AG^2$, as before.

Now, in order to determine fully the area of the heptagon, it is only requisite to know the length of AG. Thus, let $AG = 450$: then heptagon $ABCDEFG = 201004{\cdot}03$ nearly.

Cor. This method of calculating the area of a figure, may be very advantageously made use of in its division. Thus, if we wish to cut off $\frac{5}{8}$ths from the heptagon just calculated, by a line drawn from the point A, we have

$$\tfrac{5}{8}\ ABCDEFG = 0{\cdot}6203828 \,.\, AG^2.$$

Now, by the above calculations $\triangle\ BAC = 0{\cdot}1558865 \,.\, AG^2$, $\triangle\ CAD = 0{\cdot}2968328 \,.\, AG^2$, $\triangle\ DAE = 0{\cdot}6197669 \,.\, AG^2$; whence it appears, that the line of section Aa falls between AD and AE, and that $\triangle\ DAa = 0{\cdot}1676635 \,.\, AG^2$. Consequently we have $\triangle\ DAE : \triangle\ DAa = 6197669 : 1676635 = DE : Da$, or approximately $DE : DA = 920 : 257$, which is found by means of Continued Fractions. According to this proportion $\therefore$ the line DE must be divided in a, and then, when Aa is drawn, we have $ABCDa = \frac{5}{8}\,ABCDEFG$

SECTION C.

PROB. *From the same parts as are given in the foregoing problem, to find any side whatever of the polygon.*

SOLUT. Let $ABCDEFG$ (*fig.* 108) be the polygon, in which the same parts are given, as in the foregoing problem: find the sides AB, BC, CD, DE, EF, FG.

In the triangles ABG, AFG, all the angles and the side AG are given; consequently also the sides AB, FG may be found. The remaining sides can be found by the seventh solution in § XLVIII. Thus, if it is wished to determine the line DE, then the comparison with the above solution gives $a = AG$, $\alpha = DAG$, $\beta = EAG$, $\gamma = EGA$, $\delta = DGA$, and we $\therefore$ have

$$DE = AG \sqrt{\left[\left(\frac{Sin.\ DGA}{Sin.\ A}\right)^2 + \left(\frac{Sin.\ EGA}{Sin.\ B}\right)^2 - \frac{2\ Sin.\ DGA\ Sin.\ EGA\ Cos.\ (DAG - EAG)}{Sin.\ A\ Sin.\ B}\right]}$$

in which $A = 180^\circ - DAG - EAG$, $B = 180^\circ - EGA - EAG$.

REMARK. The following authors treat of Polygonometry: Lexell (De Resolutione Polygonorum Rectilineorum; Commentarii Novi. Petrop. T. 19, 20); L'Huilier (Polygonometrie, ou De la Mesure des Figures Rectilignes, &c. Geneve et Paris, 1789); Neumann (New Contributions to Practical Geometry. Munich, 1800); Däzel in the above-mentioned work; Mascheroni (Problemi di Geometrica. Milano, Anno X (1802), p. 105, et seq.); and also in a particular work (Metodo di misurari i Poligoni Piani), which appeared at Pavia in 1787, and which contains all the problems to be found in L'Huilier's Polygonometry, which appeared in 1789; I have not been able, however, to get a sight of it. On Tetragonometry, in particular, Lambert has treated in his Contributions, part II. p. 175, et seq.; J. T. Mayer (Tetragonometriæ Specimen I. Göt. 1773); Biörnsen (Introductio in Tetragonometriam ad mentem v. c. Lambert Analytice Conscripta Havniæ, 1780). There is also much matter respecting this subject amongst others to be found in Carnot's Geometrie de Position, Paris, 1803, p. 304, &c. which part was printed before the author became acquainted with the above-mentioned work of L'Huilier.

IX. A FEW IMPORTANT PROBLEMS IN PRACTICAL GEOMETRY, CHIEFLY RELATING TO THE MODE OF TREATING NEGATIVE ANGLES.

SECTION CI.

PRELIMINARIES.

1. Let A, B, C, D, E, (*fig.* 110) be a system of points; O a point, for which, in the relative position of these points in the assumed figure, and for any calculation founded upon it, we assume, $AOB = \alpha$, $BOC = \beta$, $COD = \gamma$, $DOE = \delta$. Suppose, that after having completed the calculation, we wish to apply any algebraical expression taken from it to a particular case; but that the point O has not the position supposed in the calculation with respect to the system, but is situated in O': then it is evident, that with regard to the absolute magnitude of the angles, we must put, $\alpha = AO'B$, $\beta = BO'C$, $\gamma = CO'D$, $\delta = DO'E$. Now, in order to determine, likewise, which angle, according to the change of position, is to be considered as positive, and which as negative, may be discovered by the method in § LXXXIV, with this difference, that here the corresponding radii coincide with the positions of the lines containing the angles, instead of their being as in the above section, drawn according to the direction in which they are produced. This change takes place merely for this reason, because in the present case we have to do with the angles themselves, while in § LXXXIV their adjacent angles were considered.

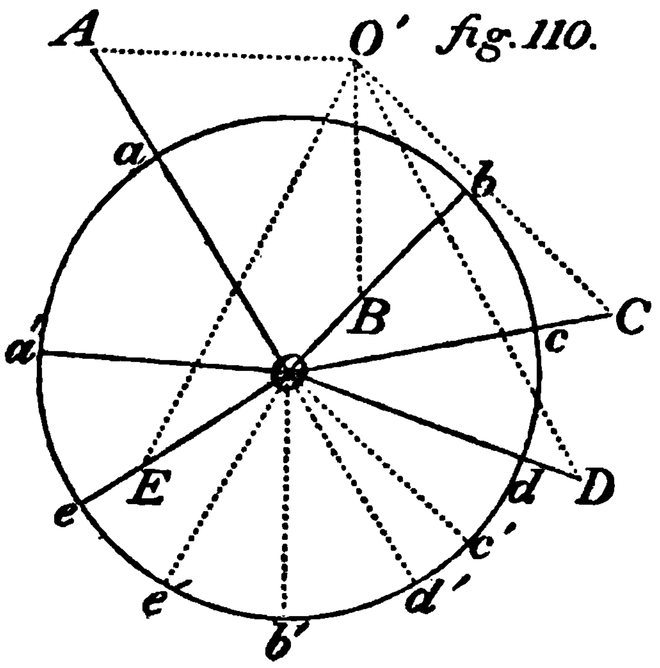
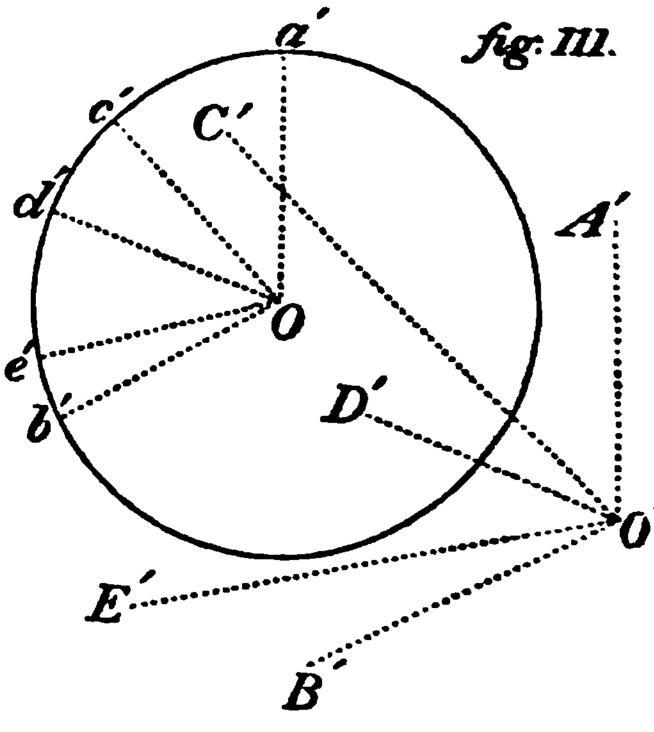


fig. 110.

2. Let $\therefore$ a circle be described about O, which cuts the lines OA, OB, OC, OD, OE, or these produced in the points a, b, c, d, e: then Oa, Ob, Oc, Od, Oe, are the corresponding radii of these lines. Let now the point O move into O', and let the lines Oa', Ob', Oc', Od', Oe', be drawn parallel to the lines $O'A$, $O'B$, $O'C$, $O'D$, $O'E$, respectively: then the first are the corresponding radii of the latter, with reference to O', and we have $a'Ob' = AO'B$, $b'Oc' = BO'C$, $c'Od' = CO'D$, $d'Oe' = DO'E$. Again, as in § LXXXIV, suppose a moving radius, which turns round the point O; this, in order to move from Oa', into Ob', must have an opposite motion from that which it has when it moves from Oa into Ob. The two angles aOb, $a'Ob'$, and $\therefore$ also AOB, $AO'B$ are consequently opposite to one another in position, and we $\therefore$ have $\alpha = -AO'B$. The same obtains of the angles bOc, $b'Oc'$, and $\therefore$ also of BOC, $BO'C$; consequently $\beta = -BO'C$. On the other hand, the moving radius has the same direction, whether it move from Oc into Od, or from Oc' into Od'; consequently γ is positive, and we have $\gamma = CO'D$. For a similar reason δ is positive, and we $\therefore$ have $\delta = DO'E$.

3. Hitherto it has been assumed, that the point O alone moves, while, on the other hand, the points A, B, C, D, E, retain their places. But even for the case where the points of the system itself change their positions with respect to each other, the above rule admits of no exception. Suppose the points A, B, C, D, E, O, have the position A', B', C', D', E', O', (*fig.* 111), so that A moves into A', B into B', and so on, and that we wish to find the values of α, β, γ, δ, accordingly: then draw the radii Oa', Ob', Oc', Od', Oe', corresponding to the lines $O'A'$, $O'B'$, $O'C'$, $O'D'$, $O'E'$, and compare *fig.* 110 with *fig.* 111. Then we get $\alpha = -a'Ob' = -A'O'B'$, $\beta = b'Oc' = B'O'C'$, $\gamma = -c'Od' = -C'O'D'$, $\delta = -d'Oe' = -D'O'E'$.

fig. 111.



It will not be necessary, in order to distinguish between

the positive and negative position of the angles, actually to draw the corresponding radii. After a little practice, the turning the compasses only will be sufficient, as every person may easily perceive without any diffuse explanation.

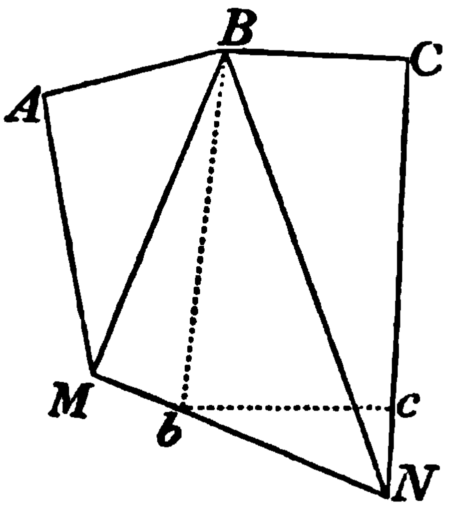
SECTION CII.

PROB. *Two inaccessible points are given in position and distance: determine the positions and distances of two other points with reference to the former, without measuring any line.*

First Solution.

Let A, B, C, (*fig.* 112) be the three points; the distances AB, BC, and the angle ABC are given: find the positions of two other points M, N merely by measuring angles.

fig: 112.



I shall assume, that all the three points A, B, C, can be seen both from M and N, and that at these last points (if we suppose MC, NA drawn), the angles AMB, BMC, ANB, BNC, can be measured. For this case, the points M, N, may be determined singly by § LIV, and the distances MA, MB, MC, NA, NB, NC, calculated. But the distance MN may also be easily computed: for in the triangle MBN, we then have the angle $MBN = ABC - ABM - CBN$, and the sides MB, NB.

Second Solution.

1. I shall now assume, that from M the points A, B, N can be seen, but not the point C, and that from N the points B, C, M, but not the point A. Then the angles AMB, BMN, BNC, CNM, can be measured. Let $AB = a$, $BC = b$, $ABC = B$, $AMB = \alpha$, $BMN = \beta$, $BNC = \gamma$, $CNM = \delta$. If, besides, the angle MAB is known, then all the rest are known. Put $\therefore$ $MAB = \phi$, and draw Bb, bc parallel to the lines CN, BC.

2. From these assumptions we obtain $BbM = CNM = \delta$, $ABM = 180^0 - (\alpha + \phi)$, $MBb = 180^0 - (\beta + \delta)$, and hence $bBC = ABC - ABM - MBb = B + \alpha + \beta + \delta + \phi - 360^0 = Ccb$; consequently $Nbc = Ccb - CNM = B + \alpha + \beta + \phi - 360^0$, $BCN = 180^0 - bBC = 540^0 - (B + \alpha + \beta + \delta + \phi)$, $CBN = 180^0 - BCN - BNC = B + \alpha + \beta + \gamma + \delta + \phi - 360^0$. If ∴ for shortness' sake, we put $B + \alpha + \beta - 360^0 = \kappa$, $B + \alpha + \beta - \gamma + \delta - 360^0 = \lambda$: then $Nbc = \kappa + \phi$, $CBN = \gamma + \phi$.

3. The triangle ABM gives

$$BM = \frac{AB \; Sin.\, MAB}{Sin.\, AMB} = \frac{a \; Sin.\, \phi}{Sin.\, \alpha},$$

and the triangle MBb,

$$Bb = \frac{BM \; Sin.\, BMN}{Sin.\, BbM} = \frac{a \; Sin.\, \phi \; Sin.\, \beta}{Sin.\, \alpha \; Sin.\, \delta};$$

further, the triangle Nbc,

$$Nc = \frac{bc \; Sin.\, Nbc}{Sin.\, CNM} = \frac{b \; Sin.\, (\kappa + \phi)}{Sin.\, \delta},$$

and the triangle BNC,

$$CN = \frac{BC \; Sin.\, CBN}{Sin.\, BNC} = \frac{b \; Sin.\, (\lambda + \phi)}{Sin.\, \gamma}.$$

4. Now $CN = Cc + Nc = Bb + Nc$: we ∴ have the equation

$$\frac{b \; Sin.\, (\lambda + \phi)}{Sin.\, \gamma} = \frac{a \; Sin.\, \beta \; Sin.\, \phi}{Sin.\, \alpha \; Sin.\, \delta} + \frac{b \; Sin.\, (\kappa + \phi)}{Sin.\, \delta}$$

or, when we expand $Sin.\, (\lambda + \phi)$, $Sin.\, (\kappa + \phi)$, divide by $Sin.\, \phi$, and substitute $Cot.\, \phi$ for $\frac{Cos.\, \phi}{Sin.\, \phi}$, we get

$$\frac{b \; (Sin.\, \lambda \; Cot.\, \phi + Cos.\, \lambda)}{Sin.\, \gamma} = \frac{a \; Sin.\, \beta}{Sin.\, \alpha \; Sin.\, \delta} + \frac{b \; (Sin.\, \kappa \; Cot.\, \phi + Cos.\, \kappa)}{Sin.\, \delta},$$

whence we obtain

$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \gamma + b\ Sin.\ \alpha\ (Sin.\ \gamma\ Cos.\ \kappa - Sin.\ \delta\ Cos.\ \lambda)}{b\ Sin.\ \alpha\ (Sin.\ \delta\ Sin.\ \lambda - Sin.\ \gamma\ Sin.\ \kappa)}.$$

EXAM. 1. Let $a = 351$, $b = 402$, $B = 167^\circ.\ 4'$, $\alpha = 33^\circ.\ 3'$, $\beta = 90^\circ.\ 51'$, $\gamma = 27^\circ.\ 19'$, $\delta = 76^\circ.\ 20'$; $\therefore$ $\kappa = -\ 69^\circ.\ 2'$, $\lambda = -\ 20^\circ.\ 1'$, and consequently $Sin.\ \kappa = -Sin.\ 69^\circ.\ 2'$, $Cos.\ \kappa = Cos\ 69^\circ.\ 2'$, $Sin.\ \lambda = -Sin.\ 20^\circ.\ 1'$, $Cos.\ \lambda = Cos.\ 20^\circ.\ 1'$. We $\therefore$ have

$$\begin{aligned} a\ Sin.\ \beta\ Sin.\ \gamma &= 161{\cdot}0590 \\ b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa &= 36{\cdot}0009 \\ b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda &= 200{\cdot}1631 \\ b\ Sin.\ \alpha\ Sin.\ \delta\ Sin\ \lambda &= -\ 71{\cdot}9193 \\ b\ Sin.\ \alpha\ Sin\ \gamma\ Sin.\ \kappa &= -\ 93{\cdot}9490 \end{aligned}$$

and $\therefore$,

$$Cot.\ \phi = \frac{161{\cdot}0590 + 36{\cdot}0009 - 200{\cdot}1631}{-\ 72{\cdot}9193 + 93{\cdot}9490}$$

$$= -\ 0{\cdot}1475627;$$

consequently $\quad \phi = 98^\circ.\ 23'.\ 39''.$

EXAM. 2. I shall assume, that the points A, B, C, M, N, have the positions A', B', C', M', N', (*fig.* 113). Here we must first of all ascertain which of the given angles, agreeably to this change in the positions, we are to consider as positive, and which as negative. In the first place, suppose two circles described about M, M'; then we shall immediately perceive, that the corresponding radii of MA, MB have an opposite position from that which the corresponding radii of $M'A'$, $M'B'$ have, and that this is likewise the case with the corresponding radii of MB, MN, compared with those of $M'B'$, $M'N'$; consequently both α and β are negative. In the

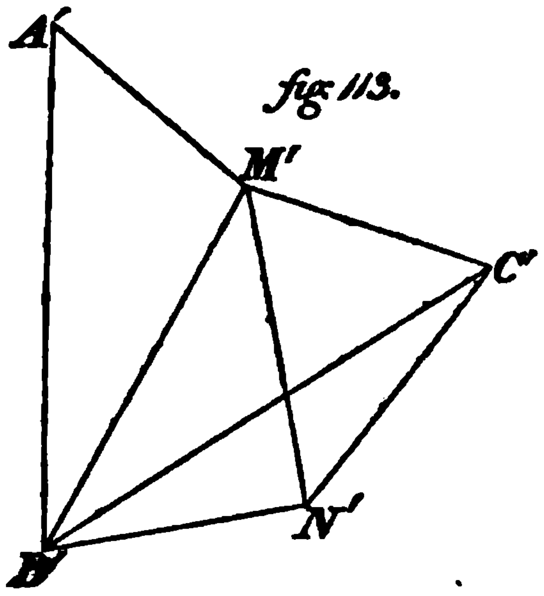


fig. 113.

same manner it may be shown, if two circles are described about B, B', that B is likewise negative. If, on the other hand, two circles are described about N, N', then it is evident, that the angles BNC, $B'N'C'$, also CNM, $C'N'M'$, have the same position, and ∴ that γ and δ remain positive.

Now, let $B = -A'B'C' = -56^\circ.\ 1'$, $\alpha = -A'M'B' = -103^\circ.\ 6'$, $\beta = -B'M'N' = -41^\circ.\ 8'$, $\gamma = B'N'C' = 144^\circ.\ 27'$, $\delta = C'N'M' = 50^\circ.\ 43'$, $a = A'B' = 980$, $b = B'C' = 1000$. Here $\kappa = B + \alpha + \beta - 360^\circ = -560^\circ.\ 15'$, $\lambda = B + \alpha + \beta - \gamma + \delta - 360^\circ = -653^\circ.\ 59'$; consequently $Sin.\ \kappa = Sin. - 560^\circ.\ 15' = -Sin.\ 560^\circ.\ 15' = -Sin.\ 200^\circ.\ 15' -Sin.\ 20^\circ.\ 15'$, $Cos.\ \kappa = Cos. - 560^\circ.\ 15' = Cos.\ 560^\circ.\ 15' = Cos.\ 200^\circ.\ 15' = -Cos.\ 20^\circ.\ 15'$, $Sin.\ \lambda = Sin. - 653^\circ.\ 59' = -Sin.\ 653^\circ.\ 59' = -Sin.\ 293^\circ.\ 59' = Sin.\ 66^\circ.\ 1'$, $Cos.\ \lambda = Cos. - 653^\circ.\ 59' = Cos.\ 653^\circ.\ 59' = Cos.\ 293^\circ.\ 59' = Cos.\ 66^\circ.\ 1'$; hence

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \gamma &= -\ 374{\cdot}8122\\
b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa &= \ \ \ 531{\cdot}2812\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda &= -\ 306{\cdot}4307\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \lambda &= -\ 688{\cdot}7940\\
b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \kappa &= -\ 196{\cdot}0000.
\end{aligned}$$

We have ∴

$$Cot.\ \phi = \frac{-\ 374{\cdot}8122 + 531{\cdot}2812 + 306{\cdot}4307}{-\ 688{\cdot}7940 + 196{\cdot}0000}$$

$$= \frac{+\ 462{\cdot}8997}{-\ 492{\cdot}7940} = -\ 0{\cdot}9393371,$$

and consequently either $\phi = 133^\circ.\ 12'.\ 30''$, or $\phi = -46^\circ.\ 47'.\ 30''$. The first of these two values for ϕ cannot be made use of here, because $B'A'M' + A'M'B'$ cannot be greater than 180°. We ∴ merely have

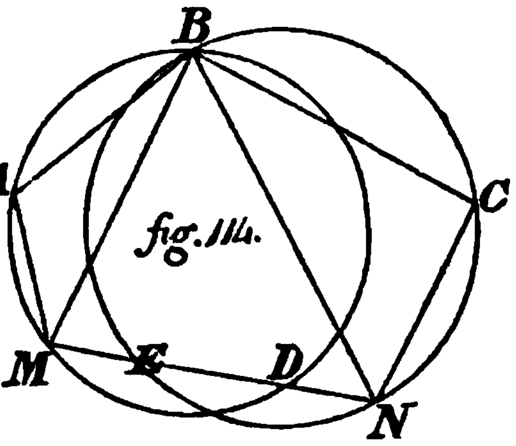
$$\phi = -\ 46^\circ.\ 47'.\ 30''.$$

Since ϕ is here negative, consequently $A'M'$, as the figure shows, must fall beyond the line $A'B'$.

When $B'A'M'$ is found, then all the remaining parts of the figure may likewise be calculated.

Cor. The two required points may also be found by the following very simple notation.

Let A, B, C (*fig.* 114), be the three given points. Upon AB as a chord, describe a circle $BDMA$, which includes the angle α, and upon BC another circle $CNEB$, which includes the angle γ; then take the arc $BD = 2\beta$, and the arc $CBE = 2\delta$, and through the points thus determined, viz., D, E, draw the line DE, which produced meets the two circles in M, N; then M, N are the two points sought.

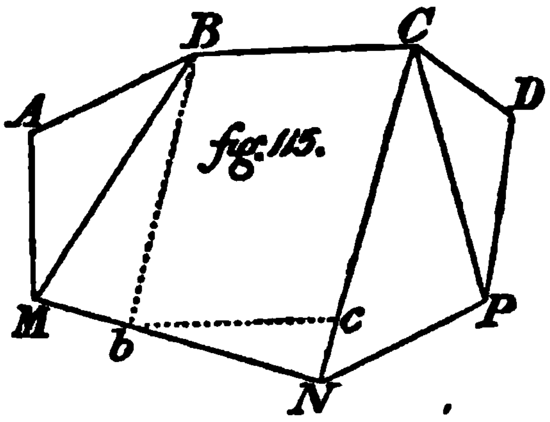


For since $BD = 2\beta$, and $CBE = 2\delta$: then $BMD = \beta$, $CNE = \delta$; also immediately from the construction itself, $AMB = \alpha$, $BNC = \gamma$.

SECTION CIII.

Prob. *Four inaccessible points are given in position and distance: determine the position of three other points in reference to these, without measuring any line.*

Solut. Let A, B, C, D (*fig.* 115) be the four given points, and M, N, P the three whose position it is required to be determined. If now from each of the points sought, three of the given ones are seen; then these may be determined singly by § 54. Since this case involves no difficulty, I shall .·. assume, that from M the three points A, B, N only are seen, and from P the three points D, C, N only; but from N only one of the given points, say C, and the two required ones M, P. On this supposition we can .·. measure the angles AMB, BMN, CNM, CNP, CPN, CPD. Further, since the four points A, B, C, D, are given in position and distance, we therefore have likewise the lines AB, BC, CD, and the angles ABC, BCD. Let .·. $AB = a$, $BC = b$, $CD = c$, $ABC = B$, $BCD = C$, $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$,



$CPN = \varepsilon$, $CPD = \zeta$. If besides we had found the angle MAB; then all the rest of the figures would be known. Put $\therefore$ $MAB = \phi$, and draw Bb parallel to CN, and bc parallel to BC.

1. Then $ABM = 180^\circ - (\alpha + \phi)$, $BbM = CNM = \gamma$, and $\therefore$ $MBb = 180^\circ - BMb - BbM = 180^\circ - (\beta + \gamma)$; consequently $Ccb = bBC = ABC - ABM - MBb = B + \alpha + \phi + \beta + \gamma - 360^\circ$, $Nbc = Ccb - CNM = B + \alpha + \phi + \gamma - 360^\circ$. Further, since in the heptagon $ABCDPNM$ the sum of all the angles $= 10\,R = 900^\circ$; therefore $CDP = 900^\circ - (AMN + MNP + NPD + BCD + ABC + MAB) = 900^\circ - (\alpha + \beta + \gamma + \delta + \varepsilon + \zeta + C + B + \phi)$. If we abbreviate by putting $B + \alpha + \beta - 360^\circ = \kappa$, $900^\circ - (\alpha + \beta + \gamma + \delta + \varepsilon + \zeta + B + C) = \lambda$; then $Nbc = \kappa + \phi$, $CDP = \lambda - \phi$.

2. In the triangle ABM,

$$BM = \frac{AB\ Sin.\ MAB}{Sin.\ AMB} = \frac{a\ Sin.\ \phi}{Sin.\ \alpha};$$

in the triangle MBb,

$$Bb = \frac{BM\ Sin.\ BMb}{Sin.\ BbM} = \frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma};$$

in the triangle Nbc,

$$Nc = \frac{bc\ Sin.\ Nbc}{Sin.\ bNc} = \frac{b\ Sin.\ (\kappa + \phi)}{Sin.\ \gamma};$$

in the triangle CDP,

$$CP = \frac{CD\ Sin.\ CDP}{Sin.\ CPD} = \frac{c\ Sin.\ (\lambda - \phi)}{Sin.\ \zeta},$$

and in the triangle CPN,

$$CN = \frac{CP\ Sin.\ CPN}{Sin.\ CNP} = \frac{c\ Sin.\ (\lambda - \phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta}.$$

3. Now since $CN = Cc + Nc = Bb + NC$: therefore we have

the equation

$$\frac{c \ Sin. \ (\lambda-\phi) \ Sin. \ \varepsilon}{Sin. \ \zeta \ Sin. \ \delta} = \frac{a \ Sin. \ \phi \ Sin. \ \beta}{Sin. \ \alpha \ Sin. \ \gamma} + \frac{b \ Sin. \ (\kappa+\phi)}{Sin. \ \gamma}.$$

Expand $Sin. \ (\lambda-\phi)$, $Sin. \ (\kappa+\phi)$, divide the equation by $Sin. \ \phi$, and put $Cot. \ \phi$ for $\frac{Cos. \ \phi}{Sin. \ \phi}$; this gives

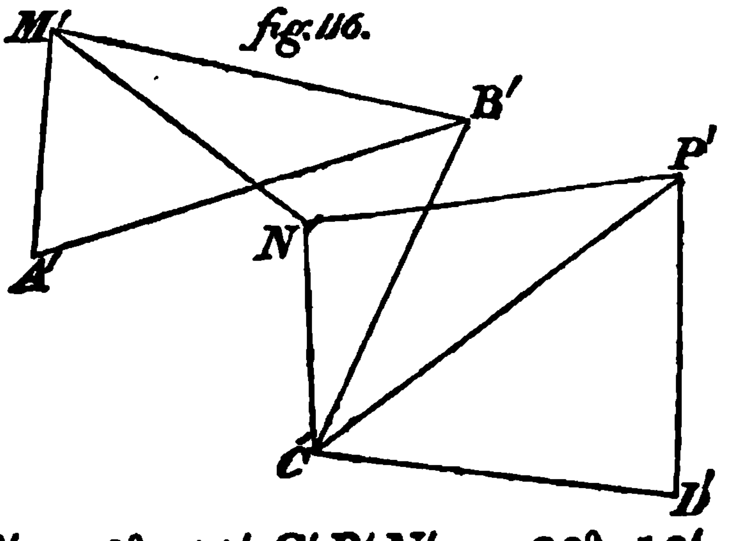
$$\frac{c \ Sin. \ \varepsilon \ (Sin. \ \lambda \ Cot. \ \phi - Cos. \ \lambda)}{Sin. \ \zeta \ Sin. \ \delta} = \frac{a \ Sin. \ \beta}{Sin. \ \alpha \ Sin. \ \gamma}$$
$$+ \frac{b \ (Sin. \ \kappa \ Cot. \ \phi + Cos. \ \kappa)}{Sin. \ \gamma},$$

and hence we obtain,

$$Cot. \ \phi =$$
$$\frac{\left[\begin{array}{c} a \ Sin. \ \beta \ Sin. \ \delta \ Sin. \ \zeta + b \ Sin. \ \alpha \ Sin. \ \delta \ Sin. \ \zeta \ Cos. \ \kappa \\ + c \ Sin. \ \alpha \ Sin. \ \gamma \ Sin. \ \varepsilon \ Cos. \ \lambda \end{array}\right]}{-b \ Sin. \ \alpha \ Sin. \ \delta \ Sin. \ \zeta \ Sin. \ \kappa + c \ Sin. \ \alpha \ Sin. \ \gamma \ Sin. \ \varepsilon \ Sin. \ \lambda},$$

whence the angle ϕ, and consequently all the others, may be found.

EXAM. A', B', C', D', (*fig.* 116) are four points, whose position is known; M', N', P' three others, whose position is sought. From M' only A', B', N', are visible, from N' only C', M', P', and from P' only C', D', N'. Let the angles measured from these points be as follow: $A'M'B' = 80°. 8'$, $B'M'N' = 24°. 55'$, $C'N'M = 124°. 16'$, $C'N'P' = 98°. 44'$, $C'P'N' = 29°. 13'$, $C'P'D' = 51°. 19'$. For the points A', B', C', D', we have $A'B' = 815$, $B'C' = 670$, $C'D' = 660$, $A'B'C' = 49°. 54'$, $B'C'D' = 73°. 57'$.



The comparison of *figs.* 115 and 116 shows immediately their relation, and then, when what has been said in § 101, is applied to every two corresponding angular points, we have

$\alpha=-80^\circ.\ 8'$, $\beta=24^\circ.\ 55'$, $\gamma=-124^\circ.\ 16'$, $\delta=-98^\circ.\ 44'$, $\varepsilon=-29^\circ.\ 13'$, $\zeta=-51^\circ.\ 19'$, $B=49^\circ.\ 54'$, $C=-73^\circ.\ 57'$; likewise $a=815$, $b=670$, $c=660$. From these data we obtain $\kappa=-365^\circ.\ 19'$, $\lambda=1282^\circ.\ 48'$; $\therefore$ $Sin.\ \kappa = Sin. -365^\circ.\ 19' = -Sin.\ 365^\circ.\ 19' = -Sin.\ 5^\circ.\ 19'$, $Cos.\ \kappa = Cos. -365^\circ.\ 19' = Cos.\ 365^\circ.\ 19' = Cos.\ 5^\circ.\ 19'$, $Sin.\ \lambda = Sin.\ 1282^\circ.\ 48' = Sin.\ 202^\circ.\ 48' = -Sin.\ 22^\circ.\ 48'$. $Cos.\ \lambda = Cos.\ 1282^\circ.\ 48' = Cos.\ 202^\circ.\ 48' = -Cos.\ 22^\circ.\ 48'$. Further, $Sin.\ \alpha = -Sin.\ 80^\circ.\ 8'$, $Sin.\ \beta = Sin.\ 24^\circ.\ 55'$, $Sin.\ \gamma = -Sin.\ 124^\circ.\ 16' = -Sin.\ 55^\circ.\ 44'$, $Sin.\ \delta = -Sin.\ 98^\circ.\ 44' = -Sin.\ 81^\circ.\ 16'$, $Sin.\ \varepsilon = -Sin.\ 29^\circ.\ 13'$, $Sin.\ \zeta = -Sin.\ 51^\circ.\ 19'$. We consequently have

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \delta\ Sin.\ \zeta &= 264{\cdot}9228\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Cos.\ \kappa &= -507{\cdot}1091\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Cos.\ \lambda &= 241{\cdot}8042\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \zeta\ Sin.\ \kappa &= 47{\cdot}1919\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon\ Sin.\ \lambda &= 101{\cdot}6451;
\end{aligned}$$

and $\therefore$

$$Cot.\ \phi = \frac{264{\cdot}9228 - 507{\cdot}1091 + 241{\cdot}8042}{-47{\cdot}1919 + 101{\cdot}6451} = \frac{-0{\cdot}3821}{54{\cdot}4532}$$
$$= -0{\cdot}0070170.$$

We consequently have, either, $\phi = 90^\circ.\ 24'.\ 7''$, or $= -89^\circ.\ 35'.\ 53''$. In the figure, for which this calculation has been made, ϕ has the second value, and $\therefore$ $B'A'M' = 89^\circ.\ 35'.\ 53''$.

In order to prove, the calculation, it is merely requisite to substitute the value here obtained for ϕ in the equation

$$\frac{c\ Sin.\ (\lambda-\phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta} = \frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma} + \frac{b\ Sin.\ (\kappa+\phi)}{Sin.\ \gamma}$$

which was found in Solution 3. Since this proof contains some important information, I shall give it here.

Since $\lambda=1282^\circ.\ 48'$, $\kappa=-365^\circ.\ 19'$, $\phi=-89^\circ.\ 35'.\ 53''$: therefore $\lambda-\phi=1372^\circ.\ 23'.\ 53''$, $\kappa+\phi=-454^\circ.\ 54'.\ 53''$; $\therefore$ $Sin.\ (\lambda-\phi)=Sin.\ 1372^\circ.\ 23'.\ 53'' = Sin.\ 292^\circ.\ 37'.\ 53'' = -Sin.\ 67^\circ.\ 36'.\ 7''$, $Sin.\ (\kappa+\phi)=Sin.-454^\circ.\ 54'.\ 53''= -Sin.\ 94^\circ.\ 54'.\ 53'' = -Sin.\ 85^\circ.\ 5'.\ 7''$. We $\therefore$ have,

$$\frac{c\ Sin.\ (\lambda - \phi)\ Sin.\ \varepsilon}{Sin.\ \zeta\ Sin.\ \delta} = \quad 386 \cdot 0368$$

$$\frac{a\ Sin.\ \phi\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ \gamma} = -\ 421 \cdot 7018$$

$$\frac{b\ Sin.\ (\kappa + \phi)}{Sin.\ \gamma} = \quad 807 \cdot 7391.$$

Thus $\therefore$ $386 \cdot 0368 = -\ 421 \cdot 7018 + 807 \cdot 7391$; and this is correct within $0 \cdot 0005$, which error arises from the incorrectness of the Tables.

When the angle ϕ has been found, it is easy to calculate all the lines of the figure. The best mode of proceeding will be, first to derive the general expressions for the required lines from *fig.* 115, and these last from *fig.* 116. Thus if we wish to calculate the line $M'N'$ (*fig.* 116), first of all we find the general expression for MN (*fig.* 115). By Solution 1, $Nbc = \kappa + \phi$, $\therefore$ $Ncb = 180^0 - bNc - Nbc = 180^0 - (\gamma + \kappa + \phi)$, and $MBb = 180^0 - (\beta + \gamma)$: we consequently have,

$$Mb = \frac{BM\ Sin.\ MBb}{Sin.\ MbB} = \frac{BM\ Sin.\ (\beta + \gamma)}{Sin.\ \gamma}$$

$$= \frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.^2\ \gamma},$$

$$Nb = \frac{bc\ Sin.\ Ncb}{Sin.\ bNc} = \frac{b\ Sin.\ (\gamma + \kappa + \phi)}{Sin.\ \gamma},$$

and $\therefore$

$$MN = Mb + Nb =$$

$$\frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.\ ^2\gamma} + \frac{b\ Sin.\ (\gamma + \kappa + \phi)}{Sin.\ \gamma}.$$

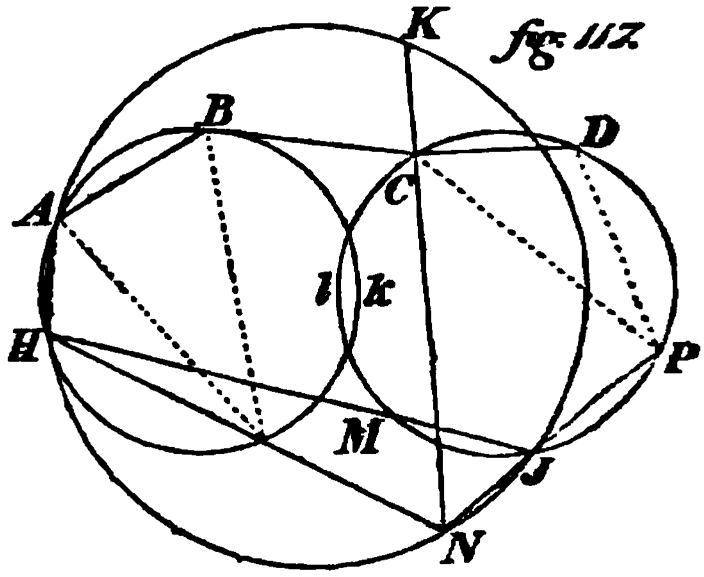
If we now apply the expression to *fig.* 116, we get $a = 815$, $b = 670$, $Sin.\ \phi = -\ Sin.\ 89^0.\ 25'.\ 53''$, $Sin.\ \gamma = -\ Sin.\ 124^0.\ 16' = -\ Sin.\ 55^0.\ 44'$, $Sin.\ (\beta + \gamma) = Sin.\ -\ 99^0.\ 21' = -\ Sin.\ 80^0.\ 39'$, $Sin.\ (\gamma + \kappa + \phi) = Sin. - 579^0.10.'53'' = - Sin. 219^0.10.'53'' = Sin. 39^0.10.'53''$. Hence we obtain,

$$\frac{a\ Sin.\ \phi\ Sin.\ (\beta + \gamma)}{Sin.\ \gamma^2} = 1177 \cdot 4171$$

$$\frac{b\ Sin.\ (\gamma + \kappa + \phi)}{Sin.\ \gamma} = -512\cdot1946,$$

and $\therefore$ $M'N'$ (*fig.* 116) $= 665\cdot2225$

Const. Upon AB, CD, (*fig.* 117) as chords, describe two circles, so that the arc BkA contain the angle α, and the arc CID the angle ζ; then assume the arc $BkH = 2\beta$, and the arc $CIJ = 2\epsilon$, draw HI, and upon this line as a chord describe a circle such, that the arc HNI may contain the angle $\gamma + \delta$: then make the arc $HK = 2\gamma$, or IK $= 2\delta$, and through K, C, draw the line KC, which produced cuts the circle in N. Now draw the lines HN, IN, which cut the other two circles in M, P: then M, N, P, are the three parts sought. Thus we have $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$, $CPN = \epsilon$, $CPD = \zeta$.

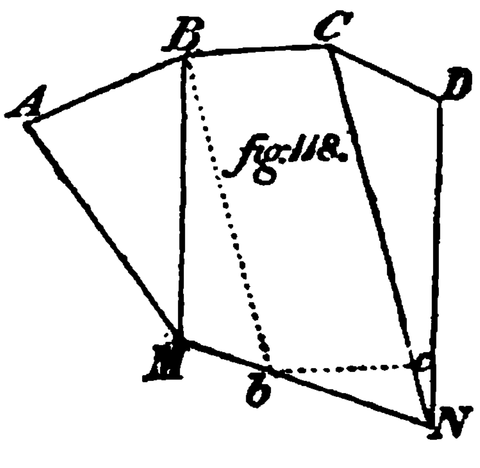


The proof of this construction is founded on the rule, that the angle at the circumference is equal to half the arc upon which it stands, and is easily found.

SECTION CIV.

Prob. *Four points are given in position and distance: required to find the positions and distances of two other points, from each of which, besides the other point sought, only two of those given are seen.*

Solut. Let A, B, C, D, (*fig.* 118), be the four given points, M, N, the two sought: from M only A, B, N, and from N only C, D, M, are seen. At M, N, measure the angles $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CND = \delta$. Further, there are given $AB = a$, $BC = b$, $CD = c$, $ABC = B$, $BCD = C$.



1. If the angle MAB is known; then we know all the rest. Put $\therefore$ $MAB = \phi$, and draw BC parallel to CN, and bc to BC. Now, if we compare *fig.* 118 with *fig.* 115, then we perceive at the first glance, that when we take the triangle CDN from the former, and the quadrilateral $CDPN$ from the latter, all the remainders, even the expressions for the angles and sides, agree. Consequently, as in the foregoing section, when for shortness-sake we put $B + \alpha + \beta - 360^0 = \kappa$, we obtain here also,

$$Bb = \frac{a \; Sin. \; \phi \; Sin. \; \beta}{Sin. \; \alpha \; Sin. \; \gamma}, \; Nc = \frac{b \; Sin. \; (\kappa + \phi)}{Sin. \; \gamma},$$

and $\therefore$

$$NC = \frac{a \; Sin. \; \phi \; Sin. \; \beta}{Sin. \; \alpha \; Sin. \; \gamma} + \frac{b \; Sin. \; (\kappa + \phi)}{Sin. \; \gamma}.$$

2. In the hexagon $ABCDNM$ (*fig.* 118) all the angles taken together $= 8\,R = 720^0$. We $\therefore$ have $CDN = 720^0 - (\alpha + \beta + \gamma + \delta + B + C + \phi)$; or, when we put $720^0 - (\alpha + \beta + \gamma + \delta + B + C) = \lambda$, $CDN = \lambda - \phi$. The triangle CDN consequently gives

$$CN = \frac{c \; Sin. \; (\lambda - \phi)}{Sin. \; \delta}.$$

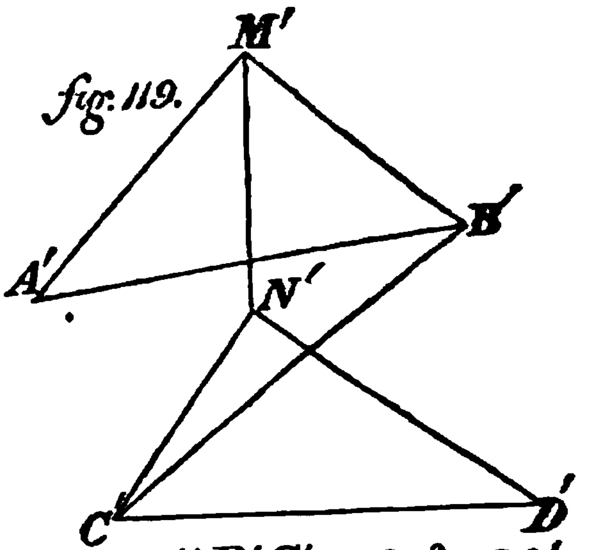
3. If we put the two expressions found for CN in 2, 3, equal to one another, we obtain the equation,

$$\frac{c \; Sin. \; (\lambda - \phi)}{Sin. \; \delta} = \frac{a \; Sin. \; \phi \; Sin. \; \beta}{Sin. \; \alpha \; Sin. \; \gamma} + \frac{b \; Sin. \; (\kappa + \phi)}{Sin. \; \gamma}$$

and this equation, when we treat it as we did a similar one in § CIII, gives

$$Cot. \; \phi = \frac{a \; Sin. \; \beta \; Sin. \; \delta + b \; Sin. \; \alpha \; Sin. \; \delta \; Cos. \; \kappa + c \; Sin. \; \alpha \; Sin. \; \gamma \; Cos. \; \lambda}{-b \; Sin. \; \alpha \; Sin. \; \delta \; Sin. \; \kappa + c \; Sin. \; \alpha \; Sin. \; \gamma \; Sin. \; \lambda}.$$

EXAM. Let A', B', C', D', (*fig.* 119), be the four given



points, and M', N', the two points whose position is sought, and of which it is assumed, that from M' only A', B', N', and from N' only C', D', M', are visible. The angles when measured are as follow: $A'M'B' = 86^\circ.\ 54'$, $B'M'N' = 46^\circ.\ 51'$, $C'N'M' = 138^\circ.\ 21'$, $C'N'D' = 87^\circ.\ 25'$. The following lines are given viz. $A'B' = 760$, $B'C' = 800$, $C'D' = 720$, and the angles $A'B'C' = 30^\circ.\ 20'$, $B'C'D' = 39^\circ.\ 34'$.

The comparison of *fig.* 119 with *fig.* 118, immediately shows their relation, and hence we get $a = 760$, $b = 800$, $c = 720$, $B = 30^\circ.\ 20'$, $C = -\ 39^\circ.\ 34'$, $\alpha = -\ 86^\circ.\ 54'$, $\beta = 46^\circ.\ 51'$, $\gamma = -\ 138^\circ.\ 21'$, $\delta = -\ 87^\circ.\ 25'$. For this example $\therefore\ \kappa = -\ 369^\circ.\ 43'$, $\lambda = 995^\circ.\ 3'$, and consequently $Sin.\ \kappa = Sin. -\ 369^\circ.\ 43' = -\ Sin.\ 369^\circ.\ 43' = -\ Sin.\ 9^\circ.\ 43'$, $Cos.\ \kappa = Cos. -\ 369^\circ.\ 43'. = Cos.\ 369^\circ.\ 43' = Cos.\ 9^\circ.\ 43'$, $Sin.\ \lambda = Sin.\ 995^\circ.\ 3' = Sin.\ 275^\circ.\ 3' = -\ Sin.\ 84^\circ.\ 57'$, $Cos.\ \lambda = Cos.\ 995^\circ.\ 3' = Cos.\ 275^\circ.\ 3' = Cos.\ 84^\circ.\ 57'$. The calculation, when continued, gives

$$\begin{aligned}
a\ Sin.\ \beta\ Sin.\ \delta &= -\ 553{\cdot}9063\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \kappa &= \quad 786{\cdot}5693\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda &\doteq \quad 42{\cdot}0580\\
b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa &= -\ 134{\cdot}6863\\
c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda &= -\ 475{\cdot}9416.
\end{aligned}$$

We $\therefore$ have

$$Cot.\ \phi = \frac{-553{\cdot}9063 + 786{\cdot}5693 + 42{\cdot}0580}{134{\cdot}6863 - 475{\cdot}9416} = \frac{+274{\cdot}7210}{-341{\cdot}2553}$$
$$= -\ 0{\cdot}8050307.$$

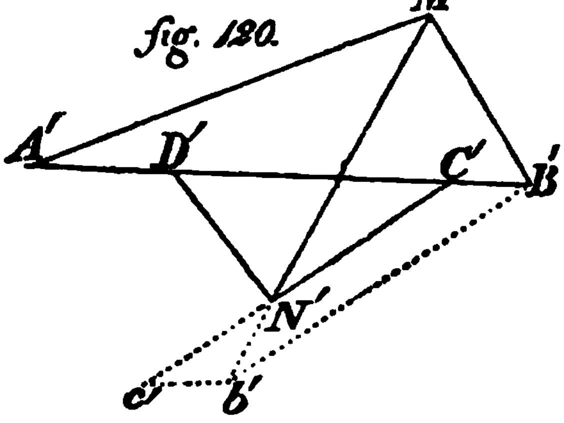
Consequently either $\phi = 128^\circ.\ 50'.\ 6''$, or $\phi = -51^\circ.\ 9'.\ 54''$. The first of these two values cannot obtain here, for otherwise $A'M'B' + M'A'B'$ would be greater than 180°. We consequently have $\phi = -\ 51^\circ.\ 9'.\ 54''$, and $\therefore\ M'A'B' = 51^\circ.\ 9'.\ 54''$.

When the angle $M'A'B'$ is found, it is easy to determine

the remaining angles of the figure which are not known. Thus $A'B'M' = 180^0 - (A'M'B + B'A'M') = 41^0.\ 56'.\ 6''$, $C'B'M' = A'B'M' + A'B'C' = 72^0.\ 16'.\ 6''$. Now, since in the quadrilateral $B'C'N'M'$ the convex angle $C'N'M' = 221^0.\ 39'$, $B'M'N = 46^0.\ 51'$, $C'B'M' = 72^0.\ 16'.\ 6''$; then $B'C'N' = 19^0.\ 13'.\ 54''$, $\therefore D'C'N' = B'C'N' + B'C'D' = 58^0.\ 47'.\ 54''$, and $C'D'N' = 180^0 - (D'C'N' + C'N'D') = 33^0.\ 47'.\ 6''$.

Exam. 2. Let A', B', C', D', (*fig.* 120) be four points in a given straight line, whose distance from each other are given, so that $AB = 840$, $BC = 110$, $CD = 500$. M' and N' are two points, whose position is sought; from M' only A', B', N', and from N' only C', D', M, are seen. According to this hypothesis, the angles, which the visible points make with each other, are measured at M', N', and we have found, that $A'M'B' = 105^0.\ 47'$, $B'M'N' = 65^0.\ 31'$, $C'N'M = 35^0.\ 54'$, $C'N'D' = 91^0.\ 35'$.

fig. 120.



By comparing *fig.* 120 with *fig.* 118, we get $a = 840$, $b = 110$, $c = 500$, $B = 0$, $C = 180^0$, $\alpha = -105^0.\ 47'$, $\beta = 65^0.\ 31'$, $\gamma = 35^0.\ 34'$, $\delta = -91^0.\ 35'$. We $\therefore$ have $\kappa = -400^0.\ 16'$, $\lambda = 636^0.\ 17'$, and consequently $Sin\ \kappa = -Sin.\ 400^0.\ 16' = -Sin.\ 40^0.\ 16'$, $Cos.\ \kappa = Cos.\ 400^0.\ 16'$, $= Cos.\ 40^0.\ 16'$, $Sin.\ \lambda = Sin.\ 276^0.\ 17' = -Sin.\ 83^0.\ 43'$, $Cos.\ \lambda = Cos.\ 276^0.\ 17' = Cos.\ 83^0.\ 43'$. Further

$$a\ Sin.\ \beta\ Sin.\ \delta = -764{\cdot}1770$$
$$b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \kappa = 80{\cdot}7395$$
$$c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda = -30{\cdot}6293$$
$$b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa = -68{\cdot}3913$$
$$c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda = -278{\cdot}1787$$

We $\therefore$ have,

$$Cot.\ \phi = \frac{-764{\cdot}1770 + 80{\cdot}7395 - 30{\cdot}6293}{68{\cdot}3913 + 278{\cdot}1787}$$
$$= -2{\cdot}0603826,$$

consequently, either $\phi = 154°.\ 6'.\ 38''$, or $\phi = -25°.\ 35'.\ 22''$. The first of these two values cannot be used here, because otherwise $B'A'M' + A'M'B' > 180°$. Therefore $B'A'M' = 25°.\ 53'.\ 22''$, whence all the rest of the figure can be determined.

Cor. We can always, if we think it advisable, from the formula already calculated for any figure, derive a particular formula for every similar figure. In order to elucidate this by an example, I shall suppose, that we wish, from *fig.* 120, to find a formula for the angle $B'A'M'$, which obtains for all similar figures. Assume, then, $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'D' = \delta'$, $B'A'M' = \phi'$; comparing it with *fig.* 118 we get $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $B = 0$, $C = 180°$. We $\therefore$ have $\kappa = -\alpha' + \beta' - 360°$, $\lambda = 720° - (-\alpha' + \beta' + \gamma' - \delta' + 180°) = 540° + \alpha' - \beta' - \gamma' + \delta'$. If these values be substituted in the equation in 3, we then obtain

$$-Cot.\ \phi' = \frac{-a\,Sin.\,\beta'\,Sin.\,\delta' + b\,Sin.\,\alpha'\,Sin.\,\delta'\,Cos.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Cos.\,\lambda}{-b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\lambda},$$

or

$$Cot.\ \phi' = \frac{-a\,Sin.\,\beta'\,Sin.\,\delta' + b\,Sin.\,\alpha'\,Sin.\,\delta'\,Cos.\,\kappa - c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Cos.\,\lambda}{b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\kappa + c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\lambda},$$

We can also find this formula immediately from the figure itself. With this view draw $B'b'$ parallel to $C'N'$, which meets $M'N'$ produced in b', and from b' draw $b'c'$ parallel to $B'C'$, which meets $C'N'$ produced in c'. Then $M'b'B = M'N'C' = \gamma'$; consequently $M'B'b' = 180° - (\beta' + \gamma')$. Further, $M'B'A' = 180° - (\alpha' + \phi')$, and $\therefore$ $A'B'b' = C'c'b' = D'C'N' = M'B'b' - M'B'A' = \alpha' + \phi' - \beta' - \gamma'$, and $N'b'c' = 180° - C'c'b' - b'N'c' = 180° - \alpha' + \beta' + \gamma' - \phi' - \gamma' = 180° - \alpha' + \beta' - \phi'$. Also $C'D'N' = 180° - D'C'N' - C'N'D' = 180° - \alpha' - \phi' + \beta' + \gamma' - \delta'$. If $\therefore$ we put $180° - \alpha' + \beta' = \kappa'$, $180° - \alpha' + \beta' + \gamma' - \delta' = \lambda'$: then $N'b'c = \kappa' - \phi'$, $C'D'N' = \lambda' - \phi'$.

Hence in the triangle $b'N'c'$, we obtain

$$N'c' = \frac{b'c' \; Sin.\; N'b'c'}{Sin.\; b'N'c'} = \frac{b \; Sin.\; (\kappa' - \phi')}{Sin.\; \gamma'},$$

in the triangle $A'M'B'$,

$$B'M' = \frac{A'B' \; Sin.\; B'A'M'}{Sin.\; M'B'A'} = \frac{a \; Sin.\; \phi'}{Sin.\; \alpha'},$$

in the triangle $M'B'b'$,

$$B'b' = \frac{M'B' \; Sin.\; B'M'b'}{Sin.\; M'b'B'} = \frac{a \; Sin.\; \phi' \; Sin.\; \beta'}{Sin.\; \alpha' \; Sin\; \gamma'},$$

in the triangle $C'N'D'$,

$$C'N' = \frac{C'D' \; Sin.\; C'D'N'}{Sin.\; C'N'D'} = \frac{c \; Sin.\; (\lambda' - \phi')}{Sin.\; \delta'}.$$

Now, since $B'b' = C'N' + N'c'$: we therefore have the equation,

$$\frac{a \; Sin.\; \phi' \; Sin.\; \beta'}{Sin.\; \alpha' \; Sin.\; \gamma'} = \frac{c \; Sin.\; (\lambda' - \phi')}{Sin.\; \delta'} + \frac{b \; Sin.\; (\kappa' - \phi')}{Sin.\; \gamma'.},$$

whence, by the usual method, we obtain

$$Cot.\; \phi' =$$

$$\frac{a Sin. \beta' Sin. \delta' + b Sin. \alpha' Sin. \delta' Cos. \kappa' + c Sin. \alpha' Sin. \gamma' Cos. \lambda'}{b \; Sin.\; \alpha' \; Sin.\; \delta' \; Sin.\; \kappa' + c \; Sin.\; \alpha' \; Sin.\; \gamma' \; Sin.\; \lambda'},$$

In order to perceive the agreement of this formula with the foregoing, it is merely necessary to express κ, λ by κ', λ'. Now, since $\kappa = \kappa' - 540$, $\lambda = 720^0 - \lambda'$; then $Sin.\; \kappa = Sin.\; (\kappa' - 540^0) = -Sin.\; (540^0 - \kappa') = -Sin.\; (180^0 - \kappa') = -Sin.\; \kappa'$, $Cos.\; \kappa = Cos.\; (\kappa' - 540^0) = Cos.\; (540^0 - \kappa') = Cos.\; (180^0 - \kappa') = -Cos.\; \kappa'$, $Sin.\; \lambda = Sin.\; (720^0 - \lambda') = Sin.\; -\lambda' = -Sin.\; \lambda'$, $Cos.\; (720^0 - \lambda') = Cos.\; -\lambda' = Cos.\; \lambda'$. If we substitute these values of $Sin.\; \kappa$, $Cos.\; \kappa$, $Sin.\; \lambda$, $Cos.\; \lambda$, in the first formula, we then obtain the second.

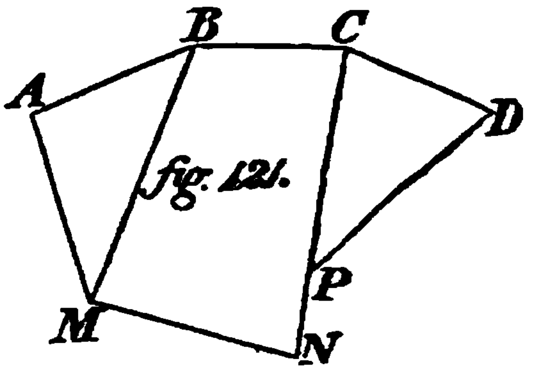
SECTION CV.

PROB. *Of the three problems solved in §§ CII, CIII, CIV, that in § CIII is the most general; it includes both the others as single cases. But there are besides some*

other remarkable cases, whose solution may be derived from it without much trouble. We shall now proceed to show how this is done.

First Case.

Let the point P (*fig.* 115) move into the line CN, as *fig.* 121 shows. For this case, in § 103, $\delta=0$, $\varepsilon=180^\circ$, and $\therefore$ $Sin.\ \delta=0$, $Sin.\ \varepsilon=0$. If we make the substitution in the formula there found for $Cot.\ \phi$, then $Cot.\ \phi = \frac{0}{0}$, which leaves its value undetermined. But we must necessarily obtain an expression of this kind, because the position of the point P in the line CN is not given.



Divide the numerator and denominator of the above expression by $Sin.\ \delta$; this gives

$$Cot.\ \phi = \frac{a\,Sin.\beta\,Sin.\zeta + b\,Sin.\alpha\,Sin.\zeta\,Cos.\kappa + c\,Sin.\alpha\,Sin.\gamma\,Cos.\lambda.\dfrac{Sin.\ \varepsilon}{Sin.\ \delta}}{-b\,Sin.\alpha\,Sin.\zeta\,Sin.\kappa + c\,Sin.\alpha\,Sin.\gamma\,Sin.\lambda.\dfrac{Sin.\ \varepsilon}{Sin.\ \delta}},$$

Now let (*fig.* 115), $CN=f$, $CP=g$. Then $\frac{Sin.\ \varepsilon}{Sin.\ \delta} = \frac{f}{g}$, and

$$Cot.\ \phi = \frac{a\,Sin.\beta\,Sin.\zeta + b\,Sin.\alpha\,Sin.\zeta\,Cos.\kappa + \dfrac{cf}{g}\,Sin.\alpha\,Sin.\gamma\,Cos.\lambda}{-\,b\,Sin.\alpha\,Sin.\zeta\,Sin.\kappa + \dfrac{cf}{g}\,Sin.\alpha\,Sin.\gamma\,Sin.\lambda};$$

in which

$$\kappa = B+\alpha+\beta-360^\circ,\ \lambda = 900^\circ-(\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+B+C).$$

This expression is always correct, wherever the point P may be situated; consequently also when, as in *fig.* 121, it is in CN. But for this case $\delta = 0$, $\varepsilon = 180^\circ$, consequently $\lambda = 720^\circ - (\alpha + \beta + \gamma + \zeta + B + C)$.

Now put (*fig.* 121) $f=g$: then P is situated in N, and the figure is transformed into *fig.* 118. We then have

$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \zeta + b\ Sin.\ \alpha\ Sin.\ \zeta\ Cos.\ \kappa + c\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda}{-b\ Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \kappa + c\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda},$$

which completely coincides with § CIV, merely by putting, suitably to the notation there used, δ for ζ.

Second Case.

Again, let the point D (*fig.* 118) move into B: then this figure is transformed into *fig.* 112, and ζ into $-\zeta$; further, $C=0$, and $c=b$. Then the expression just found is transformed into the following one:

$$Cot.\ \phi' = \frac{-a\ Sin.\ \beta'\ Sin.\ \zeta' - b\ Sin.\ \alpha\ Sin.\ \zeta\ Cos.\ \kappa' + b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \lambda}{b\ Sin.\ \alpha\ Sin.\ \zeta\ Sin.\ \kappa' + b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \lambda},$$

where $\kappa'=B+\alpha+\beta-360^\circ$, $\lambda'=720^\circ-(\alpha+\beta+\gamma+\zeta+B)$.

If this expression be compared with that found in § CII for the case, we must, in the first place, to suit the notation there used, put δ for γ, and γ for ζ. Then $\lambda'=720^\circ-(\alpha+\beta+\delta-\gamma+B)$. In the section above-mentioned, $\kappa=B+\alpha+\beta-360^\circ$, $\lambda=B+\alpha+\beta-\gamma+\delta-360^\circ$; we $\therefore$ have $\kappa'=\kappa$, $\lambda'=360^\circ-\lambda$; consequently $Sin.\ \kappa'=Sin.\ \kappa$, $Cos.\ \kappa'=Cos.\ \kappa$, $Sin.\ \lambda'=-Sin.\ \lambda$, $Cos.\ \lambda'=Cos.\ \lambda$. The substitution of these values in the above expression, after making the proper change with respect to γ and ζ, gives

$$Cot.\ \phi = \frac{a\ Sin.\ \beta\ Sin.\ \gamma + b\ Sin.\ \alpha\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Sin.\ \delta\ Cos.\ \lambda}{-b\ Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ \delta\ Sin.\ \kappa}.$$

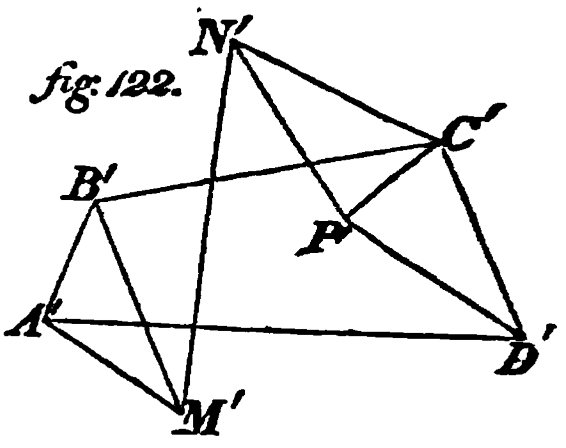
which perfectly agrees with § CII.

Third Case.

I shall now assume, that the points A, B, C, D, M, N, P, in *fig.* 115, have the positions A', B', C', D', M', N', P', (*fig.* 122), and that it is required to find for this and all

other similar figures a particular formula by which the position of M', N', P' may be determined. Let the measured angles be $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta'$, $C'P'N' = \varepsilon'$, $C'P'D' = \zeta'$, the given ones $A'B'C' = B'$, $B'C'D' = C'$. Let the required angle be $M'A'B' = \phi'$.

fig. 122.



The comparison of *figs.* 115 and 122, in reference to § CIII, gives $B = B'$, $C = C'$, $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = -\gamma'$, $\delta = \delta'$, $\varepsilon = \varepsilon'$, $\zeta = \zeta'$, $\phi = \phi'$; the substitution of these values in the formula there found for *Cot.* ϕ, consequently gives

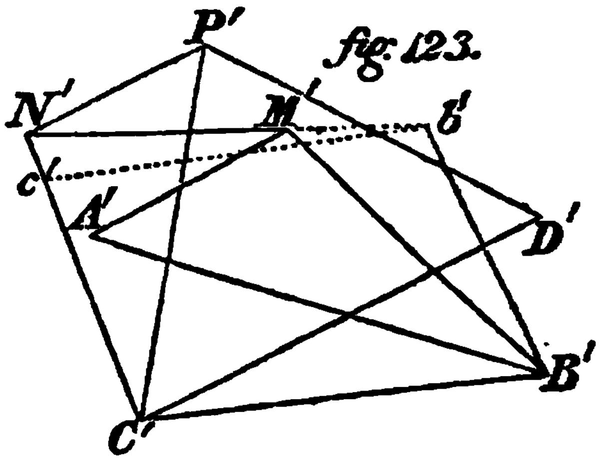
$$Cot\ \phi' = \frac{\left[\begin{array}{c} a\ Sin.\ \beta'\ Sin.\ \delta'\ Sin.\ \zeta' + b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \zeta'\ Cos.\ \kappa' \\ -\ c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \varepsilon'\ Cos.\ \lambda \end{array}\right]}{-b\ Sin.\alpha'\ Sin.\ \delta'\ Sin.\ \zeta'\ Sin.\ \kappa' - c\ Sin.\alpha'\ Sin.\gamma' Sin.\ \varepsilon'\ Sin.\ \lambda'},$$

in which $\kappa' = B' + \alpha' + \beta' - 360^\circ$, $\lambda' = 900^\circ - (\alpha' + \beta' - \gamma' + \delta' + \varepsilon' + \zeta' + B' + C')$.

Fourth Case.

Let A', B', C', D', (*fig.* 123) be the four given points, M', N', P', the three whose position is sought. We have the following measured angles: $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta'$, $C'P'N' = \varepsilon'$, $C'P'D' = \zeta'$. These angles are given, viz. $A'B'C' = B'$, $B'C'D' = C'$, and the sides $A'B' = a$, $B'C = b$, $C'D' = c$, the required angle $M'A'B' = \phi'$. Comparing this with the scheme in *fig.* 115, gives $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $\varepsilon = -\varepsilon'$, $\zeta = -\zeta'$, $B = B'$, $C = C'$, $\phi = -\phi'$. Substituting these values in the formula in § CIII, we get

fig. 123.



$$-\,Cot.\,\phi' = \frac{\left[\begin{array}{c} a\,Sin.\,\beta'\,Sin.\,\delta\,Sin.\,\zeta' - b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Cos.\,\kappa' \\ +\,c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Cos.\,\lambda' \end{array}\right]}{b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Sin.\,\kappa' + c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Sin.\,\lambda'},$$

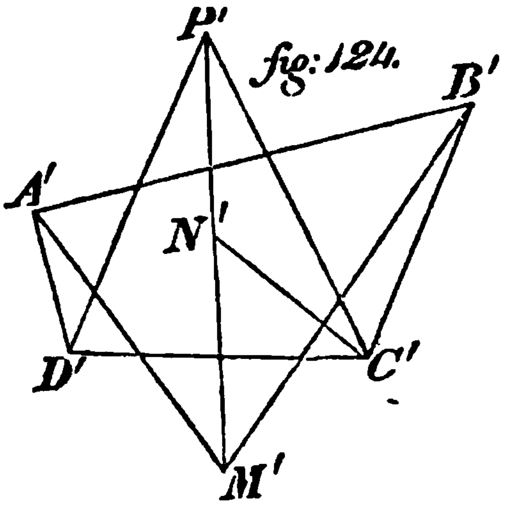
or

$$Cot.\,\phi' = \frac{\left[\begin{array}{c} -a\,Sin.\,\beta'\,Sin.\,\delta'\,Sin.\,\zeta' + b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Cos.\,\kappa' \\ -\,c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Cos.\,\lambda \end{array}\right]}{b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Sin.\,\kappa' + c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Sin.\,\lambda'},$$

in which $\kappa' = B' - \alpha' + \beta' - 360^0$, $\lambda' = 900^0 - (-\alpha' + \beta' + \gamma' - \delta' - \epsilon' - \zeta' + B' + C')$.

Fifth Case.

Let A', B', C', D', (*fig.* 124) be the four given points, and M', N', P', the three points sought; the latter are here in a direct line. Let $A'B'C' = B'$, $B'C'D' = C'$, $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'P' = \delta' = 180^0 - \gamma'$, $C'P'N' = \epsilon'$, $C'P'D' = \zeta'$; the required angle $M'A'B' = \phi'$. The comparison with *fig.* 115 gives $\alpha = \alpha'$, $\beta = -\beta'$, $\gamma = -\gamma'$, $\delta = -\delta'$, $\epsilon = -\epsilon'$, $\zeta = \zeta'$, $B = B'$, $C = C'$, $\phi = \phi'$. Consequently the substitution of these values in the formula, § CIII, gives



$$Cot.\,\phi' = \frac{\left[\begin{array}{c} a\,Sin.\,\beta'\,Sin.\,\delta\,Sin.\,\zeta' - b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Cos.\,\kappa' \\ +\,c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Cos.\,\lambda' \end{array}\right]}{b\,Sin.\,\alpha'\,Sin.\,\delta'\,Sin.\,\zeta'\,Sin.\,\kappa' + c\,Sin.\,\alpha'\,Sin.\,\gamma'\,Sin.\,\epsilon'\,Sin.\,\lambda'},$$

or, since $Sin.\,\delta' = Sin.\,\gamma'$,

$$Cot.\,\phi' = \frac{a\,Sin.\,\beta'\,Sin.\,\zeta' - b\,Sin.\,\alpha'\,Sin.\,\zeta'\,Cos.\,\kappa' + c\,Sin.\,\alpha'\,Sin.\,\epsilon'\,Cos.\,\lambda'}{b\,Sin.\,\alpha'\,Sin.\,\zeta'\,Sin.\,\kappa' + c\,Sin.\,\alpha'\,Sin.\,\epsilon'\,Sin.\,\lambda'},$$

in which

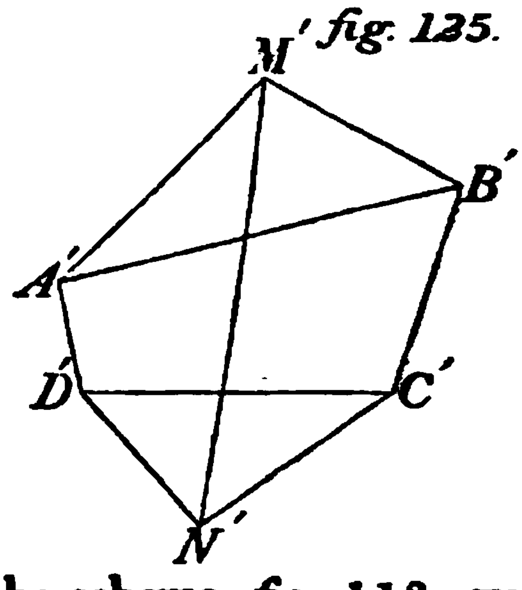
$\kappa' = B' + \alpha' - \beta' - 360^\circ$, $\lambda' = 900^\circ - (\alpha' - \beta' - \gamma' - \delta' - \varepsilon' + \zeta' + B' + C') = 1080^\circ - (\alpha' - \beta' - \varepsilon' + \zeta' + B' + C')$, because $\gamma' + \delta' = 180^\circ$. The angle $M'A'B'$, and consequently the position of the points M', F', depends ∴ for this case not on the angles γ', δ'; which, indeed, is the case.

If the points C', D', are in the line $A'B'$, then the formula here found for *Cot.* ϕ' remains the same; but the letters κ', λ', have different values, according to the order in which the points A', B', C', D', are relatively situated. Thus, if these four points be placed in the following order, viz. A', D', B', C', then $B' = 180^\circ$, $C' = 0$; consequently $\kappa' = \alpha' - \beta' - 180^\circ$, $\lambda = 900^\circ - (\alpha' - \beta' - \varepsilon' + \zeta')$. If they succeed one another in the order D', A', B', C': then $B' = 180^\circ$, $C' = 0$, and κ', λ', have the same values. But if they are in the order A', B', C', D': then $B' = 180$, $C' = 180^\circ$, and ∴ $\kappa' = \alpha' - \beta' - 180^\circ$, $\lambda' = 540^\circ - (\alpha' - \beta' - \varepsilon' + \zeta')$.

Sixth Case.

A', B, C', D', (*fig.* 125), is a given quadrilateral M', N', are two points, whose position is required to be determined. From M', only the side $A'B'$, and the point N' can be seen; from N' only the side $C'D'$, and the point M' are seen. The measured angles are $A'M'B' = \alpha'$, $B'M'N' = \beta'$, $C'N'M' = \gamma'$, $C'N'D' = \delta'$. We have given, $A'B' = a$, $B'C' = b'$, $C'D' = c$, $A'B'C' = B'$, $B'C'D' = C'$. Let the required angle $M'A'B' = \phi$. By comparing with the scheme, *fig.* 118, we get, in reference to § CIV, $B = B'$, $C = C'$, $\alpha = -\alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = -\delta'$, $\phi = -\phi'$. By the substitution of these values in the formula there found for *Cot.* ϕ, we obtain,

fig. 125.



$$-\,Cot.\ \phi' = \frac{-\,a\,Sin.\beta'\,Sin.\delta' + b\,Sin.\alpha'\,Sin.\delta'\,Cos.\kappa' - c\,Sin.\alpha'\,Sin.\gamma'\,Cos.\lambda'}{-\,b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \kappa' - c\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \lambda'}$$

or,

$$Cot.\ \phi' = \frac{-a\ Sin.\beta' Sin.\delta' + b Sin.\alpha' Sin.\delta' Cos.\kappa' - c\ Sin.\alpha' Sin.\gamma' Cos.\lambda'}{b\ Sin.\ \alpha'\ Sin.\ \delta'\ Sin.\ \kappa' + \varepsilon\ Sin.\ \alpha'\ Sin.\ \gamma'\ Sin.\ \lambda'}$$

in which

$\kappa' =$
$B' - \alpha' + \beta' - 360^{\circ}$, $\lambda' = 720^{\circ} - (-\alpha' + \beta' + \gamma' - \delta' + B' + C')$.

Again let the point D' move into A'. Then $a = 0$, the quadrilateral $A'B'C'D'$ is transformed into a triangle, and we obtain,

$$Cot.\ \phi' = \frac{b\ Sin.\ \delta'\ Cos.\ \kappa' - c\ Sin.\ \gamma'\ Cos.\ \lambda'}{b\ Sin.\ \delta'\ Sin.\ \kappa' - c\ Sin.\ \gamma'\ Sin.\ \lambda'}.$$

SECTION CVI.

We can likewise, if we choose, solve the formula for every similar figure immediately from this figure itself, as was shown in § CIV by an example. As this is a very useful practice for the young geometrician, I will here give another example, and with this view make use of the fourth case in the preceding section.

Draw, (*fig.* 123) $B'b'$ parallel to $C'N'$, which meets $M'N'$ produced in b', and from b' draw $b'c'$ parallel to $B'C'$. We then have, $B'b'M' = 180^{\circ} - C'N'M' = 180^{\circ} - \gamma'$, $M'B'b' = B'M'N' - B'b'M' = \beta' + \gamma' - 180^{\circ}$, $A'B'M' = 180^{\circ} - (A'M'B' + M'A'B') = 180^{\circ} - (\alpha' + \phi')$, $C'B'b' = A'B'C' + A'B'M' + M'B'b' = B' + \beta' + \gamma' - \alpha' - \phi' = C'c'b'$, $N'b'c' = C'c'b' - C'N'M' = B' + \beta' - \alpha' - \phi'$; further, $B'C'c = 180^{\circ} - C'c'b' = 180^{\circ} - B' - \beta' - \gamma' + \alpha' + \phi'$, $D'C'N' = B'C'c' - B'C'D' = 180^{\circ} - B' - C' - \beta' - \gamma' + \alpha' + \phi'$; $\therefore$ in the quadrilateral $D'C'N'P$, $C'D'P' = 360^{\circ} - D'C'N' - C'N'P' - N'P'D' = 360^{\circ} - D'C'N' - C'N'P' - C'P'N' - C'P'D' = 180^{\circ} + B' + C' + \beta' + \gamma' - \alpha' - \phi' - \delta' - \varepsilon' - \zeta'$. If $\therefore$, for shortness-sake, we put, $B' + \beta' - \alpha' = \kappa''$, $180^{\circ} + B' + C' + \beta' + \gamma' - \alpha'$

$-\delta' - \varepsilon' - \zeta' = \lambda''$: then $N'b'c' = \kappa'' - \phi'$, $C'D'P' = \lambda'' - \phi'$.

Consequently in the triangle $A'B'M'$, we have

$$B'M' = \frac{A'B' \text{ Sin. } M'A'B'}{\text{Sin. } A'M'B'} = \frac{a \text{ Sin. } \phi'}{\text{Sin. } \alpha'},$$

in the triangle $B'b'M'$,

$$B'b' = \frac{B'M' \text{ Sin. } B'M'b}{\text{Sin. } B'b'M'} = \frac{a \text{ Sin. } \phi' \text{ Sin. } \beta'}{\text{Sin. } \alpha' \text{ Sin. } \gamma'},$$

in the triangle $N'b'c'$,

$$N'c' = \frac{b'c' \text{ Sin. } N'b'c'}{\text{Sin. } c'N'b'} = \frac{b \text{ Sin. } (\kappa'' - \phi)}{\text{Sin. } \gamma'},$$

in the triangle $C'D'P'$,

$$C'P' = \frac{C'D' \text{ Sin. } C'D'P'}{\text{Sin. } C'P'D} = \frac{c \text{ Sin. } (\lambda'' - \phi')}{\text{Sin. } \zeta'},$$

in the triangle $C'P'N'$,

$$C'N' = \frac{C'P' \text{ Sin. } C'P'N}{\text{Sin. } C'N'P'} = \frac{c \text{ Sin. } (\lambda'' - \phi') \text{ Sin. } \varepsilon'}{\text{Sin. } \delta' \text{ Sin. } \zeta'},$$

Now $C'N' = C'c' + N'c' = B'b' + N'c'$: we $\therefore$ have the equation

$$\frac{c \text{ Sin. } (\lambda''-\phi') \text{ Sin. } \varepsilon'}{\text{Sin. } \delta' \text{ Sin. } \gamma'} = \frac{a \text{ Sin. } \phi' \text{ Sin. } \beta'}{\text{Sin. } \alpha' \text{ Sin. } \gamma'} + \frac{b \text{ Sin. } (\kappa'' - \phi')}{\text{Sin. } \gamma'}.$$

whence we obtain,

$$\text{Cot. } \phi' = \frac{\left[\begin{array}{c} a \text{ Sin. } \beta' \text{ Sin. } \delta' \text{ Sin. } \zeta' - b \text{ Sin. } \alpha' \text{ Sin. } \delta' \text{ Sin. } \zeta' \text{ Cos. } \kappa'' \\ + c \text{ Sin. } \alpha' \text{ Sin. } \gamma' \text{ Sin. } \varepsilon' \text{ Cos. } \lambda'' \end{array}\right]}{-b \text{ Sin. } \alpha' \text{ Sin. } \delta' \text{ Sin. } \zeta' \text{ Sin. } \kappa'' + c \text{ Sin. } \alpha' \text{ Sin. } \gamma' \text{ Sin. } \varepsilon' \text{ Sin. } \lambda''}.$$

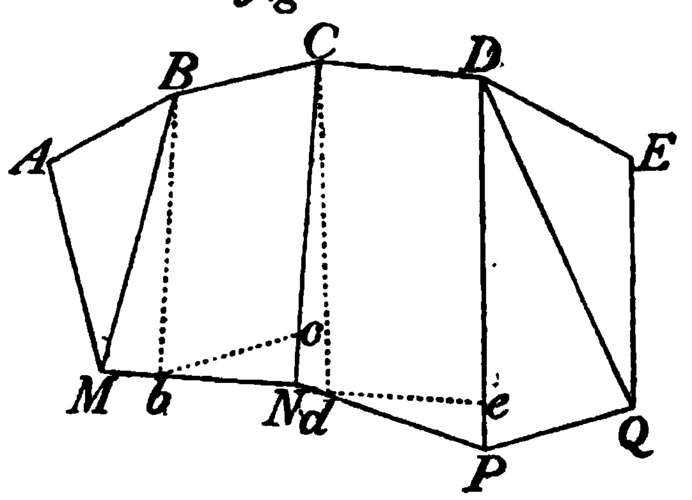
If we compare this formula with that for the fourth case of the preceding section, we find $\kappa'' = 360^\circ + \kappa'$, $\lambda'' = 1080^\circ - \lambda'$; consequently $\text{Sin. } \kappa'' = \text{Sin. } \kappa'$, $\text{Cos. } \kappa'' = \text{Cos. } \kappa'$, $\text{Sin. } \lambda'' = - \text{Sin. } \lambda'$, $\text{Cos. } \lambda'' = \text{Cos. } \lambda'$, whence the equality of both formulæ is evident.

SECTION CVII.

Prob. *Five inaccessible points are given in position and distance: required to determine the position of four other points.*

Solut. Let A, B, C, D, E, (*fig.* 126), be the five points, M, N, P, Q, the four required ones. From M, only A, B, N, can be seen, from N, only C, M, P, from P, only D, N, P, and from Q, only D, E, P. Let the given lines and angles be $AB = a$, $BC = b$, $CD = c$, $DE = d$, $ABC = B$, $BCD = C$, $CDE = D$; the measured angles $AMB = \alpha$, $BMN = \beta$, $CNM = \gamma$, $CNP = \delta$, $DPN = \varepsilon$, $DPQ = \zeta$, $DQP = \eta$, $DQE = \theta$. Further, let the required angle be $MAB = \phi$.

fig. 126.



1. Draw Bb parallel to CN, bc to BC, cd to DP, and de to CD. Then $BbM = CNM = \gamma$, $CdN = DPN = \varepsilon$; consequently $MBb = 180^\circ - (BMb + BbM) = 180^\circ - (\beta + \gamma)$, $NCd = 180^\circ - (\delta + \varepsilon)$. Further, in the quadrilateral $ABbM$, $ABb = 360^\circ - (AMb + BbM + MAB) = 360^\circ - (\alpha + \beta + \gamma + \phi)$, and in the pentagon $ABCNM$, $BCN = 540^\circ - (ABC + AMN + CNM + MAB) = 540^\circ - (B + \alpha + \beta + \gamma + \phi)$. We consequently have $bBC = ABC - ABb = B + \alpha + \beta + \gamma + \phi - 360^\circ = Ccb$, $dCD = BCD - BCN - NCd = B + C + \alpha + \beta + \gamma + \delta + \varepsilon + \phi - 720^\circ = Ded$, and ∴ $Nbc = Ccb - bNC = B + \alpha + \beta + \phi - 360^\circ$; $Pde = Ded - dPD = B + C + \alpha + \beta + \gamma + \delta + \phi - 720^\circ$. Further, in the nonagon $ABCDEQPNM$, the angle $DEQ = 1260^\circ - (ABC + BCD + CDE + AMN + MNP + NPQ + MAB) = 1260^\circ - (B + C + D + \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \eta +$

$\theta + \phi$). Now, if for the sake of brevity, we put, $B + \alpha + \beta - 360^\circ = \kappa$, $B + C + \alpha + \beta + \gamma + \delta - 720^\circ = \lambda$, $1260^\circ - (B + C + D + \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \eta + \theta) = \mu$: then $Nbc = \kappa + \phi$, $Pde = \lambda + \phi$, $DEQ = \mu - \phi$.

2. In the triangle ABM,

$$BM = \frac{AB \text{ Sin. } MAB}{\text{Sin. } AMB} = \frac{a \text{ Sin. } \phi}{\text{Sin. } \alpha};$$

in the triangle MBb,

$$Bb = \frac{BM \text{ Sin. } BMb}{\text{Sin. } BbM} = \frac{a \text{ Sin. } \phi \text{ Sin. } \beta}{\text{Sin. } \alpha \text{ Sin. } \gamma};$$

in the triangle Nbc,

$$Nc = \frac{bc \text{ Sin. } Nbc}{\text{Sin. } bNc} = \frac{b \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \gamma};$$

consequently $Cn = Cc + Nc = Bb + Nc =$

$$\frac{a \text{ Sin. } \phi \text{ Sin. } \beta}{\text{Sin. } \alpha \text{ Sin. } \gamma} + \frac{b \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \gamma}.$$

Further, in the triangle NCd,

$$Cd = \frac{CN \text{ Sin. } CNd}{\text{Sin. } CdN} = \frac{CN \text{ Sin. } \delta}{\text{Sin. } \varepsilon},$$

and in the triangle Pde,

$$Pe = \frac{de \text{ Sin. } Pde}{\text{Sin. } dPe} = \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon};$$

consequently $DP = De + Pe = Cd + Pe =$

$$\frac{CN \text{ Sin. } \delta}{\text{Sin. } \varepsilon} + \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon} =$$

$$\frac{a \text{ Sin. } \phi \text{ Sin. } \beta \text{ Sin. } \delta}{\text{Sin. } \alpha \text{ Sin. } \gamma \text{ Sin. } \varepsilon} + \frac{b \text{ Sin. } (\kappa + \phi) \text{ Sin. } \delta}{\text{Sin. } \gamma \text{ Sin. } \varepsilon} + \frac{c \text{ Sin. } (\lambda + \phi)}{\text{Sin. } \varepsilon}.$$

3. But an expression may be found for DP. Thus in the triangle DEQ,

$$DQ = \frac{DE \text{ Sin. } DEQ}{\text{Sin. } DQE} = \frac{d \text{ Sin. } (\mu - \phi)}{\text{Sin. } \theta},$$

and in the triangle DPQ,

$$DP = \frac{DQ\ Sin.\ DQP}{Sin.\ DPQ} = \frac{d\ Sin.\ (\mu - \phi)\ Sin.\ \eta}{Sin.\ \theta\ Sin.\ \zeta}.$$

4. If we equate the expressions found for DP in 2 and 3, we then obtain the equation,

$$\frac{a\ Sin.\ \phi\ Sin.\ \beta\ Sin.\ \delta}{Sin.\ \alpha\ Sin.\ \gamma\ Sin.\ \varepsilon} + \frac{b\ Sin.\ (\kappa + \phi)\ Sin.\ \delta}{Sin.\ \gamma\ Sin.\ \varepsilon},$$

$$+ \frac{c\ Sin.\ (\lambda + \phi)}{Sin.\ \varepsilon} = \frac{d\ Sin.\ (\mu - \phi)\ Sin.\ \eta}{Sin.\ \theta\ Sin.\ \zeta},$$

or, when we solve $Sin.\ (\kappa + \phi)$, $Sin.\ (\lambda + \phi)$, $Sin.\ (\mu - \phi)$ divide by, $Sin.\ \phi$, and substitute $Cot.\ \phi$ for $\frac{Cos.\ \phi}{Sin.\ \phi}$,

$$\frac{a\ Sin.\ \beta\ Sin.\ \delta}{Sin.\ a\ Sin.\ \gamma\ Sin.\ \varepsilon} + \frac{b\ Sin.\ \delta\ (Sin.\ \kappa\ Cot.\ \phi + Cos.\ \kappa)}{Sin.\ \gamma\ Sin.\ \varepsilon} +$$

$$\frac{c\ (Sin.\ \lambda\ Cot.\ \phi + Cos.\ \lambda)}{Sin.\ \varepsilon} = \frac{d\ Sin.\ \eta\ (Sin.\ \mu\ Cot.\ \phi - Cos.\ \mu)}{Sin.\ \theta\ Sin.\ \zeta}.$$

Hence we obtain,

$$Cot.\ \phi = \frac{\left[\begin{array}{l} a\ Sin.\beta\ Sin.\delta\ Sin.\zeta\ Sin.\theta + b\ Sin\ \alpha Sin.\delta\ Sin.\zeta\ Sin.\theta\ Cos.\kappa + \\ c Sin.\alpha Sin.\gamma Sin.\zeta Sin.\theta Cos.\lambda + d Sin.\alpha Sin.\gamma Sin.\varepsilon Sin.\eta Cos.\mu \end{array}\right]}{\left[\begin{array}{c} -b Sin.\alpha Sin.\delta Sin.\zeta Sin.\theta Sin.\kappa - c Sin.\alpha Sin\ \gamma Sin\ \zeta Sin.\theta Sin.\lambda \\ + d\ Sin.\ \alpha\ Sin.\gamma\ Sin.\ \varepsilon\ Sin.\ \eta\ Sin.\ \mu \end{array}\right]}$$

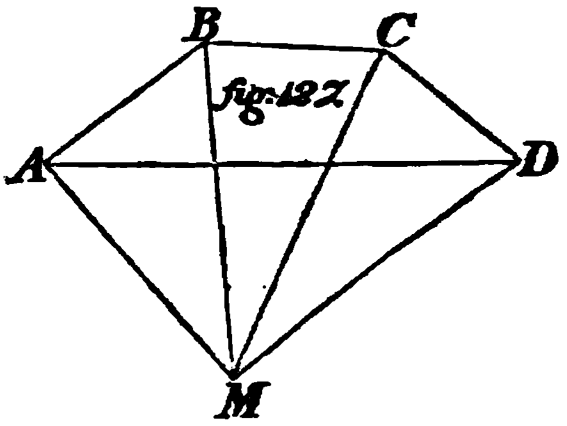
REMARK. In a similar manner we can generally solve the problem when n points are given, to determine the position of $n - 1$ other points.

SECTION CVIII.

PROB. *In a quadrilateral, two of its opposite sides, together with the angles which they make with one of the other two sides, are given. This quadrilateral is seen from any one position, and from hence the apparent distances of the angular points from one another*

are measured; find from these data the other two sides, and the distance of the position from each angular point.

SOLUT. Let $ABCD$ (*fig.* 127) be the quadrilateral, M the position, the given sides $AB = a$, $CD = b$, and angles $ABC = B$, $BCD = C$. The measured angles $ABM = \alpha$, $BMC = \beta$, $CMD = \gamma$.



1. If the angle MAB is known, we then likewise have all the rest of the figure. Assume $\therefore MAB = \phi$. Then $ABM = 180^0 - (\alpha + \phi)$, $MBC = ABC - ABM = B + \alpha + \phi - 180^0$, $BCM = 180^0 - (MBC + BMC) = 360^0 - (B + \alpha + \beta + \gamma + \phi)$. Further, in the pentagon $MABCD$, $MDC = 540^0 - (B + C + \alpha + \beta + \gamma + \phi)$. For the sake of brevity, put $B + \alpha - 180^0 = \kappa$, $360^0 - (B + \alpha + \beta) = \lambda$, $540^0 - (B + C + \alpha + \beta + \gamma) = \mu$; this gives, $MBC = \kappa + \phi$, $BCM = \lambda - \phi$, $MDC = \mu - \phi$,

2. In the triangle ABM,

$$BM = \frac{AB \text{ Sin. } MAB}{\text{Sin. } AMB} = \frac{a \text{ Sin. } \phi}{\text{Sin. } \alpha},$$

in the triangle BMC,

$$MC = \frac{BM \text{ Sin. } MBC}{\text{Sin. } BCM} = \frac{a \text{ Sin. } \phi \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \alpha \text{ Sin. } (\lambda - \phi)}.$$

Further, in the triangle MCD,

$$MC = \frac{CD \text{ Sin. } MDC}{\text{Sin. } CMD} = \frac{b \text{ Sin. } (\mu - \phi)}{\text{Sin. } \gamma}.$$

3. If the two values found for MC are equated, we then obtain,

$$\frac{a \text{ Sin. } \phi \text{ Sin. } (\kappa + \phi)}{\text{Sin. } \alpha \text{ Sin. } (\lambda - \phi)} = \frac{b \text{ Sin. } (\mu - \phi)}{\text{Sin. } \gamma},$$

or,

$$a \text{ Sin. } \gamma \text{ Sin. } (\kappa + \phi) = b \text{ Sin. } \alpha \text{ Sin. } (\lambda - \phi) \text{ Sin. } (\mu - \phi).$$

4. Now $Sin.\ \phi\ Sin.\ (\kappa+\phi) = \frac{1}{2}[Cos.\ \kappa - Cos.\ (\kappa+2\phi)]$, $Sin.\ (\lambda-\phi)\ Sin.\ (\mu-\phi) = \frac{1}{2}[Cos.\ (\lambda-\mu) - Cos.\ (\lambda+\mu-2\phi)]$; we consequently have

$$a\ Sin.\ \alpha [Cos.\ (\lambda+\mu-2\phi)] = b\ Sin.\ \alpha [Cos.\ (\lambda-\mu) - Cos.\ (\lambda+\mu-2\phi)].$$

or,

$$a\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Cos.\ (\lambda-\mu) = a\ Sin.\ \gamma\ Cos.\ (\kappa+2\phi) - b\ Sin.\ \alpha\ Cos.\ (\lambda+\mu-2\phi).$$

Now since $Cos.\ (\kappa+2\phi) = Cos.\ \kappa\ Cos.\ 2\phi - Sin.\ \kappa\ Sin.\ 2\phi$, $Cos.\ (\lambda+\mu-2\phi) = Cos.\ (\lambda+\mu)\ Cos.\ 2\phi + Sin.\ (\lambda+\mu)\ Sin.\ 2\phi$: therefore

$$\begin{gathered} a\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Cos.\ (\lambda-\mu) = \\ [a\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Cos.\ (\lambda+\mu)]\ Cos.\ 2\phi \\ -[a\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ (\lambda+\mu)]\ Sin.\ 2\phi. \end{gathered}$$

5. Divide this equation by $a\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ (\lambda+\mu)$, and put

$$\frac{a\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Cos.\ (\lambda+\mu)}{a\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ (\lambda+\mu)} = Tan.\ \omega;$$

this gives

$$Tan.\ \omega\ Cos.\ 2\phi - Sin.\ 2\phi = \frac{a\ Sin.\ \gamma\ Cos.\ \kappa - b\ Sin.\ \alpha\ Cos.\ (\lambda-\mu)}{a\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ (\lambda+\mu)},$$

or, since $Tan.\ \omega\ Cos.\ 2\phi - Sin.\ 2\phi =$

$$\frac{Sin.\ \omega\ Cos.\ 2\phi - Cos.\ \omega\ Sin.\ 2\phi}{Cos.\ \omega} = \frac{Sin.\ (\omega-2\phi)}{Cos.\ \omega},$$

$$Sin.\ (\omega-2\phi) = \frac{a\ Sin.\ \gamma\ Cos.\ \kappa\ Cos.\ \omega - b\ Sin.\ \alpha\ Cos.\ (\lambda-\mu)\ Cos.\ \omega}{a\ Sin.\ \gamma\ Sin.\ \kappa + b\ Sin.\ \alpha\ Sin.\ (\lambda+\mu)};$$

whence ϕ may be determined.

EXAM. Let $a=360$, $b=300$, $B=140^{\circ}.\ 26'$, $C=136^{\circ}.\ 22'$, $\alpha=36^{\circ}.\ 20'$, $\beta=29^{\circ}.\ 41'$, $\gamma=27^{\circ}.\ 38'$. Here $\kappa=-3^{\circ}.\ 14'$, $\lambda=153^{\circ}.\ 33'$, $\mu=169^{\circ}.\ 33'$.

$$a\ Sin.\ \gamma\ Cos.\ \kappa \qquad = \quad 166{\cdot}7063$$
$$b\ Sin.\ \alpha\ Cos.\ (\lambda + \mu) = \quad 142{\cdot}1396$$
$$a\ Sin.\ \gamma\ Sin.\ \kappa \qquad = - \quad 9{\cdot}4176$$
$$b\ Sin.\ \alpha\ Sin.\ (\lambda + \mu) = - 106{\cdot}7214\,;$$

consequently,

$$Tan.\ \omega = \frac{166{\cdot}7063 - 142{\cdot}1396}{-9{\cdot}4176 - 106{\cdot}7214} = -0{\cdot}2115284$$

and $\therefore$

$$\omega = 168^\circ.\ 3'.\ 23''.$$

We consequently have,

$$a\ Sin.\ \gamma\ Cos.\ \kappa\ Cos.\ \omega \qquad = - 163{\cdot}0974$$
$$b\ Sin.\ \alpha\ Cos.\ (\lambda - \mu)\ Cos.\ \omega = - 167{\cdot}1602,$$

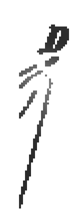
and $\therefore$

$$Sin.\ (\omega - 2\phi) = \frac{-163{\cdot}0974 + 167{\cdot}1602}{-9{\cdot}4176 - 106{\cdot}7214} = -0{\cdot}0349822$$
$$\omega - 2\phi = -2^\circ.\ 0'.\ 10''.$$
$$\phi = 85^\circ.\ 1'.\ 46''.$$

SECTION CIX.

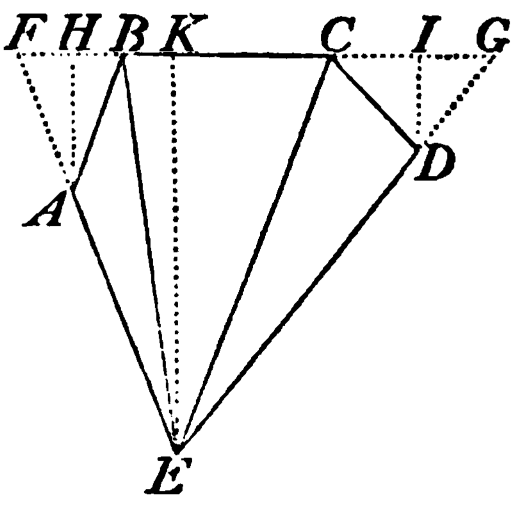
Prob. *Four objects are seen from four stations, and at each of these stations the apparent distances of the objects from one another are measured: from hence determine the position of all the eight points.*

Solut. the four objects A, B, C, D (*fig.* 128) are seen from the four stations E, F, G, H, and there the angles $AEB = \alpha$, $BEC = \beta$, $CED = \gamma$, $AFB = \alpha'$, $BFC = \beta'$, $CFD = \gamma'$, $AGB = \alpha''$, $BGC = \beta''$, $CGD = \gamma''$, $AHB = \alpha'''$, $BHC = \beta'''$, $CHD = \gamma'''$, are measured.



1. Join the points A, B, C, D, (*fig.* 129) by the lines AB, BC, CD; find first an equation for the point E. With this view, produce the lines BC, EA, ED, till they meet in F, G, and draw the perpendiculars AH, DI, EK. Since angles only are here given, consequently we cannot determine the actual magnitude of the lines belonging to the figure, but merely their proportion to one another. We can ∴ assume one of these lines, say BC, for unity, and put $BC = 1$. Now if besides the lines AB, CD, and the angles ABC, BCD, or the adjacent angles ABF, DCG, are known; then the position of the four points A, B, C, D, and consequently also the situation of the points E, F, G, H, are determined. Assume ∴ $AB = x$, $CD = y$, $ABF = \phi$, $DCG = \psi$, and moreover, for shortness' sake, $BAE = A$, $CDE = D$, which last two angles vanish further on in the calculation. We then have $AFB = BAE - ABF = A - \phi$, $CGD = CDE - DCG = D - \psi$, $CBE = AFB + AEB = A - \phi + \alpha$, $BCE = CGD + CED = D - \psi + \gamma$.

fig. 129.



2. In the triangle ABE,

$$BE = \frac{AB \text{ Sin. } BAE}{\text{Sin. } AEB} = \frac{x \text{ Sin. } A}{\text{Sin. } \alpha},$$

and in the triangle CED,

$$CE = \frac{CD \text{ Sin. } CDE}{\text{Sin. } CED} = \frac{y \text{ Sin. } D}{\text{Sin. } \gamma};$$

consequently in the right-angled triangle BEK,

$$BK = BE \text{ Cos. } CBE = \frac{y \text{ Sin. } A \text{ Cos. } (A - \phi + \alpha)}{\text{Sin. } \gamma}.$$

and in the right-angled triangle CEK,

$$CK = CE \text{ Cos. } BCE = \frac{y \text{ Sin. } D \text{ Cos. } (D - \psi + \gamma)}{\text{Sin. } \gamma}.$$

Now since $BK+CK=BC=1$: we then have the equation

$$\frac{x\ Sin.\ A\ Cos.\ (A-\phi+\alpha)}{Sin.\ \alpha}+\frac{y\ Sin.\ D\ Cos.\ (D-\psi+\gamma)}{Sin.\ \gamma}=1,$$

or,

$$x\ Sin.\ \gamma\ Sin.\ A\ Cos.\ (A-\phi+\alpha)+y\ Sin.\ \alpha\ Sin.\ D\times Cos.\ (D-\psi+\gamma)=Sin.\ \alpha\ Sin.\ \gamma.$$

3. But $Cos.\ (A-\phi+\alpha)=Cos.\ [A-(\phi-\alpha)]=Cos.\ A\ Cos.\ (\phi-\alpha)+Sin.\ A\ Sin.\ (\phi-\alpha)$, $Cos.\ (D-\psi+\gamma)=Cos.\ [D-(\psi-\gamma)]=Cos.\ D\ Cos.\ (\psi-\gamma)+Sin.\ D\times Sin.\ (\psi-\gamma)$. Substitute these values in the obtained equation, and at the same time put $1-Cos.\ ^2A$ for $Sin.\ ^2A$, $1-Cos.\ ^2D$ for $Sin.\ ^2D$; this gives, $Sin.\ \alpha\ Sin.\ \gamma=$

$$x\ Sin.\ \gamma\ Cos.\ A\ [Sin.\ A\ Cos.\ (\phi-\alpha)-Cos.\ A\ Sin.\ (\phi-\alpha)]+x\ Sin.\ \gamma\ Sin.\ (\phi-\alpha)+y\ Sin.\ \alpha\ Cos.\ D\ [Sin.\ D\ Cos.\ (\psi-\gamma)-Cos.\ D\ Sin.\ (\psi-\gamma)]+y\ Sin.\ \alpha\ Sin.\ (\psi-\gamma)$$

$$=x\ Sin.\ \gamma\ Cos.\ A\ Sin.\ (A-\phi+\alpha)+x\ Sin.\ \gamma\ Sin.\ (\phi-\alpha)+y\ Sin.\ \alpha\ Cos.\ D\ Sin.\ (D-\psi+\gamma)+y\ Sin.\ \alpha\ Sin.\ (\psi-\gamma)$$

$$=x\ Sin.\ \gamma\ Cos.\ A\ Sin.\ CBE+x\ Sin.\ \gamma\ Sin.\ (\phi-\alpha)+y\ Sin.\ \alpha\ Cos.\ D\ Sin.\ BCE+y\ Sin.\ \alpha\ Sin.\ (\psi-\gamma).$$

4. In the triangle BEC we have

$$BC=1=\frac{BE\ Sin.\ \beta}{Sin.\ BCE}=\frac{x\ Sin.\ A\ Sin.\ \beta}{Sin.\ \alpha\ Sin.\ BCE},$$

and $\therefore$

$$Sin.\ BCE=\frac{x\ Sin.\ A\ Sin.\ \beta}{Sin.\ \alpha}.$$

In like manner

$$BC=1=\frac{CE\ Sin.\ \beta}{Sin.\ CBE}=\frac{y\ Sin.\ D\ Sin.\ \beta}{Sin.\ \gamma\ Sin.\ CBE},$$

and $\therefore$

$$Sin.\ CBE=\frac{y\ Sin.\ D\ Sin.\ \beta}{Sin.\ \gamma}.$$

Substitute these values of *Sin. BCE, Sin. CBE* in the equation in 3; this gives $Sin.\ \alpha\ Sin.\ \gamma =$

$$xy\ Sin.\ \beta\ Sin.\ D\ Cos.\ A + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + xy\ Sin.\ \beta\ Sin.\ A\ Cos.\ D + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma)$$

$$= xy\ Sin.\ \beta\ Sin.\ (A + D) + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma)$$

or, since $A + D = 540^0 - (AED + ABC + BCD) = 180^0 - (E - \phi - \psi)$, when, for the sake of brevity, we put $AED = E$,

$$Sin.\ \alpha\ Sin.\ \gamma = xy\ Sin.\ \beta\ Sin.\ [E - (\phi + \psi)] + x\ Sin.\ \gamma\ Sin.\ (\phi - \alpha) + y\ Sin.\ \alpha\ Sin.\ (\psi - \gamma).$$

5. In order to separate the unknown magnitudes from the known, solve this equation; this gives

$$Sin.\ \alpha\ Sin.\ \gamma = xy\ Sin.\ \beta\ [Sin.\ E\ Cos.\ (\phi + \psi) - Cos.\ E \times Sin.\ (\phi + \psi)] + x\ Sin.\ \gamma\ (Sin.\ \phi\ Cos.\ \alpha - Cos.\ \phi\ Sin.\ \alpha) + y\ Sin.\ \alpha\ (Sin.\ \psi\ Cos.\ \gamma - Cos.\ \psi\ Sin.\ \gamma)$$

$$= xy\ Sin.\ \beta\ [Sin.\ E\ (Cos.\ \phi\ Cos.\ \psi - Sin.\ \phi\ Sin.\ \psi) - Cos.\ E\ (Sin.\ \phi\ Cos.\ \psi + Cos.\ \phi\ Sin.\ \psi)] + x\ Sin.\ \gamma\ (Sin.\ \phi\ Cos.\ \alpha - Cos.\ \phi\ Sin.\ \alpha) + y\ Sin.\ \alpha\ (Sin.\ \psi\ Cos.\ \gamma - Cos.\ \psi\ Sin.\ \gamma).$$

6. If we put $x\ Sin.\ \phi = p$, $x\ Cos.\ \phi = q$, $y\ Sin.\ \psi = r$, $y\ Cos.\ \psi = s$; then this equation is transformed into the following one:

$$(qs - pr) Sin.\ \beta\ Sin.\ E - (ps + qr)\ Sin.\ \beta\ Cos.\ E + p\ Sin.\ \gamma\ Cos.\ \alpha + r\ Sin.\ \alpha\ Cos.\ \gamma - (q + s + 1)\ Sin.\ \alpha\ Sin.\ \gamma = 0,$$

or

$$(qs - pr) \frac{Sin.\ \beta\ Sin.\ E}{Sin.\ \alpha\ Sin.\ \gamma} - (ps + qr) \frac{Sin.\ \beta\ Cos.\ E}{Sin.\ \alpha\ Sin.\ \gamma} + p\ Cot.\ \alpha + r\ Cot.\ \gamma - (q + s + 1) = 0,$$

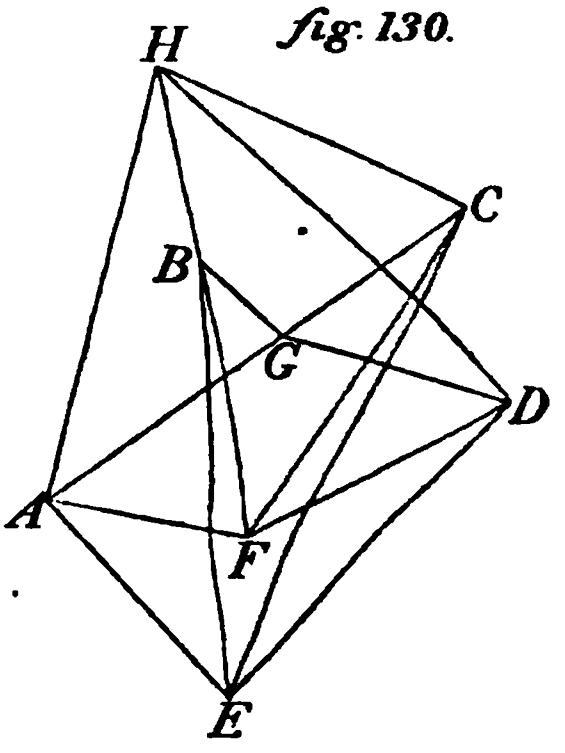
where $E = \alpha + \beta + \gamma$.

Likewise each of the other stations *F, G, H* gives a similar equation. We ∴ have four equations in all, whence

the value of $qs - pr$, $ps + qr$, p, r, $q + s + 1$, and hence again the values of p, q, r, s, may be determined. If these last are found, then we also have the values of x, y, ϕ, ψ. A complete solution of the above four literal equations is given by Professor Pfleiderer in the Archives for Pure and Practical Mathematics, 10th Number, p. 190. In practice, however, it is much easier to calculate with the given numbers themselves; how this is done will be seen by the following example.

EXAM. I shall assume, that the points E, F, G, H have the position in *fig.* 130. Let $\alpha = AEB = 39^\circ.\ 38'$, $\beta = BEC = 30^\circ.\ 54'$, $\gamma = CED = 23^\circ.\ 21'$, $\alpha' = AFB = 67^\circ.\ 23'$, $\beta' = BFC = 44^\circ.\ 44'$, $\gamma' = CFD = 28^\circ.\ 54'$, $\alpha'' = AGB = 75^\circ.\ 48'$, $\beta'' = BGC = 105^\circ.51'$, $\gamma'' = CGD = 50^\circ.23'$, $\alpha''' = -AHB = -25^\circ.\ 47'$, $\beta''' = -BHC = -55^\circ.\ 2'$, $\gamma''' = CHD = 15^\circ.\ 30'$. Here $E = \alpha + \beta + \gamma = 93^\circ.\ 53'$, $F = \alpha' + \beta' + \gamma' = 141^\circ.\ 1'$, $G = \alpha'' + \beta'' + \gamma'' = 232^\circ.\ 2'$, $H = \alpha''' + \beta''' + \gamma''' = -65^\circ.\ 19'$. Therefore

fig. 130.



$$\frac{Sin.\ \beta\ Sin.\ E}{Sin.\ \alpha\ Sin.\ \gamma} = 2{\cdot}0266000$$

$$\frac{Sin.\ \beta\ Cos.\ E}{Sin.\ \alpha\ Sin.\ \gamma} = -0{\cdot}1375675$$

$$Cot.\ \alpha = 1{\cdot}2073615$$

$$Cot.\ \gamma = 2{\cdot}3164076$$

$$\frac{Sin.\ \beta'\ Sin.\ F}{Sin.\ \alpha'\ Sin.\ \gamma'} = 0{\cdot}9924784$$

$$\frac{Sin.\ \beta'\ Cos.\ F}{Sin.\ \alpha'\ Sin.\ \gamma'} = -1{\cdot}2263381$$

$$Cot.\ \alpha' = 0{\cdot}4166012$$

$$Cot.\ \gamma' = 1{\cdot}8114969$$

$$\frac{Sin.\ \beta'' \ Sin.\ G}{Sin.\ \alpha' \ Sin.\ \gamma''} = -1{\cdot}0155390$$

$$\frac{Sin.\ \beta'' \ Sin.\ G}{Sin.\ \alpha'' \ Sin.\ \gamma''} = -0{\cdot}7924751$$

$$Cot.\ \alpha'' = 0{\cdot}2530389$$

$$Cot.\ \gamma'' = 0{\cdot}8277620$$

$$\frac{Sin.\ \beta''' \ Sin.\ H}{Sin.\ \alpha''' \ Sin.\ \gamma'''} = -6{\cdot}4057632$$

$$\frac{Sin.\ \beta''' \ Cos.\ H}{Sin.\ \alpha''' \ Sin.\ \gamma'''} = 2{\cdot}9440655$$

$$Cot.\ \alpha''' = -2.0701359$$

$$Cot.\ \gamma''' = 3{\cdot}6058835.$$

We consequently have the four following equations:

$$2{\cdot}0266000\,(qs-pr) + 0\,1375675\,(ps+qr) + 1{\cdot}2073615\,p + 0{\cdot}3164076\,r - (q+s+1) = 0$$

$$0{\cdot}9924784\,(qs-pr) + 1{\cdot}2263381\,(ps+qr) + 0{\cdot}4166012\,p + 1{\cdot}8114969\,r - (q+s+1) = 0$$

$$-1\,0155390\,(qs-pr) + 0{\cdot}7924751\,(ps+qr) + 0{\cdot}2530389\,p + 0{\cdot}8277620\,r - (q+s+1) = 0$$

$$-6{\cdot}4057632\,(qs-pr) - 2{\cdot}9440655\,(ps+qr) - 2{\cdot}0701359\,p + 3{\cdot}6058835\,r - (q+s+1) = 0.$$

If the second, third, and fourth be subtracted from the first, we then obtain

$$1{\cdot}0341216\,(qs-pr) - 1{\cdot}0887706\,(ps+qr) + 0{\cdot}7907603\,p + 0{\cdot}5049107\,r = 0.$$

$$3{\cdot}0421390\,(qs-pr) - 0{\cdot}6549076\,(ps+qr) + 0{\cdot}9543226\,p + 1{\cdot}4886456\,r = 0.$$

$$8{\cdot}4323632\,(qs-pr) + 3{\cdot}0816330\,(ps+qr) + 3{\cdot}2774974\,p - 1{\cdot}2894759\,r = 0.$$

Divide the coefficients of each equation by the coefficient of the first term; this gives

$$(qs-pr) - 1{\cdot}0528458\,(ps+qr) + 0{\cdot}7646686\,p + 0{\cdot}4882508\,r = 0$$

$$(qs - pr) - 0{\cdot}2152786\,(ps + qr) + 0{\cdot}3137012\,p + 0{\cdot}4893417\,r = 0$$

$$(qs - pr) + 0{\cdot}3654531\,(ps + qr) + 0{\cdot}3886808\,p - 0{\cdot}1529199\,r = 0$$

and when the two first equations are subtracted from the last,

$$1{\cdot}4182989\,(ps + qr) - 0{\cdot}3759878\,p - 0{\cdot}6411707\,r = 0$$
$$0{\cdot}5807317\,(ps + qr) + 0{\cdot}0749796\,p - 0{\cdot}6422616\,r = 0.$$

Divide again each equation by the coefficient of the first term; this gives

$$(ps + qr) - 0{\cdot}2650977\,p - 0{\cdot}4520702\,r = 0$$
$$(ps + qr) + 0{\cdot}1291123\,p - 1{\cdot}1059524\,r = 0,$$

and when these equations are subtracted from one another,

$$0{\cdot}3942100\,p - 0{\cdot}6538822\,r = 0$$

By means of this equation we successively get the four following equations:

$$\begin{aligned} r &= 0\,6028762\,p \\ ps + qr &= 0{\cdot}5376401\,p \\ qs - pr &= -0{\cdot}4929713\,p \\ q + s + 1 &= 1{\cdot}6787747\,p \end{aligned}$$

From the first and second equations we obtain

$$s + 0{\cdot}6028762\;q = 0{\cdot}5376401.$$

This equation combined with the fourth, gives

$$\begin{aligned} q &= 4{\cdot}2273334\;p - 3{\cdot}8719414 \\ s &= -\,2{\cdot}5485587\,p + 2{\cdot}8719414. \end{aligned}$$

If we now substitute the values of q, r, s, in the third equation, we then obtain

$$11{\cdot}3764835\,p^2 - 22{\cdot}5014950\,p + 11{\cdot}1199888 = 0$$

or,

$$p^2 - 1{\cdot}9778955\,p + 0{\cdot}977450 = 0\,;$$

and this equation gives

$$p = 0{\cdot}9889477 \pm 0{\cdot}0237403,$$

consequently either $p = 1{\cdot}0126880$, or $p = 0{\cdot}9652074$. The two last figures of these values are uncertain.

If we take the first value of p, we find

$$p = 1{\cdot}0126880 = x \; Sin. \; \phi$$
$$q = 0{\cdot}4090284 = x \; Cos. \; \phi$$
$$r = 0{\cdot}6105255 = y \; Sin. \; \psi$$
$$s = 0{\cdot}2910466 = y \; Cos. \; \psi.$$

'rom the first and second equations, we obtain

$$Tan. \; \phi = \frac{1{\cdot}0126880}{0{\cdot}4090284} = 2{\cdot}4758378,$$

and from the third and fourth,

$$Tan. \; \psi = \frac{0{\cdot}6105255}{0{\cdot}2910466} = 2{\cdot}0976898;$$

consequently

$$\phi = 68^0.\; 0'.\; 21''{\cdot}7, \; \psi = 64^0.\; 30'.\; 43''{\cdot}8,$$

and

$$x = \frac{p}{Sin. \; \phi} = 1{\cdot}09217, \; y = \frac{r}{Sin. \; \psi} = 0{\cdot}67635.$$

If we take the second value of p, we have

$$p = 0{\cdot}9652074 = x \; Sin. \; \phi$$
$$q = 0{\cdot}2083121 = x \; Cos. \; \phi$$
$$r = 0{\cdot}5819006 = y \; Sin. \; \psi$$
$$s = 0{\cdot}4120537 = y \; Cos. \; \psi$$

and $\therefore$

$$Tan. \; \phi = \frac{0{\cdot}9652074}{0{\cdot}2083121} = 4{\cdot}6334677$$

$$Tan. \; \psi = \frac{0{\cdot}5819006}{0{\cdot}4120537} = 1{\cdot}4121960$$

$$\phi = 77^0.\; 49'.\; 16''{\cdot}1, \; \psi = 54^0.\; 41'.\; 49''{\cdot}3$$

$$x = \frac{p}{Sin. \; \phi} = 0{\cdot}98743, \; y = \frac{r}{Sin. \; \psi} = 0{\cdot}71301.$$

Consequently in each of these problems, two different positions of the eight points are possible. Which is the right one, must $\therefore$ be determined from other circumstances.

When the situations of the four points A, B, C, D, are determined by means of the angles and lines ϕ, ψ, x, y, when calculated, it is easy, by the problem in § LIV, to determine the position of the points E, F, G, H.

REMARK. The problem here solved, incontestably the most important in practical geometry, was invented by the celebrated Lambert; it may be seen in his Contributions II. p. 186, &c. In the calculation of his example, he has only made use of four decimal places; ∴ the expressions there found for P, Q, R, S, or, according to my notation, for p, q, r, s, are not in one case correct as far as this number of decimal places. Thus, according to Lambert, p. 193, $P = 0{\cdot}5690$, $Q = -0{\cdot}3443$ ($-0{\cdot}3543$ is an error of the press), $R = -0{\cdot}4998$, $S = -0{\cdot}3028$; but more properly, $P = 0{\cdot}5698$, $Q = -0{\cdot}3447$, $R = -0{\cdot}5005$, $S = -0{\cdot}3024$. For the angles ϕ, ψ (according to the notation used here), we should only, from these data, have found the degrees correctly, which, in determining the other parts of the figure, would cause great mistakes. It will be advisable, ∴, to perform the calculation throughout with six decimal places.

To this section also belongs the first treatise of the excellent work of Mr. Hauptmann, by Hügenin (Mathematical Contributions for the Formation of the young Geometrician, Königsberg, 1803), which the reader will certainly peruse with advantage and pleasure.

X. ON MAXIMA AND MINIMA, AS FAR AS THIS SUBJECT BELONGS TO ELEMENTARY GEOMETRY.

SECTION CX.

DEFINITION.

A magnitude is said to be a maximum, when it is the greatest of all those which are similar to it; a minimum, when it is the least.

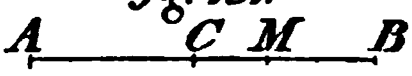
The diameter of a circle is the greatest of all the lines which can be drawn from one point in the circumference of a circle to another, and consequently, in reference to these lines it is a maximum. Further, of all the lines which can be drawn from a given point to a given line, the perpendicular is the least, and consequently, in reference to these lines, it is a minimum.

SECTION CXI.

PROB. *To divide a given line, so that the rectangle contained by the two parts may be a maximum.*

SOLUT. Let the given line AB (*fig.* 131) be divided in M, so that $AM \times MB$ is greater than any other rectangle which can be contained by any other two parts of this line.

fig. 131.



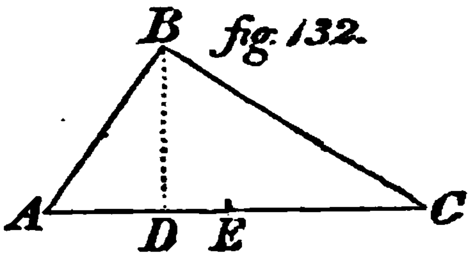
Bisect the line AB in C, and assume $AC = CB = \frac{1}{2} AB = a$, $CM = x$. Then $AM = a + x$, $BM = a - x$, and $AM \times MB = a^2 - x^2$. But the expression $a^2 - x^2$ is evidently the greatest, when $x = 0$; consequently the point M falls in C, and $\therefore AC \times CB$ is the greatest rectangle.

Cor. When ∴ two lines, viz. P, Q, have a given sum; then the rectangle contained by these two lines is the greatest when $P = Q$. Consequently of all the rectangles of a given circumference, the square contains the greatest area.

SECTION CXII.

Prob. *Amongst all the angles which are upon the same base, and have the same circumference, to find that which contains the greatest area.*

Solut. Let the triangle ABC (*fig.* 132) be a maximum; the given base $AC = a$, and with respect to the given circumference $AB + BC = b$.



1. Draw the perpendicular BD: then, since the base AC is determined, this is a maximum. Bisect AC in E, and put $ED = x$, $BD = y$, $AB = z$. Then $AD = \frac{1}{2}a - x$, $CD = \frac{1}{2}a + x$; consequently, since $AB^2 = AD^2 + BD^2$, $BC^2 = CD^2 + BD^2$,

$$z^2 = (\tfrac{1}{2}a - x)^2 + y^2$$
$$(b - z)^2 = (\tfrac{1}{2}a + x)^2 + y^2.$$

2. If the first equation be subtracted from the second, then we obtain

$$b^2 - 2bz = 2ax$$
$$z = \tfrac{1}{2}b - \frac{ax}{b}.$$

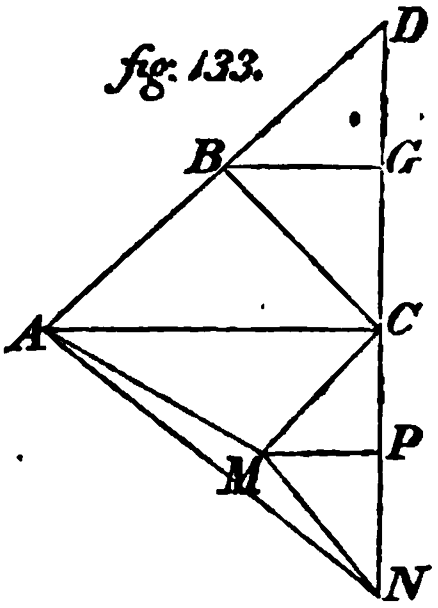
3. Substitute this value of z in the first equation in 1; this gives

$$y^2 = \frac{b^2 - a^2}{b^2}(\tfrac{1}{4}b^2 - x^2).$$

4. Hence it follows, that $y = BD$ is the greatest, when $x = 0$, and consequently when the perpendicular BD bisects the line AC. But in this case, $AB = BC$; consequently, of all the triangles of the same circumference and on the same base, the isosceles triangle contains the greatest area.

COR. This result may likewise be proved geometrically in the following way.

Let ABC (*fig.* 133) be an isosceles triangle, AMC any other triangle on the same base AC, and $AM + MC = AB + BC$: prove that $\triangle ABC > \triangle AMC$.



With B as a centre, and radius $AB = BC$, suppose a circle described, which meets AB produced in D. Draw DC; then the angle ACD, being in a semicircle, $= R$. Produce the perpendicular DC; make $MN = MC$, and draw AN. From the points B, M, draw BG, MP perpendicular to DN.

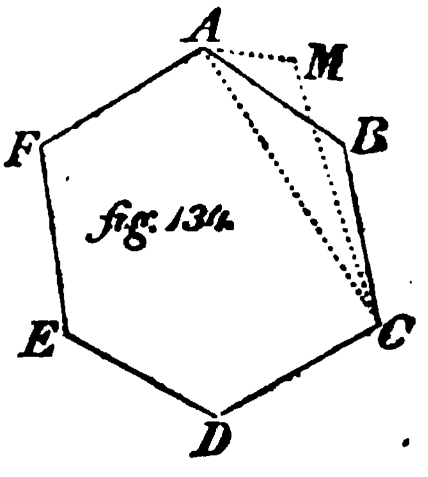
Since $BC = BD$, and $MN = MC$; then $AB + BC = AD$, and $AM + MC = AM + MN$; consequently $AD = AM + MN$, and $\therefore AD > AN$; and since $CD^2 = AD^2 - AC^2$, $CN^2 = AN^2 - AC^2$, consequently $CD > CN$. Now since $CG = \frac{1}{2} CD$, $CP = \frac{1}{2} CN$: then likewise $CG > CP$. But CG is the altitude of the $\triangle ABC$, and CP the altitude of the $\triangle AMC$; consequently the triangle ABC has a greater altitude, and $\therefore$ also a greater area, than the triangle AMC.

SECTION CXIII.

From the foregoing section, the following general and important rule is deduced:

Of all polygons of equal circumference, and of the same number of sides, that one which is equilateral has the greatest area.

For let $ABCDEF$ (*fig.* 134) be the greatest polygon. If AB be not equal to BC; then upon AC as a base describe an isosceles triangle AMC, whose circumference is equal to that of ABC, in which consequently $AM + MC = AB + BC$. Then by the foregoing section, $\triangle AMC > ABC$, and consequently also the polygon $AMCDEF$, which has the same number of sides, and the same circumference as



the polygon $ABCDEF$, is greater than this last. Therefore the polygon $ABCDEF$ is not the greatest of all the polygons having the same circumference and the same number of sides. Consequently $AB = BC$. But, likewise for the same reason, $BC = CD$, $CD = DE$, &c.; consequently the polygon which is a maximum, is equilateral.

SECTION CXIV.

PROB. *Two sides of a triangle are given: find the angle contained by these sides, when the triangle is a maximum.*

SOLUT. Let (ABC *fig.* 132) be the greatest of all the triangles, which have the same sides AB, BC.

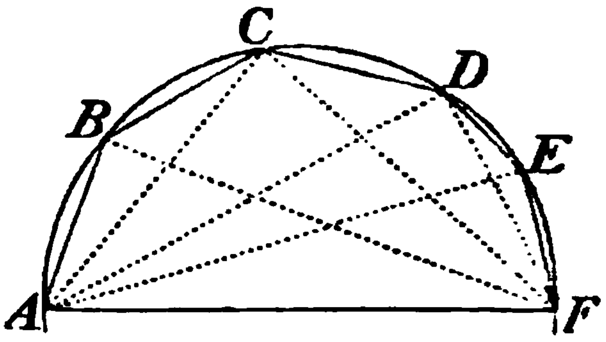
Since $\triangle ABC = \frac{1}{2} AB \cdot BC$ *Sin.* ABC, and the lines AB, BC, are given; then that triangle is the greatest, for which *Sin.* ABC is the greatest. But *Sin.* ABC is the greatest, when ABC is a right angle; consequently the greatest triangle is that, the two given sides of which contain a right angle.

SECTION CXV.

PROB. *All the sides but one of a polygon are given: find the conditions under which the polygon is a maximum.*

SOLUT. Let $ABCDEF$ (*fig.* 135), be the greatest of all the polygons, which are composed of the sides AB, BC, CD, DE, EF, and an undetermined side AF. Draw the diagonals AD, DF: then ADF is a right angle. For if ADF be not a right angle, then, by the foregoing section, retaining the parts $ABCD$, DEF, the triangle ADF, and consequently also the polygon $ABCDEF$ may be enlarged; which is contrary to the supposition, that this polygon is a

fig. 135.

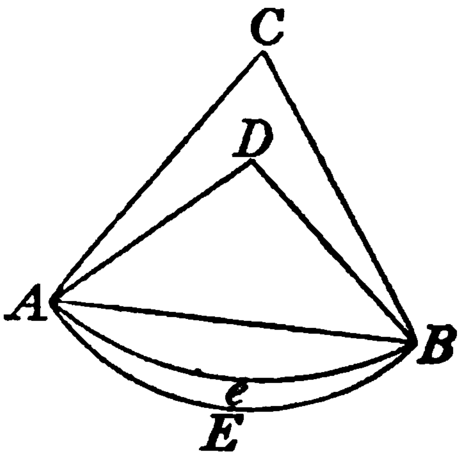


maximum. In like manner it may be proved, that *ABF*, *ACF*, *AEF*, are right angles. Consequently the points *A*, *B*, *C*, *D*, *E*, *F*, must be in a semicircle, whose diameter is *AF*.

COR. That there can be only one polygon, which, under this condition, can be constructed from the given sides, the following observations will demonstrate.

In the first place it is evident, that when two arcs *AEB*, *AeB* (*fig.* 136), have the same chord *AB*, the chord to which the given radius belongs has a smaller angle at the centre, and consequently, when *C*, *D*, are the centres of these two arcs, *ADB* > *ACB*.

fig. 136.



Hence it immediately follows, that when once a semicircle (*fig.* 135) is found, in which the given sides *AB*, *BC*, *CD*, *DE*, *EF*, are exactly contained, it is not possible to find another which satisfies these conditions. For if the radius of the other semicircle be greater or less than the radius of the first; then, in the first case, the angles at the centre, which belong to the arcs *AB*, *BC*, &c., would be less than before, and in the second case greater. In both cases these angles cannot together be greater than two right angles, which is the condition.

It matters not in what order the sides *AB*, *BC*, &c. follow each other in the semicircle, since in each series of these sides, the sum of the arcs, which are by them cut off, is always equal to half the circumference; likewise the area of the polygon remains the same, and this last, because the sum of the segments *AB*, *BC*, &c. remains the same.

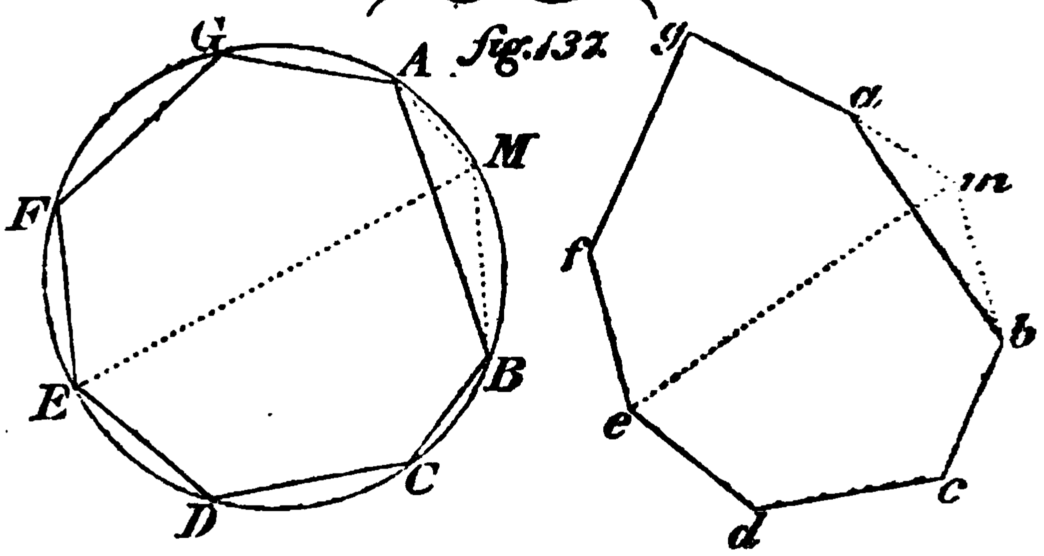
SECTION CXVI.

From the foregoing section the following rule is deduced:

PROB. *Amongst all the polygons which consist of a certain number of given sides, that one about which a circle can be described is the greatest.*

For let *ABCDEFG* (*fig.* 137) be a polygon described in

fig. 132



a circle, and $abcdefg$ another, about which no circle can be described, and whose sides are equal to those of the first, so that $ab = AB$, $bc = BC$, $cd = CD$, and so on. Draw the diameter EM, and the lines AM, BM; upon $ab = AB$, describe the triangle abm, similar and equal to ABM, and draw em. Then by the foregoing section, the polygon $EFGAM$ is greater than the polygon $efgam$, unless indeed this last polygon can be described in a circle, whose diameter is em, in which case, as was shown in the above section, its area is equal to that of the former. For the same reason, and under the same condition, the polygon $EDCBM$, is also greater than $edcbm$. The polygon $AMBCDEFG$ is likewise necessarily greater than $ambcdefg$; for it cannot be equal to it, for otherwise about the whole polygon $ambcdefg$ a circle might be described; which is contrary to the hypothesis. If, now, we take from the above two polygons the equal triangles ABM, abm: then it follows, that the polygon $ABCDEFG$ is greater than the polygon $abcdefg$.

SECTION CXVII.

From §§ CXIII, CXVI, the following rule is deduced:

Of all the polygons of the same circumference and of the same number of sides, the regular polygon is the greatest.

For by § CXIII, the polygon which is a maximum, is equilateral, and by the preceding section, it can be described in a circle; ∴ it is regular.

SECTION CXVIII.

From the foregoing section the following rule is also deduced:

The circle is greater than every rectilineal figure, which has the same circumference.

For it was there proved, that the regular polygon is greater than every other of the same circumference and the same number of sides. If, now, it can be proved, that the circle is always greater than a regular polygon of the same circumference, the truth of the rule follows at once.

Let ∴ r be the radius of a circle, p its circumference, q its area. Assume any regular polygon of the same circumference with the circle; let the perpendicular drawn from its centre to any of its sides $= h$; the area $= q'$. Further, suppose a polygon similar to it described at the circle; let its circumference $= p'$, and the area $= q''$. Then, as may be easily seen,

$$q = \tfrac{1}{2}rp, \quad q' = \tfrac{1}{2}hp, \quad q'' = \tfrac{1}{2}rp'.$$

Because of the similarity of the above two polygons, we also have

$$q'' : q' = r^2 : h^2.$$

If in the proportion for q'', q', their values are substituted, we then obtain

$$\tfrac{1}{2}rp' : \tfrac{1}{2}hp = r^2 : h^2,$$

or $$p' : p = r : h$$

and ∴ $$p' = \frac{pr}{h};$$

consequently $$q'' = \tfrac{1}{2}rp' = \frac{pr^2}{2h}$$

$$q'q'' = \tfrac{1}{4}p^2r^2 = q^2.$$

We ∴ have $$q' : q = q : q''.$$

Now since the circle is always less than the polygon described about it, consequently $q'' > q$, and ∴ also $q > q'$. Q. E. D.

SECTION CXIX.

Prob. *In a line given in position, to find a point, such, that when two straight lines are drawn to this point from two given points with this line, their sum is a minimum.*

Solut. Let MN (*fig.* 138) be the line given in position, C a point in it, A, B, the two given points; so that when the lines AB, BC, are drawn, $AC + BC$ is a minimum.

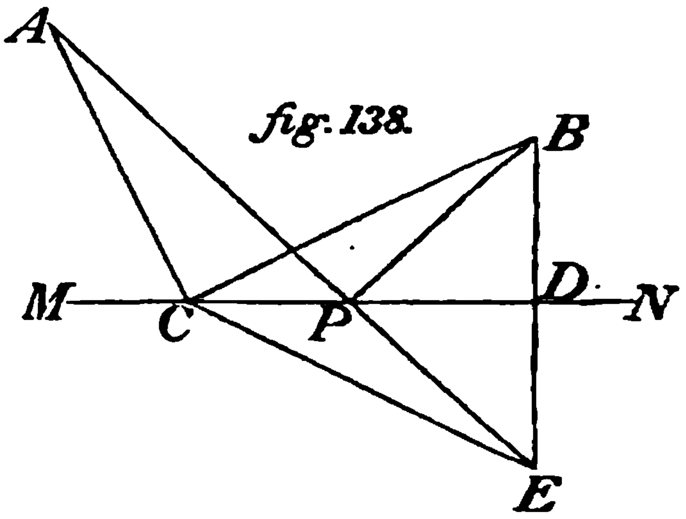


fig. 138.

1. From one of the given points, say B, draw BD perpendicular to MN, produce it, and make $DE = BD$; draw CE, AE, and from P, in which AE, MN, intersect each other, draw the line BP.

2. Since $BD = DE$, $CD = CD$, $CDB = CDE = R$; then $\triangle CDB$ is similar and equal to $\triangle CDE$, and $\therefore$ $CB = CE$; consequently $AC + CB = AC + CE$. The sum of the lines AC, CE, and consequently also the sum of the lines AC, CB, will be the least, when the point C falls in P; $\therefore$ $AP + PB$ is a minimum, and consequently P is the point sought.

Cor. Since $BD = DE$, $PD = PD$, $BDP = EDP$; therefore $\triangle BDP$ is similar and equal to $\triangle EDP$, and $\therefore$ $\angle BPD = \angle DPE$. But $DPE = APM$; consequently $APM = BPD = BPN$. The required point is where the lines AP, BP, make equal angles with the line MN.

SECTION CXX.

Prob. *Two points without a given circle are given; find a point in its circumference such, that when straight lines are drawn from it to the given points, the sum of these two lines is a minimum.*

SOLUT. Let (*fig.* 139) C be the centre of a circle, M a point on the convex side of its circumference, such, that the angle $AMC = BMC$, and P any other point in the circumference. To M draw the tangent DE; further, draw the lines AP, BP, and from the point Q, in which AP, DE, intersect each other, draw BQ.

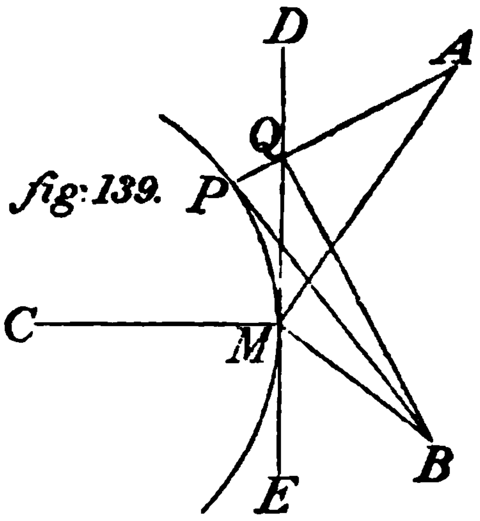


fig: 139.

Since $AMC = BMC$, $CMD = CME = R$: then $AMD = BME$. But $BQE < BME$, $AQD > AMD$; consequently $AQD > BQE$, and $\therefore$ by the foregoing section, $AQ + BQ > AM + BM$. Now $AP + BP > AQ + BQ$ (*Euc. I.* 21); much more $\therefore$ is $AP + BP > AM + BM$. Since this is true, wherever the point P is assumed; consequently M is the point, for which the sum of the lines AM, BM, is a minimum.

REMARK. Although the property by which the point M is determined, for which the sum of the lines AM, BM is a minimum, is extremely simple; yet this point cannot be determined in any way by Elementary Geometry. A very elegant solution of this problem is given by Robert Simson, which, on account of conic sections being made use of in it, does not belong to this subject.

SECTION CXXI.

PROB. *In a given triangle to determine a point, which is such, that when lines are drawn from it to the angular points of the triangle, the sum of these lines is a minimum.*

SOLUT. Let ABC (*fig.* 140) be the given triangle; P the point for which $PA + PB + PC$ is a minimum. From the point C, with the radius CP, suppose a circle described; then $PA + PB$ must necessarily be less than the sum of the lines, which can be drawn from the points A, B, to any other point of the circumference of this circle; because if this be not the case, the sum of the three lines

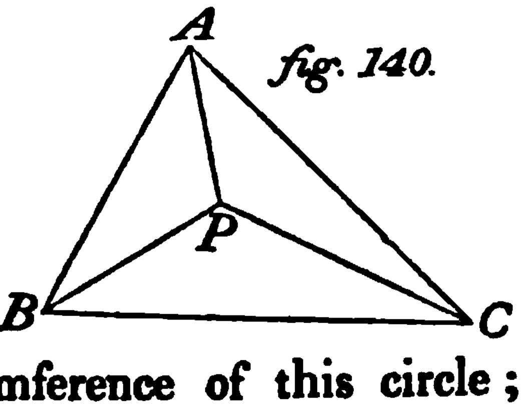


fig. 140.

PA, PB, PC, could not be a minimum. Therefore, by the foregoing section, the angle $APC = BPC$. In like manner it may be shown, that $APB = APC$; $\therefore$ $APB = APC = BPC$, and consequently each angle $= 120^\circ$.

If one of the angles of the triangle be greater than 120°; then the problem is impossible.

REMARK. Many more problems relating to Maxima and Minima may be solved by means of Elementary Geometry; there are a great many cases of this kind to be found in a work on this subject, by L. Huilier, entitled: "De Relatione mutua Capcitate et Terminorum Figurarum, geometrice considerata: seu de Maximis et Minimis: Varsarviæ, 1782." *Also in his work on Polygonometry, already quoted, p. 174.* But however ingenious such elementary solutions for single cases may be, yet they are by no means suited to the establishment of general rules for the treatment of this subject. How far the Differential Calculus, and especially the Calculus of Variations, invented by the celebrated Lagrange (which alone is sufficient to immortalise its inventor) is applicable to this subject, is reserved to the following Collections.

XI LOCI.

SECTION CXXII.

Definition.

When several points have any one property in common, and all are in a straight or crooked line, then this line is called the *Locus* of these points, or of each of them; and a Plane Locus, when the line in which the points lie, is a straight line. The following examples will serve to elucidate what has been said.

Let the base of a triangle, together with its area, be given; find its vertex. The properties which are here required of the triangles sought, apply to an endless number of triangles, whose vertices are all in a straight line, which is parallel to the given base. This line is consequently the Locus of the required vertex.

Let the base and the vertical angle of a triangle be given; required to find its vertex. Here the given properties are evidently not sufficient for the determination of the triangle and its vertex; for there is an endless number of points, which satisfy the conditions of the problem; and all these points are in a circular arc, which has the given base of the required triangle for a chord. Consequently this circular arc is the *Locus* of the required vertex.

SECTION CXXIII.

Prob. *From a given point there is a line drawn, whose extremity touches another straight line given in position: find the Locus of the point which divides the first line in a given proportion.*

Solut. From a given point C (*fig.* 141), let any line

CH be drawn to a line AB given in position, and let this line be divided according to a certain proportion in I: represent the Locus of the point I.

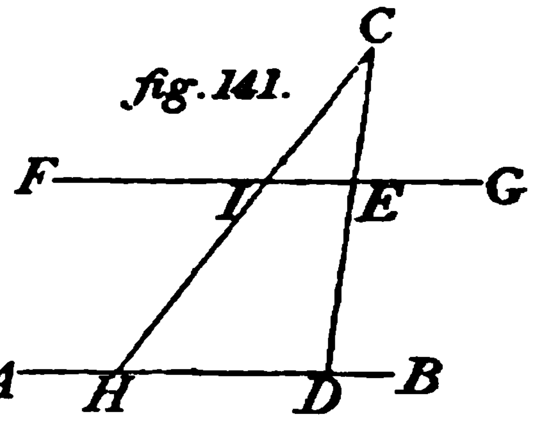


fig. 141.

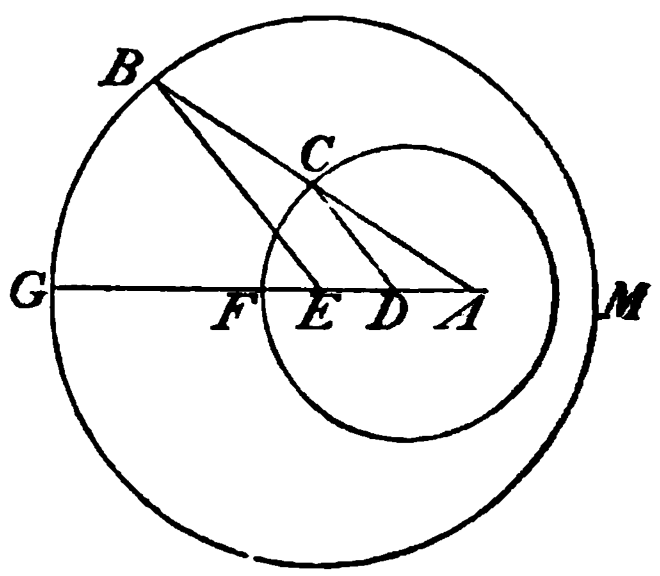
Draw any line CD, cut it in the given proportion in E, and through this point draw the line FG parallel to AB: then this line is the *Locus* sought. For if any line CH be drawn, which cuts FG in I, and touches AB in H; then $CH : CI = CD : CE =$ the given proportion.

SECTION CXXIV.

PROB. *From a given point a straight line is drawn, whose extremity touches the circumference of a given circle: find the Locus of the point which divides this straight line in a given proportion.*

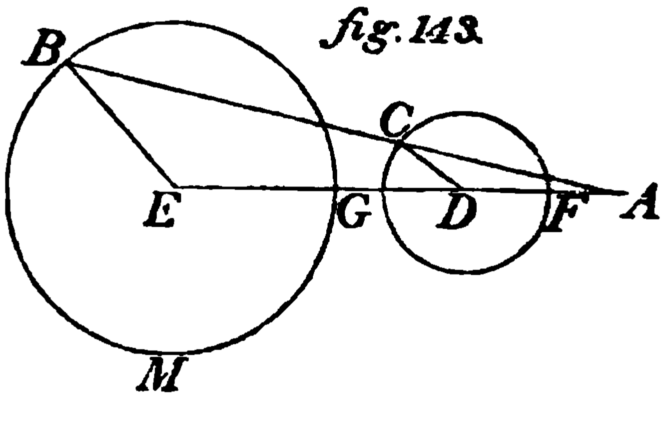
SOLUT. Let A (*figs.* 142, 143) be the given point, BGM the given circle, E its centre, AB any straight line drawn from the point A, and this divided in C, so that $AB : AC = m : n$; find the Locus of the point C.

fig. 142.



Draw the line AE, which meets the circumference of the given circle in G; determine the point D, so that $AE : AD = m : n$; and when this has been found, determine the point F, so that $EG : DF = m : n$. Then from D, with the radius DF, describe a circle: this circle is the required Locus.

fig. 143.



From the point A draw any line AB, which meets the circumferences of the two circles

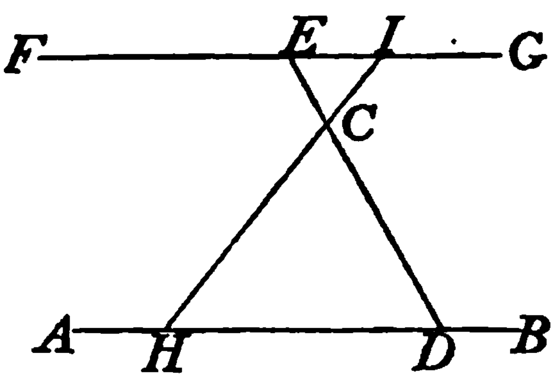
in B, C, and draw CD, BE to their centres. Now since $AE : AD = m : n$, $EG : DF = m : n$; then $AE : AD = EG ; DF = BE : CD$, consequently BE is parallel to CD, and $\therefore$, $AB : AC = AE : AD = m : n$, as required.

SECTION CXXV.

Prob. *From a line given from a certain point, let two parts be cut off by this point in opposite directions, so that these parts may have a given proportion, and that the extremity of one part may meet a line given in position: find the Locus of the extremity of the other part.*

Solut. Let C (*fig.* 144) be the given point, AB the straight line given in position, HI any line passing through C which is not given in position, and $CH : CI = m : n$; the extremity H of the line CH touches the line AB: find the Locus of the point I.

fig. 144.

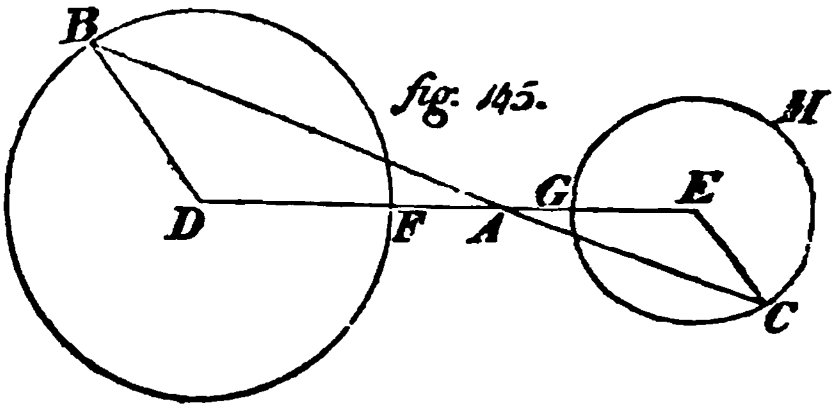


From C to AB draw any line CD, make $CD : CE = m : n$, and through the point E draw the parallel FG: this is the Locus sought. Through C draw any line HI, which meets the lines AB, FG, in H, I; then $CH : CI = CD : CE = m : n$, which was sought.

SECTION CXXVI.

Prob. *In a straight line passing through a given point, two parts are cut off from this point in opposite directions, which have a given proportion; the extremity of one part touches a given circle: find the Locus of the extremity of the other part.*

Solut. Let CGM (*fig.* 145) be the given circle, E its



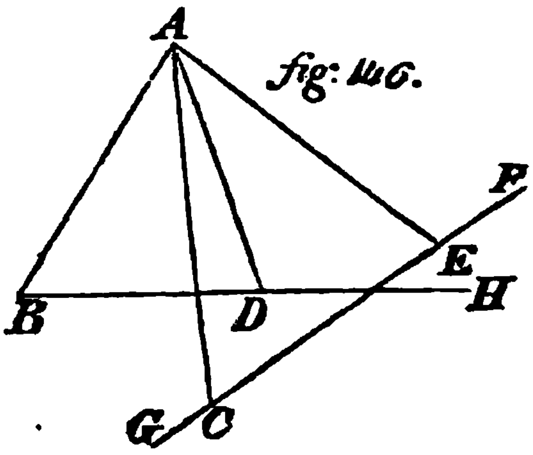
centre, BC any line passing through a given point A, which meets this circle in C; let AB, AC be two parts, whose proportion to one another is constantly $= m : n$; find the Locus of the point B.

In AE determine a point D, so that $AD : AE = m : n$; then a point F, so that $DF : EG = m : n$; then from D, with a radius DF, describe a circle: this is the required Locus. For if any line BC is drawn through A, and from B, C, in which it meets the two circles, the lines BD, CE be drawn; then, since $AD : AE = m : n$, and $DF : EG = m : n$, likewise $AD : AE = DF : EG = BD : CE$; consequently BD is parallel to CE, and $\therefore AB : AC = AD : AE = m : n$, as required.

SECTION CXXVII.

PROB. *Two straight lines drawn from a given point, and having a given proportion to one another, contain a given angle; the extremity of one line touches a straight line given in position: find the Locus of the extremity of the other line.*

SOLUT. Let A (*fig.* 146) be the fixed point, from which the lines AB, AC are drawn, which contain a given angle BAC, and have a given proportion to one another; the point B meets the line BH; find the Locus of the point C.



From A towards BH, draw any line AD, make the angle DAE equal to the given one, and take AE, so that AD has the given proportion to AE; through the point E draw the line FG, forming the angle $AEG = ADB$: this line is the required Locus. Draw any two

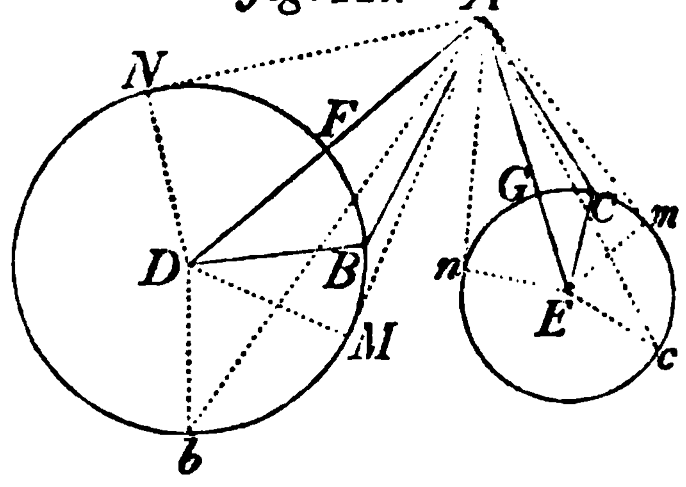
lines AB, AC, forming the given angle $BAC = DAE$; then in the triangles ABD, ACE, the angle $BAD = CAE$ and $ADB = AEC$; consequently triangle ABD is similar to the triangle ACE, and $AB : AC = AD : AE =$ the given proportion, as required.

SECTION CXXVIII.

Prob. *Two straight lines which are drawn from a given point, and have a given proportion, contain a given angle; the extremity of one line touches a given circle: find the Locus of the extremity of the other line.*

Solut. Let A (*fig.* 147) be a given point, D the centre of a circle BMb, and from A let any line AB be drawn to the circumference of this circle; to AB apply an angle BAC of a given magnitude, and upon AC determine the point C, so that $AB : AC$ may be equal to a given proportion $m : n$.

fig. 147.



Draw the line AD, which cuts the circumference of the given circle in F, make the angle DAE equal to the given one, and in AE determine the point E, so that $AD : AE = m : n$, and then the point G, so that likewise $DF : EG = m : n$. If now from the point E, with a radius EG, a circle is described; then this circle will be the required Locus.

For let AB, AC be any two lines drawn to the circumferences of both circles, containing the angle BAC equal to the given one; and draw DB, EC: then $BAC = DAE$, and consequently also, $DAB = EAC$. Further, since $AD : AE = m : n$, and $DF : EG = m : n$; then $AD : AE = DF : EG = DB : EC$. The angles DAB, EAC, are ∴ similar (*Euc. VI.* 7), if it can be shown, that ABD, ACE, are always at the same time obtuse, right, or acute, angles. Assume this to be the case: then $AD : AE = AB : AC$. But $AD : AE = m : n$; consequently also $AB : AC = m : n$, and ∴ the circle, whose centre is E, is the required position.

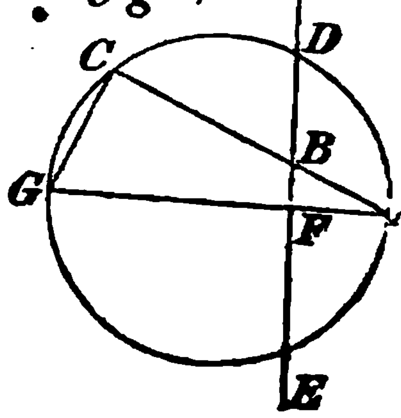
But that the two angles ABD, ACE have the above property, can be proved in the following way. From A draw AM, AN, Am, An tangents to the two circles; and to the points where they touch the circles, the radii DM, DN, Em, En. It may be easily proved, that the triangles DAM, EAm, and consequently also the triangles DAN, EAn, which are equal to them, are similar to one another; for $AD : AE = DM : Em (=m : n)$, and $AMD = AmE = R$. Therefore the angle $DAM = DAN = EAm = EAn$, and consequently also $DAE = MAm = NAn$. If $\therefore$ the point B is situated in M or N, then the point C is in m or n. First suppose that the point B falls in F, and consequently the point C in G; then $ABD = ACE = 2\,R$. Now if the point B remove from F to M, the point C at the same time removes from G to m, the angles ABD, ACE constantly become less, but continue to be obtuse, till B moves into M, and C into m, in which case both are right angles. Beyond M, m, these angles become acute, as AbD, AcE, when b, c are two corresponding points, and remain so, till B, C remove into N, n, where they are right angles, and then again become obtuse. Consequently the condition assumed in the proof, is fulfilled.

SECTION CXXIX.

Prob. *From a line which is drawn from a given point, two parts are cut off, so that their rectangle contained by them has a given magnitude. The extremity of one part meets a straight line given in position: determine the Locus of the extremity of the other part.*

Solut. From the line AC (*fig.* 148), which is drawn from the given point A, two parts AB, BC are cut off such, that the rectangle $AB \times AC$ has a given area $= q$. The point B meets the line DE: determine the Locus of the point C.

fig. 148.



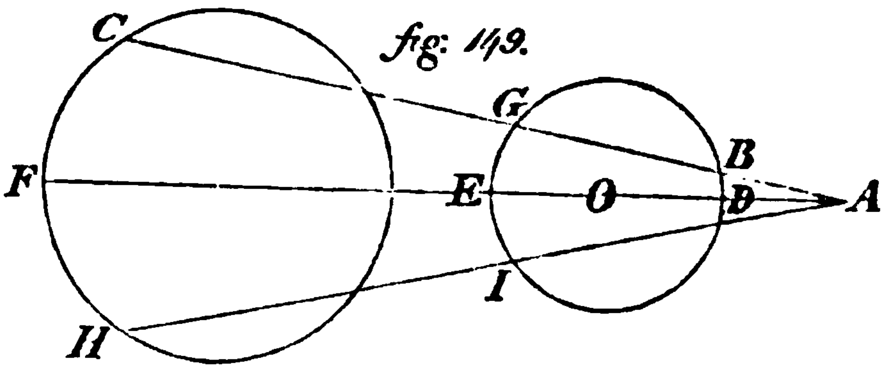
From A draw AF perpendicular to DE; in this line, or this same one produced, determine the point G, so that $AF \times AG = q$; upon AG, as a diameter,

describe a circle: this circle is the position sought. From A draw any line AC, and then CG. Then ACG, as an angle in a semicircle, is a right angle, and $\therefore$ $AFB = ACG$; consequently, because the angle CAG is common to both, $\triangle ACG$ is similar to $\triangle AFB$. We $\therefore$ have $AG : AB = AC : AF$, and consequently $AB \times AC = AG \times AF = q$, as required.

SECTION CXXX.

PROB. *From a line drawn from a given point, two parts are cut off, such, that the rectangle contained by the two parts have a given magnitude. The extremity of one part meets the circumference of a given circle: determine the Locus of the extremity of the other.*

SOLUT. From the given point A (*fig.* 149), a line AC is



drawn, and from it two parts AB, AC are cut off, so that $AB \times AC$ has an area $= q$. The point B meets a given circle $BDEG$: determine the Locus of the point C.

From A, through the centre O of the given circle, draw the line AO, which cuts its circumference in D, E, and in this line determine a point F such, that $AF \times AD = q$. Then find by § CXXVI, the Locus of a point H, such, that when AH is drawn, and $AH : AI$ is made equal to the given proportion $AF : AE$, the point I meets the given circle. Let the circle CFH be this Locus; then, I assert, this circle is also the Locus of the point C, for which $AC \times AB = q$.

If from A any line AC is drawn, which touches the circumferences of both circles in C and G: then, by the construction, $AC : AG = AF : AE$, and $\therefore$ likewise $AC \times AB : AG \times AB = AF \times AD : AE \times AD$. Now (*Euc.*

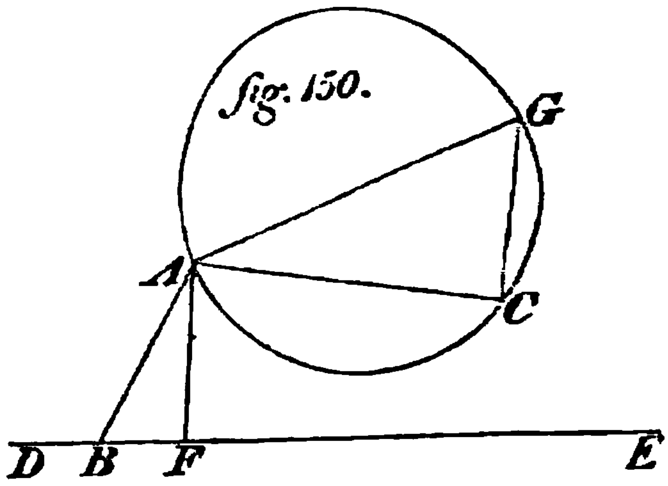
III. 36) $AG \times AB = AE \times AD$; consequently also $AC \times AB = AF \times AD = q$, as required.

Cor. Hence it also follows, that when the point B meets the convex or the concave side of the circle opposite to A, the point C, in the first case, is situated on the concave side, and in the second, on the convex side, of the circle sought.

SECTION CXXXI.

Prob. *From a given point, two straight lines containing a given angle are drawn, and from these two parts are cut off, such, that the rectangle contained by them has a given area. The extremity of one part meets a straight line given in position; find the Locus of the extremity of the other part.*

Solut. From the given point A (*fig.* 150), draw two lines AB, AC, so that the angle BAC has a given magnitude, and the rectangle $AB \times AC$ has a given area $= q$; the point B lies in the straight line DE; find the Locus of the point C.

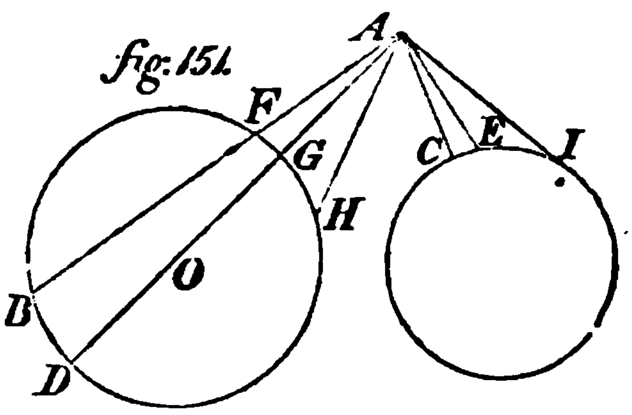


Draw AF perpendicular to DE, make the angle FAG equal to the given one, and $FA \times AG = q$; upon AG as a diameter, describe a circle: then this circle is the Locus sought. From A draw any two lines AB, AC, containing the given angle, the first to DE, the other to the circumference of the circle, and moreover draw the line CG. Then from the construction, $BAC = FAG$, and $\therefore$ likewise $BAF = CAG$. Now since also ACG as an angle in a semicircle, is a right angle; consequently $ACG = AFB$, and $\therefore \triangle ACG$ is similar to $\triangle ABF$. We consequently have $AB : AG = AF : AC$, and $\therefore AB \times AC = AG \times AF = q$, as required.

SECTION CXXXII.

PROB. *From a given point two straight lines containing a given angle, are drawn, such, that the rectangle contained by them has a given area. The extremity of one line is situated in the circumference of a given circle: find the Locus of the extremity of the other line.*

SOLUT. Let BDH (*fig.* 151) be a given circle, O its centre; AB, AC, any two lines drawn from A, which contain a given angle $BAC = \alpha$, and which are such, that the rectangle contained by them has a given area $= q$.



Draw AO, which cuts the given circle in D, G, make $DAE = \alpha$, and in AE determine the point E, so that $AD \times AE = q$. Now let I be a point, which is determined by these means, that when any line AH is drawn to the circle, it makes the angle $HAI = \alpha$, and the proportion $AH : AI$ equal to the given proportion $AG : AE$. The Locus of this point is ∴ a circle, which may be found by § CXXVIII. This circle is also the Locus of the point C.

According to the construction, for every two lines drawn from A to the two circles, which contain an angle $= \alpha$, viz. AF, AC, $AF : AC = AG : AE$; consequently also $AF : AB : AC : AB = AG \times AD : AE \times AD$. Now $AF \times AB = AG \times AD$; consequently likewise $AC \times AB = AE \times AD = q$, as required.

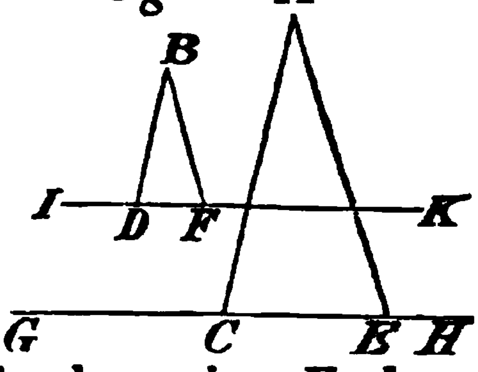
SECTION CXXXIII.

PROB. *From two given points two parallel lines are drawn, which have a given proportion. The extremity*

of one meets a straight line given in position: find the Locus of the extremity of the other.

Solut. Let A, B (*fig.* 152) be two given points, and AC, BD, two parallel lines; the point C is in the given line GH: find the Locus of the point D, when for each two of these lines, the proportion $AC : BD$ is equal to the given proportion $m : n$.

fig. 152.



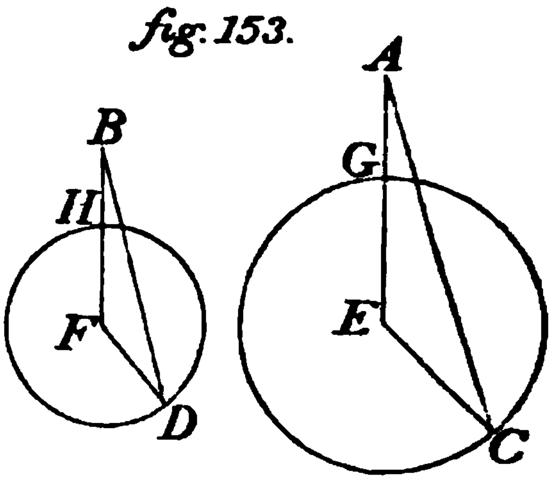
From A to GH draw any line AE, and from B draw BF parallel to AE; make $AE : BF = m : n$, and through the point F thus determined, draw IK parallel to GH: then this line is the required Locus. From A, B, to the lines GH, IK, draw any two parallels AC, BD: then, since AE is parallel to BF, AC parallel to BD, CE parallel to DF, the three angles of the triangle ACE are equal to the three angles of the triangle BDF, and $\therefore$ $AC : BD = AE : BF = m : n$, as required.

SECTION CXXXIV.

Prob. *From two given points, two parallel lines are drawn of a given proportion; the extremity of one is in the circumference of a given circle: find the Locus of the extremity of the other.*

Solut. Let A, B (*fig.* 153) be two given points; AC, BD, two parallel lines, which have the given proportion $m : n$; the point C is in the circumference of the circle, whose centre is E: find the Locus of the point D.

fig. 153.



Draw the line AE and BF parallel to it, and determine the point F, so that $AE : BF = m : n$; then make $EG : FH = m : n$, and by these means determine the point H. From F, with the radius FH, describe a circle: then this circle is the required

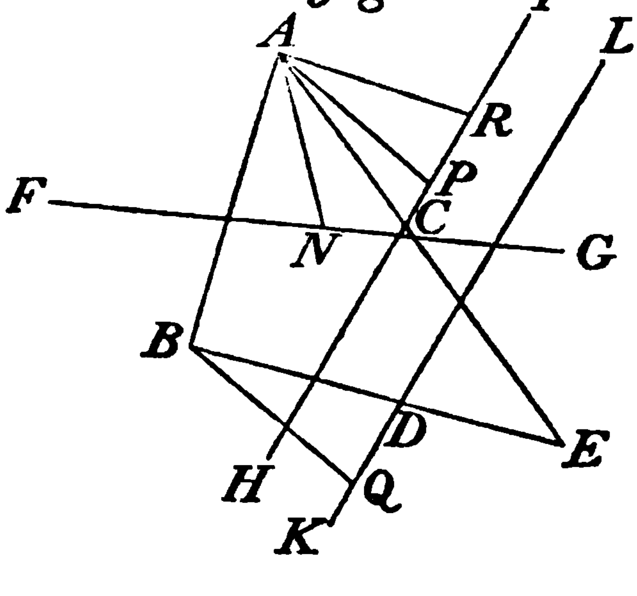
Locus. Draw any two parallel lines AC, BD, to the circumference of the circles, and the radii EC, FD; then the angle $EAC = FBD$; further, $AE : BF = m : n$, and $EC : FD = EG : FH = m : n$; $\therefore AE : BF = EC : FD$. Moreover, as is easily seen, the angles AEC, BFD, are at the same time acute, rectangular, or obtuse. Therefore triangle AEC is similar to triangle BFD, and consequently $AC : BD = AE : BF = m : n$, as required.

SECTION CXXXV.

Prob. *From two given points two straight lines are drawn, which contain a given angle; in each of these a part is cut off from the given point, such, that these parts have a given proportion. The extremity of one part meets a straight line given in position: find the Locus of the extremity of the other.*

Solut. Let A, B (*fig.* 154), be two given points, and AE, BE, any two lines, which contain a given angle $AEB = \alpha$. From these lines two parts AC, BD, are cut off, such, that $AC : BD = m : n$. The point C is in the line FG; determine the Locus of the point D.

fig. 154.



From A to FG draw any line AN, make $NAP = \alpha$, and $AN : AP = m : n$, and then determine, by § CXXVII, the Locus of the point P. Let the straight line HI be this Locus. Make the line BQ equal and parallel to AP; further, through the point Q draw the line KL parallel to HI: then this line is the Locus sought. Thus, if any two lines AE, BE are drawn, which contain the given angle α, the parts AC, BD cut off by the lines FG, KL, have the given proportion $m : n$.

Make the angle $CAR = \alpha$, and $AC : AR = m : n$; then, by the construction, the point R is in the line HI. Now

since AP is parallel to BQ, PR parallel to QD, and because the angles AEB, CAR, are equal, then likewise BD is parallel to AR; further, $AP = BQ$, then $\triangle APR$ is similar and equal to $\triangle BQD$, and $AR = BD$. But $AC : AR = m : n$; consequently also $AC : BD = m : n$, as required.

REMARK. To those of my readers who wish to know more of the subject of Plane Loci, I beg to recommend a work already quoted once, viz. the Translation of the Plane Loci of Apollonius, by Camerer. To the ancients these Loci were the chief means of solving problems; but the great improvement in Analysis has rendered them in a great measure unnecessary now-a-days. They may, however, be considered as useful in preparing the student for the higher geometry, and this is the reason why they have not been altogether passed over in this Collection.

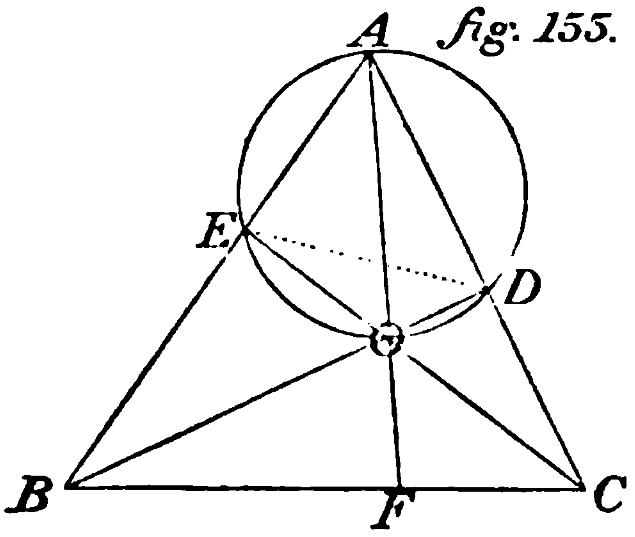
XII. MISCELLANEOUS PROBLEMS.

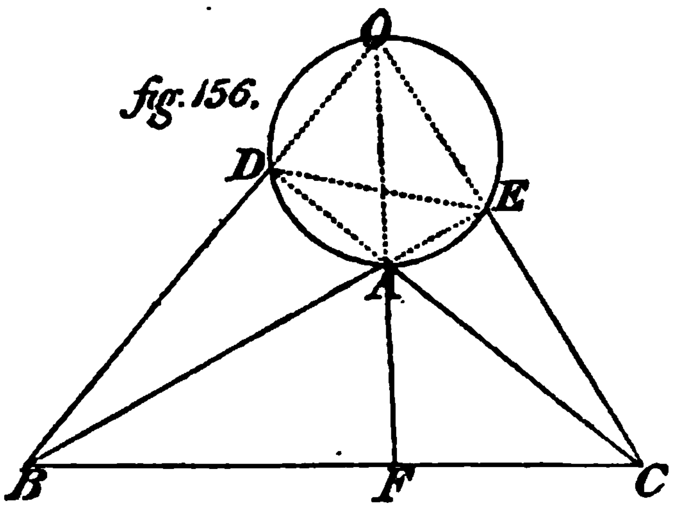
SECTION CXXXVI.

AUXILIARY RULE.

When from the three angular points of a triangle, perpendiculars are drawn to the opposite sides; these perpendiculars intersect each other in one and the same point.

Proof. Let ABC (*figs.* 155, 156) be any triangle; BD,





CE, two perpendiculars to AC, AB, or these produced, and O the point in which BD, CE, intersect each other: it remains to be proved, that when AO is drawn, and produced, it cuts the side BC at right angles in F. Draw DE.

The triangles BOE, COD, are similar, for $BOE = COD$, and $BEO = CDO = R$. We $\therefore$ have $BO : CO = OE : OD$, and since likewise the angle $BOC = EOD$: then $\triangle\, BOC$ is similar to $\triangle\, EOD$, and angle $CBD = DEO$. Further, since AEO, ADO, are right angles: then $ADO + AEO = 2\,R$; consequently a circle may be described

about the quadrilateral $ADOE$, and we have $\angle DEO = \angle DAO$ (*Euc.* III. 27) and $\therefore CBD = DAO$. The triangles CAF, CBD, consequently have the equal angles CAF, CBD, and the common one ACB; $\therefore$ likewise $AFC = BDC = R$.

Q. E. D.

SECTION CXXXVII.

PROB. *In a triangle, two perpendiculars, drawn from the angular points to the opposite sides, are given: find its sides, angles, and area.*

SOLUT. In the triangle ABC (*figs.* 155, 156), the perpendiculars given are $BD = a$, $CE = b$, $AF = c$, and the sides sought, are $AB = x$, $AC = y$, $BC = z$. Because the triangles ABD, ACE, are similar, we have $BD : AB = CE : AC$, or $a : x = b : y$, and $\therefore$

$$y = \frac{bx}{a}.$$

In like manner, from the similar triangles CBE, ABF, we find

$$z = \frac{bx}{c}.$$

2. We $\therefore$ have

$$\triangle ABC = \tfrac{1}{4}\sqrt{(x+y+z)(x+y-z)(x+z-y)(y+z-x)}$$
$$= \frac{x^2}{4a^2c^2}\sqrt{(ac+bc+ab)(ac+bc-ab)(ac+ab-bc)(bc+ab-ac)},$$

or when, for the sake of brevity, we put

$(ac+bc+ab)(ac+bc-ab)(ac+ab-bc)(bc+ab-ac) = K$

we get,

$$\triangle ABC = \frac{x^2\sqrt{K}}{4a^2c^2}.$$

3. But likewise $\triangle ABC = \frac{1}{2} AB \times CE = \frac{1}{2} bx$; we

consequently have the equation

$$\frac{x^2\sqrt{K}}{4a^2c^2} = \tfrac{1}{2}bx,$$

whence we obtain

$$x = \frac{2a^2bc^2}{\sqrt{K}}.$$

Hence we obtain the two remaining sides of the triangle: thus,

$$y = \frac{bx}{a} = \frac{2ab^2c^2}{\sqrt{K}}$$

$$z = \frac{bx}{c} = \frac{2a^2b^2c}{\sqrt{K}}.$$

4. For the angles of the triangle, we find

$$Sin.\ BAC = \frac{a}{x} = \frac{\sqrt{K}}{2abc^2}$$

$$Sin.\ ABC = \frac{c}{x} = \frac{\sqrt{K}}{2a^2bc}$$

$$Sin.\ ACB = \frac{c}{y} = \frac{\sqrt{K}}{2ab^2c}.$$

5. Further,

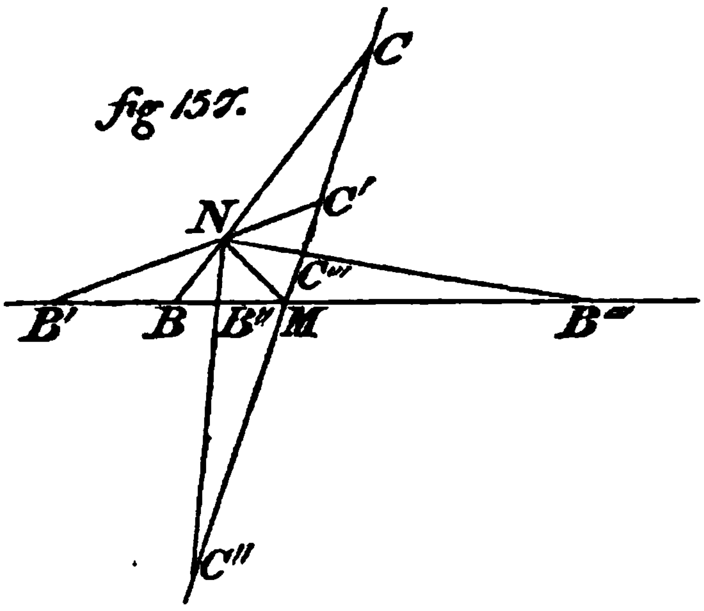
$$\triangle\ ABC = \tfrac{1}{2}bx = \frac{a^2b^2c^2}{\sqrt{K}}.$$

SECTION CXXXVIII.

Prob. *Two lines intersect each other at a given angle; this angle is bisected by a line, in which a point is given: it is required through this point to draw a line, such, that the part contained between the two first lines intersecting each other, may have a given magnitude.*

Solut. The lines $B'B'''$, CC'' (*fig.* 157), intersect each

other in M, forming the angle $BMC = \alpha$; this angle is bisected by the line MN, and $MN = a$; draw a line through N, so that the line contained between the two first lines, viz. $BC = b$.



1. Since $BMO = \alpha$: therefore $BMN = CMN = \frac{1}{2}\alpha$. If the angle BNM is also known, then the problem is solved. Let ∴ $BNM = \phi$; $MCN = \phi - \frac{1}{2}\alpha$, $MBN = 180^0 - (\phi + \frac{1}{2}\alpha)$. We ∴ have in the triangle BMN,

$$BN = \frac{MN \; Sin. \; BMN}{Sin. \; MBN} = \frac{a \; Sin. \; \frac{1}{2} \alpha}{Sin. \; (\phi + \frac{1}{2} \alpha)},$$

and in the triangle CMN,

$$CN = \frac{MN \; Sin. \; CMN}{Sin. \; MCN} = \frac{a \; Sin. \; \frac{1}{2} \alpha}{Sin. \; (\phi - \frac{1}{2} \alpha)},$$

Now, since $BC = BN + CN$; we ∴ have the equation,

$$b = \frac{a \; Sin. \; \frac{1}{2} \alpha}{Sin. \; (\phi + \frac{1}{2} \alpha)} = \frac{a \; Sin. \; \frac{1}{2} \alpha}{Sin. \; (\phi - \frac{1}{2} \alpha)},$$

or

$$b \; Sin. \; (\phi + \tfrac{1}{2} \alpha) \; Sin. \; (\phi - \tfrac{1}{2} \alpha) = a \; Sin. \; \tfrac{1}{2} \alpha \; [Sin. \; (\phi + \tfrac{1}{2}\alpha) + Sin. \; (\phi - \tfrac{1}{2} \alpha)]$$

2. But we have $Sin. \; (\phi + \frac{1}{2} \alpha) \; Sin. \; (\phi - \frac{1}{2} \alpha) = Sin.^2 \phi - Sin.^2 \frac{1}{2} \alpha$, and $Sin. \; (\phi + \frac{1}{2} \alpha) + Sin. \; (\phi - \frac{1}{2} \alpha) = 2 \; Sin. \; \phi \; Cos. \; \frac{1}{2} \alpha$; consequently the foregoing equation is transformed into the following one:

$$b \; Sin.^2 \phi - b \; Sin.^2 \tfrac{1}{2} \alpha = 2 a \; Sin. \; \tfrac{1}{2} \alpha \; Cos. \; \tfrac{1}{2} \alpha \; Sin. \; \phi,$$

or, since $2 \; Sin. \; \frac{1}{2} \alpha \; Cos. \; \frac{1}{2} \alpha = Sin. \; \alpha$, into

$$Sin.^2 \phi - \frac{a \; Sin. \; \alpha}{b} \; Sin. \; \phi = Sin.^2 \tfrac{1}{2} \alpha;$$

and hence we obtain

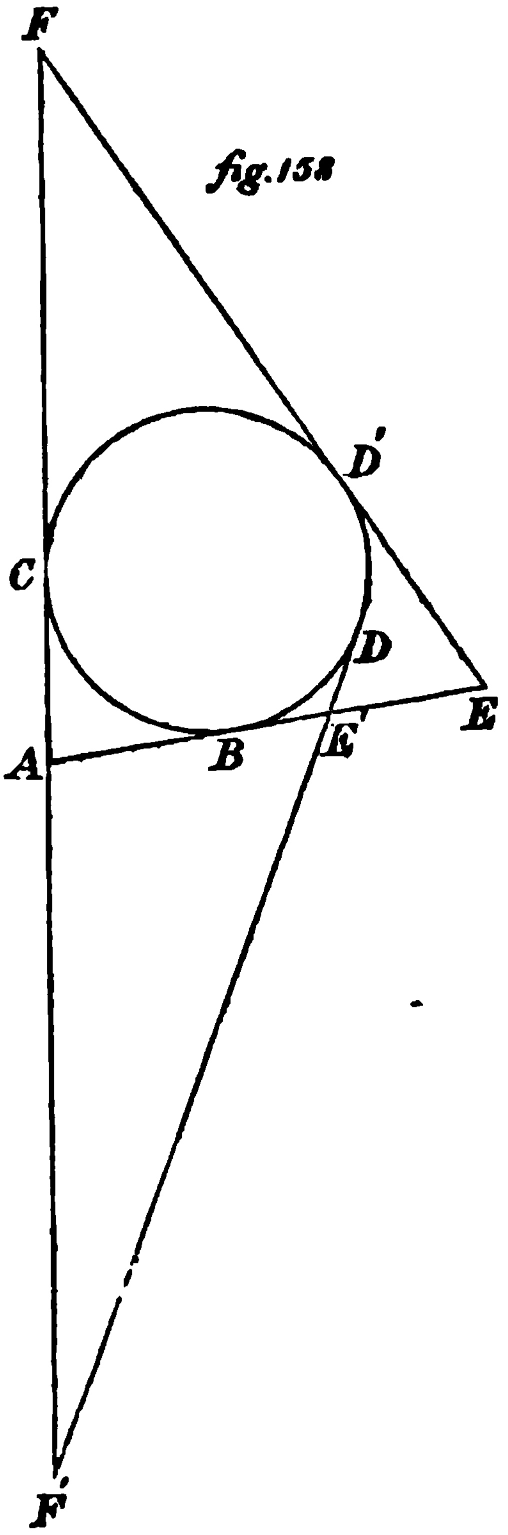
$$Sin.\ \phi = \frac{a\ Sin.\ \alpha \pm \sqrt{(a^2\ Sin.^2\ \alpha + 4b^2\ Sin.^2\ \frac{1}{2}\ \alpha)}}{2\ b}.$$

REMARK. The problem in § LXXXIII is only a particular case of this more general one. What was there said of the four solutions of the problem there given, also applies to this one; for each sine has two angles. The figure shows the four different positions which the line BC can have.

SECTION CXXXIX.

PROB. *Two tangents are drawn to a given circle: find a third tangent such, that the part contained between the two first has a given magnitude.*

SOLUT. Let BDC (*fig.* 158) be a given circle; AB, AC, two of its tangents, so that $AB = AC = a$, and the angle $BAC = \alpha$; draw a third tangent EF, which cuts the two first in E, F, so that $EF = b$.



1. If DE is known, then the problem is solved; for it would only be necessary in this case to make $BE = DE$, and from the point E so determined to draw a tangent to the circle. Put ∴ $DE = x$; this gives $DF = b - x$, $AE = AB + BE = a + x$, $AF = AC + CF = AC + DF = a + b - x$.

2. In the triangle AEF,

$$EF^2 = AE^2 + AF^2 - 2\ AE\ .\ AF\ Cos.\ EAF$$

and ∴

$$b^2 = (a+x)^2 + (a+b-x)^2 - 2(a+x)(a+b-x) \times Cos.\ \alpha$$

or

$$x^2 - bx + (a^2 + ab)\frac{1 - Cos.\ \alpha}{1 + Cos.\ \alpha} = 0.$$

From this equation we further obtain

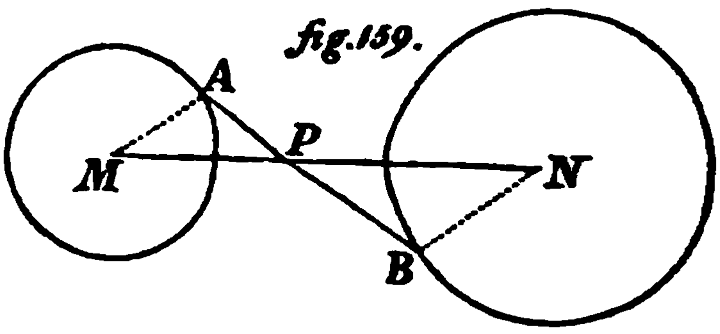
$$x = \tfrac{1}{2} b \pm \sqrt{\left[\tfrac{1}{4} b^2 - (a^2 + ab)\frac{1 - Cos.\,\alpha}{1 + Cos.\,\alpha}\right]}$$
$$- \tfrac{1}{2} b \pm \sqrt{[\tfrac{1}{4} b^2 - 2\,(a^2 + ab)\ Tan.^2\ \tfrac{1}{2}\,\alpha]},$$

The two expressions just found for x, are always positive, as the figure indeed shows. Thus the line EF can also have the position $E'F'$, and then DE' has the second value.

SECTION CXL.

PROB. *Two circles, and a point in the line which joins their centres, are given: through this point draw a line which meets the circumferences of both circles, such, that the parts included between these circumferences and the given point have a given proportion.*

SOLUT. Let M, N (*fig.* 159) be the centres of the two given circles, and P a point in the line MN; further, let $MP = a$, $NP = b$, the radius $MA = r$, the radius $NB = R$: through P draw a line AB to the circumferences of the two circles, so that $AP : BP = m : n$.



1. If the line AP is determined, the problem is solved. Put ∴ $AP = x$: this gives $BP = \frac{nx}{m}$. Therefore in the triangle AMP,

$$Cos.\ APM = \frac{MP^2 + AP^2 - MA^2}{2\,MP\,.\,AP} = \frac{a^2 + x^2 - r^2}{2\,ax}$$

and in the triangle BNP,

$$Cos.\ BPN = \frac{NP^2 + BP^2 - NB^2}{2\,NP\,.\,BP} = \frac{b^2 + \frac{n^2x^2}{m^2} - R^2}{\frac{2\,bnx}{m}}$$

$$= \frac{m^2b^2 + n^2x^2 - m^2R^2}{2\,mnbx}.$$

2. Now since $Cos.\ APM = Cos.\ BPN$: we obtain the following equation :

$$\frac{a^2 + x^2 - r^2}{2ax} = \frac{m^2b^2 + n^2x^2 - m^2R^2}{2\,mnbx},$$

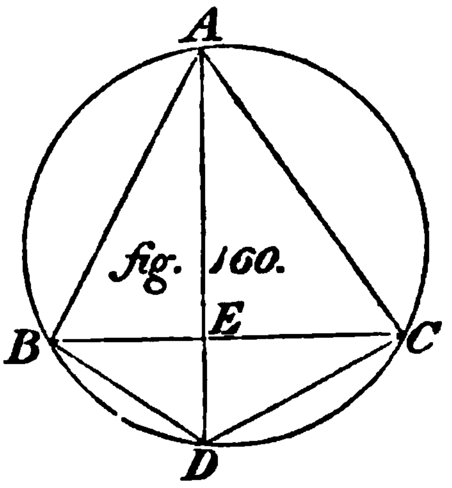
and hence

$$x = \sqrt{\frac{am^2\,(b^2 - R^2) - bmn\,(a^2 - r^2)}{bmn - an^2}}$$

SECTION CXLI.

Prob. *From two given points in the circumference of a given circle, to draw two chords, which contain a given angle, and have a given proportion.*

Solut. Let $ABDC$ (*fig.* 160) be a given circle, whose radius $= r$; let A, B, be two points in the circumference of the circle, from which two chords AB, BC, are drawn, so that $AEC = \alpha$, and $AD : BC = m : n$. Further, let $AB = a$.



1. If we know the angles BAD, DAC: we then can draw the chords AD, DC. Put $\therefore BAD = \phi$, $DAC = \psi$. Hence we find $CBD = CAD = \psi$, $ABC = AEC - BAD = \alpha - \phi$, $ACB = 180^0 - (AEC + CAD) = 180^0 - (\alpha + \psi)$, $BAC = BAD + DAC = \phi + \psi$, $ABD = ABC + CBD = \alpha - \phi + \psi$.

2. By the first principles of Trigonometry, $AB = 2r \times Sin.\ ACB = a$, $AD = 2r\ Sin.\ ABD$, $BC = 2r\ Sin.\ BAC$, and $\therefore AD : BC = Sin.\ ABD : Sin.\ BAC = m : n$. Now, if for ACB, ABD, BAC, we put their values from 1, we then obtain the two following equations:

$$2r\ Sin.\ (\alpha + \psi) = a$$
$$m\ Sin.\ (\phi + \psi) = n\ Sin.\ (\alpha - \phi + \psi).$$

3. In order to determine from hence the angles ϕ, ψ,

expand *Sin.* $(\alpha+\psi)$ in the first equation; this gives

$$2\,r\,(Sin.\,\alpha\,Cos.\,\psi + Cos.\,\alpha\,Sin.\,\psi) = a$$

or

$$2\,r\,Cos.\,\alpha\,Sin.\,\psi = a - 2\,r\,Sin.\,\alpha\,Cos.\,\psi\,;$$

and when the square root is extracted from both sides of the equation, and $1 - Cos.^2\,\phi$ is substituted for $Sin.^2\,\phi$,

$$4\,r^2\,Cos.^2\,\alpha - 4\,r^2\,Cos.^2\,\alpha\,Cos.^2\,\psi = a^2 - 4\,a\,r\,Sin.\,\alpha\,Cos.\,\psi + 4\,r^2\,Sin.^2\,\alpha\,Cos.^2\,\psi$$

or

$$4\,r^2\,Cos.^2\,\psi - 4\,a\,r\,Sin.\,\alpha\,Cos.\,\psi = 4\,r^2\,Cos.^2\,\alpha - a^2,$$

whence we obtain

$$Cos.\,\psi = \frac{a\,Sin.\,\alpha}{2\,r} \pm \frac{Cos.\,\alpha}{2r}\sqrt{(4\,r^2 - a^2)}\,;$$

and this equation serves to determine the angle ψ.

4. Multiply this equation by $2\,r\,Sin.\,\alpha$; this gives

$$2\,r\,Sin.\,\alpha\,Cos.\,\psi = a\,Sin.^2\,\alpha \pm Sin.\,\alpha\,Cos.\,\alpha\,\sqrt{(4\,r^2 - a^2)}.$$

But from 3,

$$2\,r\,Sin\,\alpha\,Cos.\,\psi + 2\,r\,Cos.\,\alpha\,Sin.\,\psi = a\,;$$

if $\therefore$ we subtract the first from the second, we then obtain

$$2\,r\,Cos.\,\alpha\,Sin.\,\psi = a\,Cos.^2\,\alpha \mp Sin.\,\alpha\,Cos.\,\alpha\,\sqrt{(4\,r^2 - a^2)}$$

and

$$Sin.\,\psi = \frac{a\,Cos.\,\alpha}{2\,r} \mp \frac{Sin.\,\alpha}{2\,r}\sqrt{(4\,r^2 - a^2)}.$$

5. In order now to determine the angle ϕ, also expand the second equation in 2; this gives

$$m\,[Sin.\,\phi\,Cos.\,\psi + Cos.\,\phi\,Sin.\,\psi] = n\,[Sin.\,(\alpha + \psi)\,Cos.\,\phi - Cos.\,(\alpha + \psi)\,Sin.\,\phi],$$

and hence we obtain

$$\frac{Sin.\,\phi}{Cos.\,\phi} = Tan.\,\phi = \frac{n\,Sin.\,(\alpha + \psi) - m\,Sin.\,\psi}{m\,Cos.\,\psi + n\,Cos.\,(\alpha + \psi)}.$$

But from the first equation in 2, we obtain $Sin. (\alpha + \psi) = \frac{a}{2r}$; $\therefore Cos. (\alpha + \psi) = \frac{\sqrt{(4r^2 - a^2)}}{2r}$, and when we substitute these values in the expression for $Tan. \phi$,

$$Tan. \phi = \frac{na - 2mr\, Sin. \psi}{2mr\, Cos. \psi + n\sqrt{(4r^2 - a^2)}}.$$

In this expression it is only necessary to substitute for $Sin. \psi$ and $Cos. \psi$ the values found in 3, 4, and we also get the angle ϕ.

Remark. When general expressions for the angles ϕ, ψ, are not treated of, but merely the actual calculation of a single case; then the following method will be the easiest. Thus from the first equation in 2, we obtain $Sin. (\alpha + \psi) = \frac{a}{2r}$; hence, when numbers only are considered, the angle $\alpha + \psi$, and consequently also ψ may be determined. After ψ has been determined in this way, the equation in 5, viz.

$$Tan. \phi = \frac{n\, Sin. (\alpha + \psi) - m\, Sin. \psi}{m\, Cos. \psi + n\, Cos. (\alpha + \psi)}$$

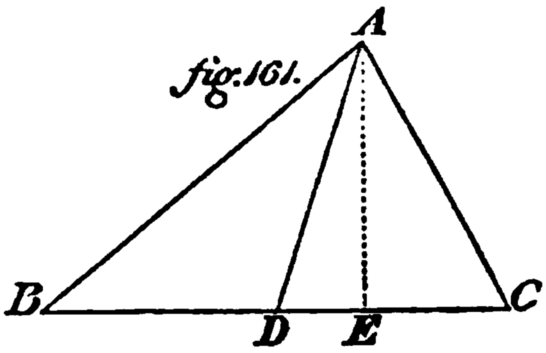
immediately gives the angle ϕ.

SECTION CXLII.

Prob. *The base of a triangle, the difference of the two angles, at this base, and also the line drawn from the vertical angle of the triangle to the centre of the base, are given: find the triangle.*

Solut. Let ABC (*fig.* 161) be the required triangle, the base $BC = a$, D the middle of BC, and $AD = b$; further, let $ACB - ABC = \alpha$.

fig. 161.



1. If the angle ADC is known; then in each of the triangles CAD, DAB, two sides, and the angle contained by them, are given, consequently these triangles

themselves, and $\therefore$ also the whole triangle BAC. Put $\therefore$ $ADC=\phi$, and draw the perpendicular AE. Then $AE = b\,Sin.\,\phi$, $DE=b\,Cos.\,\phi$, $BE=\frac{1}{2}a+b\,Cos.\,\phi$, $CE=\frac{1}{2}a-b\,Cos.\,\phi$; consequently

$$Tan.\,ABC = \frac{AE}{BE} = \frac{b\,Sin.\,\phi}{\frac{1}{2}a + b\,Cos.\,\phi}$$

$$Tan.\,ACB = \frac{AE}{CE} = \frac{b\,Sin.\,\phi}{\frac{1}{2}a - b\,Cos.\,\phi}.$$

2. By

$$Tan.\,(ACB - ABC) = Tan.\,\alpha = \frac{Tan.\,ACB - Tan.\,ABC}{1 + Tan.\,ACB\,Tan.\,ABC}.$$

If in this for $Tan.\ ACB$, $Tan.\ ABC$, we substitute their values found in 1, we obtain the equation

$$Tan.\,\alpha = \frac{2\,b^2\,Sin.\,\phi\,Cos.\,\phi}{\frac{1}{4}a^2 - b^2\,(Cos.^2\,\phi - Sin.^2\,\phi)},$$

or, since $2\,Sin.\,\phi\,Cos.\,\phi = Sin.\,2\,\phi$, $Cos.^2\,\phi - Sin.^2\,\phi = Cos.\,2\,\phi$.

$$Tan.\,\alpha = \frac{b^2\,Sin.\,2\,\phi}{\frac{1}{4}a^2 - b^2\,Cos.\,2\,\phi}.$$

3. In the equation last found, substitute $\frac{Sin.\,\alpha}{Cos.\,\alpha}$ for $Tan.\,\alpha$; by these means it is transformed into the following one:

$$b^2\,(Sin.\,2\,\phi\,Cos.\,\alpha + Cos.\,2\,\phi\,Sin.\,\alpha) = \tfrac{1}{4}a^2\,Sin.\,\alpha,$$

or

$$b^2\,Sin.\,(2\,\phi + Cos.\,\alpha) = \tfrac{1}{4}a^2\,Sin.\,\alpha;$$

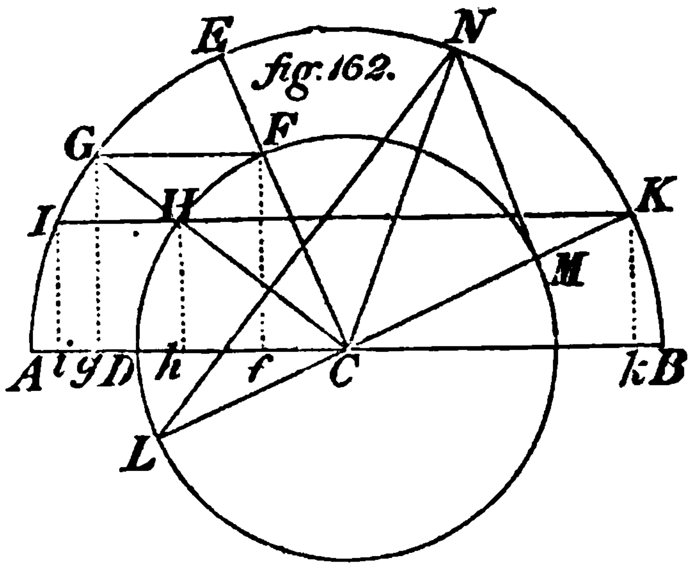
and the equation gives

$$Sin.\,(2\,\phi + \alpha) = \frac{a^2\,Sin.\,\alpha}{4\,b^2}.$$

Hence we may now determine $2\,\phi + \alpha$, and consequently also ϕ.

Cor. From this analytical solution we derive the following tolerably easy construction.

From any point C (*fig.* 162), with a radius $AC = b$, describe the semicircle ANB, and with the radius $CD = \frac{1}{2}a$, the circle LFM; make $ACE = \alpha$, and from the point F, in which the line CE cuts the circle, draw FG parallel to AB, which meets the semicircle in G. Draw CG, and through the point H, in which the circle is cut by this line, draw IK parallel to AB till it meets the semicircle; then draw KC, cutting the circle in L, M; bisect the angle ECK by CN, and from the point N, where it meets the semicircle, draw the lines NL, NM: then LNM is the triangle sought.



Draw the perpendiculars Ff, Gg, Hh, Ii, Kk; then $Gg = Ff = \frac{1}{2}a \text{ Sin. } \alpha$. Further, $CG : CH = Gg : Hh$, or $b : \frac{1}{2}a = \frac{1}{2}a \text{ Sin. } \alpha : Hh$, and $\therefore Hh = \frac{a^2 \text{ Sin. } \alpha}{4b} = Kk$, and $\text{Sin. } BCK = \frac{Kk}{CK} = \frac{a^2 \text{ Sin. } \alpha}{4b^2} = \text{Sin. } ACK$. But likewise by the analytical solution, $\text{Sin. } (2\phi + \alpha) = \frac{a^2 \text{ Sin. } \alpha}{4b^2}$; $\therefore 2\phi + \alpha = ACK$, and $\phi = \frac{1}{2} ECK = NCM$. The rest is self-evident.

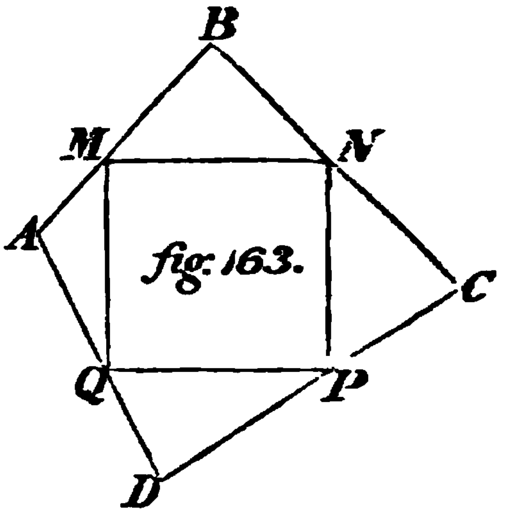
Besides the angle ACK, likewise the angle $ACI = 2\phi + \alpha$, $\therefore$ the angle $ICE = -2\phi$. But since there was no negative angle required here for ϕ, consequently in this case $KCN = \phi$ only.

SECTION CXLIII.

Prob. *To describe a square in a given quadrilateral.*

Solut. Let $ABCD$ (*fig.* 163) be the given quadrilateral, $MNPQ$ the square sought.

1. Since the quadrilateral is given: then also are its four angles, A, B, C, D; and the sides AB, BC are known. Let ∴ $AB = a$, $BC = b$. If now we knew the sides of the square, and only one of the angles which its sides make with the sides of the quadrilateral, then all the rest would be determined, and we could describe the square. Put ∴ $MN = x$, $BMN = \phi$.



2. From ϕ and the given angles, all the other angles o. the figure may be found. For since $QMN = R$: then $AMQ = 90^\circ - \phi$, and ∴ $AQM = 180^\circ - A - AMQ = 90^\circ - (A - \phi)$. Further, $BNM = 180^\circ - (B + \phi)$, and ∴, because MNP is a right angle, $CNP = B + \phi - 90^\circ$, and $CPN = 180^\circ - C - CNP = 270^\circ - (B + C + \phi)$. The other angles are not required in this case.

3. The triangle BMN gives

$$BM = \frac{MN \text{ Sin. } BNM}{\text{Sin. } MBN} = \frac{x \text{ Sin. } (B + \phi)}{\text{Sin. } B},$$

$$BN = \frac{MN \text{ Sin. } BMN}{\text{Sin. } MBN} = \frac{x \text{ Sin. } \phi}{\text{Sin. } B}.$$

The triangle MAQ gives

$$AM = \frac{MQ \text{ Sin } AQM}{\text{Sin. } MAQ} = \frac{x \text{ Cos. } (A - \phi)}{\text{Sin. } A},$$

and the triangle NCP,

$$CN = \frac{NP \text{ Sin. } CPN}{\text{Sin. } NCP} = - \frac{x \text{ Cos. } (B + C + \phi)}{\text{Sin. } C}.$$

4. Now since $AM + BM = AB = a$, $BN + CN = BC = b$; we then have the two equations

$$\frac{x \text{ Cos. } (A - \phi)}{\text{Sin. } A} + \frac{x \text{ Sin. } (B + \phi)}{\text{Sin. } B} = a,$$

$$\frac{x \text{ Sin. } \phi}{\text{Sin. } B} - \frac{x \text{ Cos. } (B + C + \phi)}{\text{Sin. } C} = b,$$

and hence, by eliminating x, we obtain

$$a\,Sin.\,A\,Sin.\,C\,Sin.\,\phi - a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C+\phi)$$
$$= b\,Sin.\,B\,Sin.\,C\,Cos.(A-\phi) + b\,Sin.\,A\,Sin.\,C\,Sin.\,(B+\phi).$$

5. If we solve $Cos.\,(B + C + \phi)$, $Cos.\,(A - \phi)$, $Sin.\,(B + \phi)$, and take away ϕ from the remaining magnitudes; this gives

$$Sin.\,\phi \begin{bmatrix} a\,Sin.\,A\,Sin.\,C + a\,Sin.\,A\,Sin.\,B\,Sin.\,(B+C) \\ -b\,Sin.\,A\,Sin.\,B\,Sin.\,C - b\,Sin.\,A\,Sin.\,C\,Cos.\,B \end{bmatrix}$$
$$= Cos.\,\phi \begin{bmatrix} b\,Sin.\,A\,Sin.\,B\,Sin.\,C + b\,Sin.\,B\,Sin.\,C\,Cos.\,A \\ + a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C) \end{bmatrix}$$

and hence we obtain

$$Tan.\,\phi = \frac{\begin{bmatrix} b\,Sin.\,A\,Sin.\,B\,Sin.\,C + b\,Sin.\,B\,Sin.\,C\,Cos.\,A \\ + a\,Sin.\,A\,Sin.\,B\,Cos.\,(B+C) \end{bmatrix}}{\begin{bmatrix} a\,Sin.\,A\,Sin.\,C + a\,Sin.\,A\,Sin.\,B\,Sin.\,(B+C) \\ - b\,Sin.\,A\,Sin.\,B\,Sin.\,C - b\,Sin.\,A\,Sin.\,C\,Cos.\,B \end{bmatrix}}$$

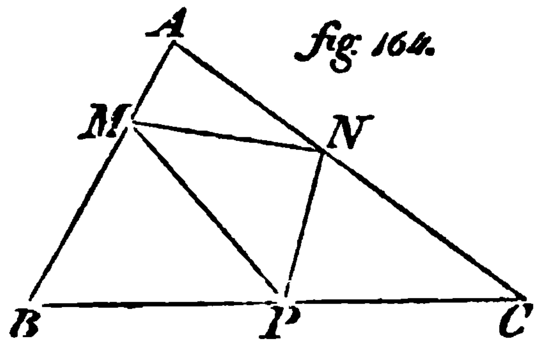
or, if the numerator and denominator be divided by $Sin.\,A$,

$$Tan.\,\phi = \frac{Sin.\,B\,[b\,Sin.\,C + b\,Sin.\,C\,Cot.\,A + a\,Cos.\,(B+C)]}{a\,Sin.\,C + a\,Sin.\,B\,Sin.(B+C) - b\,Sin.\,B\,Sin.\,C - b\,Sin.\,C\,Cos.\,B}.$$

SECTION CXLIV.

PROB. *In a given triangle to describe another given triangle.*

SOLUT. Let ABC (*fig.* 164) be the given triangle, in which the triangle MNP, which is also given, is so inscribed, that its angular points touch the sides of the former.



1. Since the two triangles are given; we then likewise know their angles and sides. Let ∴ $BAC = A$, $ABC = B$, $ACB = C$, $NMP = m$, $MNP = n$, $AB = a$, $MN = f$,

$MP = g$. If we know the angle AMN; then the problem is solved. Let $\therefore$ $ANM = \phi$. From this and the given angles we may now determine all the rest. Thus we have $ANM = 180^\circ - (A+\phi)$, $BMP = 180^\circ - (NMP + AMN) = 180^\circ - (m + \phi)$, $BPM = 180^\circ - BMP - MBP = m - B + \phi$.

2. Hence we further get,

$$AM = \frac{MN \text{ Sin. } ANM}{\text{Sin. } MAN} = \frac{f \text{ Sin. } (A + \phi)}{\text{Sin. } A},$$

$$BM = \frac{MP \text{ Sin. } BPM}{\text{Sin. } MBP} = \frac{g \text{ Sin. } (m - B + \phi)}{\text{Sin. } B}.$$

3. Now since $AB = AM + BM$: we consequently have the equation

$$\frac{f \text{ Sin. } (A + \phi)}{\text{Sin. } A} + \frac{g \text{ Sin. } (m - B + \phi)}{\text{Sin. } B} = a,$$

or, when we expand $\text{Sin. } (A + \phi)$, $\text{Sin. } (m - B + \phi)$, and arrange the expressions properly,

$$[f \text{ Sin. } A \text{ Sin. } B + g \text{ Sin. } A \text{ Sin. } (m - B)] \text{ Cos. } \phi + [f \text{ Sin. } B \text{ Cos. } A + g \text{ Sin. } A \text{ Cos. } (m - B)] \text{ Sin. } \phi = a \text{ Sin. } A \text{ Sin. } B,$$

and when we divide this equation by $\text{Sin. } A$,

$$[f \text{ Sin. } B + g \text{ Sin. } (m - B)] \text{ Cos. } \phi + [f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)] \text{ Sin. } \phi = a \text{ Sin. } B$$

4. Divide this equation by $f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)$, and put

$$\frac{f \text{ Sin. } B + g \text{ Sin. } (m - B)}{f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)} = \text{Tan. } \mu;$$

$\therefore$ find an angle μ such, that its tangent is equal to the expression on the left side of this equation (a method which has already been frequently used for shortening the calculation): we then obtain

$$\text{Tan. } \mu \text{ Cos. } \phi + \text{Sin. } \phi = \frac{a \text{ Sin. } B}{f \text{ Sin. } B \text{ Cot. } A + g \text{ Cos. } (m - B)}$$

or,

$$Sin.\,\mu\, Cos.\,\phi + Cos.\,\mu\, Sin.\,\phi = \frac{a\ Sin.\ B\ Cos.\ \mu}{f Sin.\, B\, Cot.\, A + g Cos.\, (m - B)}$$

or lastly

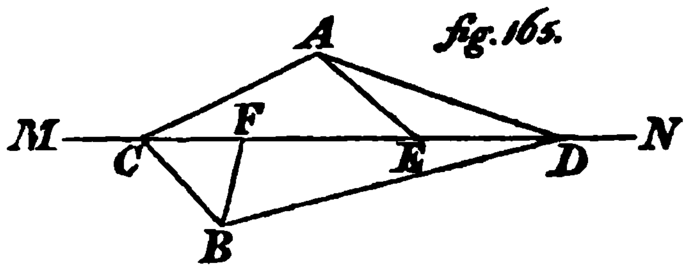
$$Sin.\ (\phi + \mu) = \frac{a\ Sin.\ B\ Cos.\ \mu}{f\ Sin.\ B\ Cot.\ A\ + g\ Cos.\ (m - B)}.$$

From this equation we may now very easily determine $\phi + \mu$, and consequently likewise ϕ.

SECTION CXLV.

PROB. *Two points and a straight line are given: find two points in this line, such, that when lines are drawn from these to the former, the angles which these lines contain at the given points, have a given magnitude.*

SOLUT. Let A, B, (*fig.* 165), be the given points, MN a line given in position; it is required to find two points C, D, in it, such, that when the lines AC, AD, BC, BD, are drawn, the angle $CAD = \alpha$, and $CBD = \beta$.



1. From A, B, to MN, draw the lines AE, BF, containing the angles $AED = CAD = \alpha$, $BFD = CBD = \beta$. The lines FE, AE, BF, may then be considered as known, and let $\therefore$ $FE = a$, $AE = b$, $BF = c$. In order to determine the points C, D, put $DE = x$.

2. Now it is easily seen, that $\triangle\, AED$ is similar to $\triangle\, CAD$, and $\triangle\, BFD$ is similar to $\triangle\, CBD$: we $\therefore$ have $CD : AD = AD : DE$, $CD : BD = BD : DF$, and consequently

$$CD = \frac{AD^2}{DE},\ CD = \frac{BD^2}{DF},$$

or, since $DE = x$, $DF = a + x$, $AD^2 = b^2 + x^2 - 2\,bx \times$ $Cos.\,\alpha$, $BD^2 = c^2 + (a+x)^2 - 2c\,(a+x)\,Cos.\,\beta$; therefore

$$CD = \frac{b^2 + x^2 - 2bx\,Cos.\,\alpha}{x}$$

$$CD = \frac{c^2 + (a+x)^2 - 2c\,(a+x)\,Cos.\,\beta}{a+x}.$$

3. If these two expressions for CD are put equal to one another; we then obtain, after the usual reductions, the following equation:

$$(a + 2b\,Cos.\,\alpha - 2c\,Cos.\,\beta)\,x^2 + (a^2 + c^2 - b^2 + 2ab\,Cos.\,\alpha - 2ac\,Cos.\,\beta)\,x = ab^2$$

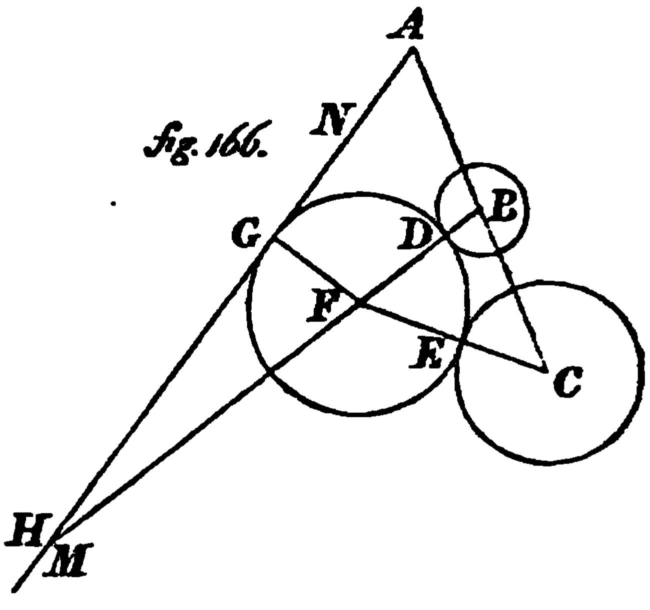
whence x may be determined.

SECTION CXLVI.

PROB. *The position and magnitudes of two circles are given, and also the position of a straight line: describe a circle, which touches these two circles of the straight line.*

SOLUT. Let B, C, (*fig.* 166) be the centres of the two given circles, F the centre of the required circle, which touches these two circles in D, E, and also the straight line MN given in position in G.

fig. 166.



1. Through the centres B, C, draw a line BC, which meets MN, in A; further, draw the lines FB, FC, FG, the two first of which necessarily pass through the points of contact D, E, (*Euc. III.* 12), and the last is perpendicular to MN. Since the circles, whose centres are B, C, and the line MN have a given position; consequently the angle MAC, the lines AB, BC are given. Put $\therefore MAC = \alpha$, $AB = a$, $BC = b$. Further, let the radius $BD = r$, and the radius

$CE = R$. Now draw the lines BF, CF, and produce the former, till it meets MN in H. If we can now determine the angle CBH, and the radius FD, we can then also find BF: $\therefore$ we have the centre and the radius of the required circle, and consequently the problem is solved. Let $\therefore$ $CBH = \phi$, $FD = x$.

2. From these data we obtain $FB = r + x$, $FC = R + x$, $AHB = \phi - \alpha$. We $\therefore$ have

$$BH = \frac{AB \text{ Sin. } BAH}{\text{Sin. } AHB} = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)},$$

consequently

$$FH = BH - BF = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)} - (r + x).$$

But likewise

$$FH = \frac{GF}{\text{Sin. } AHB} = \frac{x}{\text{Sin. } (\phi - \alpha)};$$

we $\therefore$ have the equation

$$\frac{x}{\text{Sin. } (\phi - \alpha)} = \frac{a \text{ Sin. } \alpha}{\text{Sin. } (\phi - \alpha)} - (r + x),$$

whence we obtain

$$x = \frac{a \text{ Sin. } \alpha - r \text{ Sin. } (\phi - \alpha)}{1 + \text{Sin. } (\phi - \alpha)}.$$

3. Further, in the triangle BFC,

$$CF^2 = BF^2 + BC^2 - 2\, BC \,.\, BF \text{ Cos. } CBF$$

or

$$(x + R)^2 = (x + r)^2 + b^2 - 2\, b\, (x + r) \text{ Cos. } \phi.$$

Hence we obtain

$$x = \frac{r^2 + b^2 - R^2 - 2\, br \text{ Cos. } \phi}{2\, (R - r + b \text{ Cos. } \phi)}.$$

4. If we equate the two expressions found for x in 2, 3, we then obtain, after the usual reduction, the equation

$$2\, b\, (r + a \text{ Sin. } \alpha) \text{ Cos. } \phi + (R^2 - 2\, R\, r + r^2 - b^2) \text{ Sin. } (\phi - \alpha)$$
$$= r^2 + b^2 - R^2 - 2\, a\, (R - r) \text{ Sin. } \alpha.$$

Now $Sin. (\phi - \alpha) = Sin. \phi \; Cos. \alpha - Cos. \phi \; Sin. \alpha$, $R^2 - 2Rr + r^2 - b^2 = (R - r)^2 - b^2 = (R - r + b)\,(R - r - b)$. If these values be substituted in the equation already found, it is transformed into the following one:

$$[2b(r + a \, Sin. \alpha) - (R - r + b)(R - r - b) \, Sin. \alpha] \, Cos. \phi + (R - r + b)(R - r - b) \, Cos. \alpha \; Sin. \phi = r^2 + b^2 - R^2 - 2a(R - r) \, Sin. \alpha.$$

5. Divide the equation by $(R - r + b)\,(R - r - b)\, Cos. \alpha$, and put

$$\frac{2b\,(r + a \, Sin. \alpha) - (R - r + b)\,(R - r - b)\, Sin. \alpha}{(R - r + b)\,(R - r - b)\, Cos. \alpha} = Tan. \mu ;$$

this gives

$$Tan. \mu \; Cos. \phi + Sin. \phi = \frac{r^2 + b^2 - R^2 - 2a(R - r)\, Sin. \alpha}{(R - r + b)\,(R - r - b)\, Cos. \alpha}$$

and when both sides of the equation are multiplied by $Cos. \mu$, and $Sin. (\phi + \mu)$ substituted for $Sin. \mu \; Cos. \phi + Cos. \mu \; Sin. \phi$

$$Sin. (\phi + \mu) = \frac{[r^2 + b^2 - R^2 - 2a(R - r)\, Sin. \alpha]\, Cos. \mu}{(R - r + b)\,(R - r - b)\, Cos. \alpha}.$$

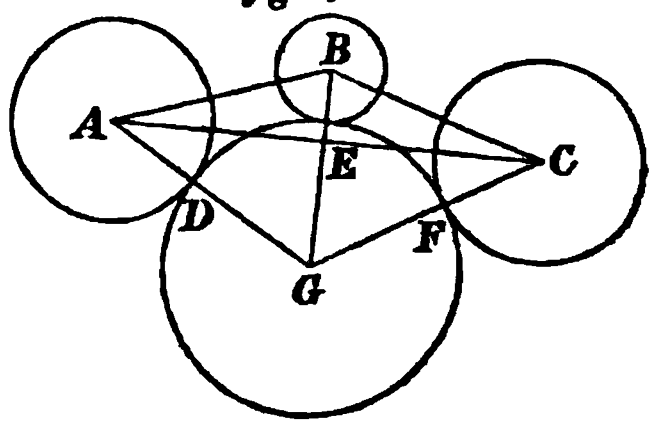
Hence $\phi + \mu$, and consequently also ϕ, may be determined.

SECTION CXLVII.

PROB. *The positions and magnitudes of three circles are given: describe a circle, which touches these circles.*

SOLUT. Let the centres of the given circles be A, B, C (*fig.* 167), and G the centre of the circle, which touches the given ones in D, E, F. With regard to the radii of the given circles, let $AD = R$, $BE = r$, $CF = \rho$; the radius of the required circle $GD = GE = GF = x$. Since the three circles are given in position, consequently the straight lines, which connect

fig. 167.



their centres, and the angle which these lines contain, are likewise given. Let ∴ $AB = a$, $BC = b$, and the angle $ABC = B$.

1. To the centre G of the required circle, draw the lines AG, BG, CG, consequently these pass through the points of contact. If we now put the unknown angle $ABG = \phi$; then from the angles AGB, BGC, we have the two following equations:

$$AG^2 = AB^2 + BG^2 - 2\,AB\,.\,BG\ Cos.\ \phi$$
$$CG^2 = BC^2 + BG^2 - 2\,BC\,.\,BG\ Cos.\ (B - \phi);$$

or, since $AG = R + x$, $BG = r + x$, $CG = \rho + x$,

$$(R + x)^2 = a^2 + (r + x)^2 - 2\,a\,(r + x)\ Cos.\ \phi$$
$$(\rho + x)^2 = b^2 + (r + x)^2 - 2\,b\,(r + x)\ Cos.\ (B - \phi).$$

From the first of these two equations, we get

$$x = \frac{a^2 + r^2 - R^2 - 2\,a\,r\ Cos.\ \phi}{2\,(R - r + a\ Cos.\ \phi)},$$

and from the second,

$$x = \frac{b^2 + r^2 - \rho^2 - 2\,b\,r\ Cos.\ (B - \phi)}{2\,[\rho - r + b\ Cos.\ (B - \phi)]}.$$

2. If the two expressions found for x are equated, we then obtain, after the proper reduction, the following equation:

$$a\,(r^2 - 2\,r\,\rho + \rho^2 - b^2)\ Cos.\,\phi - b\,(R^2 - 2\,Rr + r^2 - a^2) \times$$
$$Cos.\ (B - \phi) = (b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho)$$

or, since $r^2 - 2\,r\rho + \rho^2 - b^2 = (r - \rho)^2 - b^2 = (r - \rho + b) \times (r - \rho - b)$, and $R^2 - 2\,Rr + r^2 - a^2 = (R - r)^2 - a^2 = (R - r + a)\,(R - r - a)$,

$$a\,(r - \rho + b)\,(r - \rho - b)\ Cos.\ \phi - b\,(R - r + a)\,(R - r - a)$$
$$Cos.\,(B - \phi) = (b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho).$$

Now expand $Cos.\,(B - \phi)$, and divide the whole equation by $b\,(R - r + a)\,(R - r - a)\ Sin.\ B$. This gives, when $Cot.\ B$ is substituted for $\frac{Cos.\ B}{Sin.\ B}$,

$$\left[\frac{a\,(r - \rho + b)\,(r - \rho - b)}{b\,(R - r + a)\,(R - r - a)\,Sin.\,B} - Cot.\ B\right] Cos.\,\phi - Sin.\,\phi$$
$$= \frac{(b^2 + r^2 - \rho^2)\,(R - r) + (a^2 + r^2 - R^2)\,(r - \rho)}{b\,(R - r + a)\,(R - r - a)\ Sin.\ B}.$$

3. Now put

$$\frac{a(r-\rho+b)(r-\rho-b)}{b(R-r+a)(R-r-a)\,Sin.\,B}-Cot.\,B=Tan.\,\mu$$

then μ is a known angle, and we have

$$Tan.\,\mu\,Cos.\,\phi - Sin.\,\phi =$$

$$\frac{(b^2+r^2-\rho^2)(R-r)+(a^2+r^2-R^2)(r-\rho)}{b(R-r+a)(R-r-a)\,Sin.\,B},$$

and when both sides of this equation are multiplied by $Cos.\,\phi$, and $Sin.\,(\mu-\phi)$ substituted for $Sin.\,\mu\,Cos.\,\phi - Cos.\,\mu\,Sin.\,\phi$,

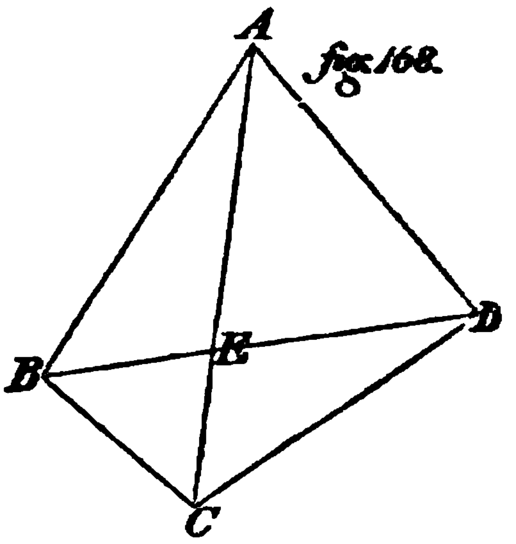
$$Sin.(\mu-\phi)=\frac{[(b^2+r^2-\rho^2)(R-r)+(a^2+r^2-R^2)(r-\rho)]Cos.\mu}{b(R-r+a)(R-r-a)}.$$

Hence $\mu-\phi$, and consequently also ϕ, may be determined.

SECTION CXLVIII.

PROB. *The four sides of a quadrilateral, and its two diagonals, are given: find the segments of these diagonals.*

SOLUT. In the quadrilateral $ABCD$ (*fig.* 168), the four sides AB, BC, CD, AD, and the two diagonals AC, BD, are given; find the segments AE, CE, BE, DE. Let $AB=a$, $BC=b$, $CD=c$, $AD=d$, $AC=f$, $BD=g$, $AE=x$, $BE=y$, and $\therefore CE=f-x$, $DE=g-y$.



1. From the four triangles BAD, BCD, ABC, ADC,

$$Cos.\,ABD=\frac{AB^2+BD^2-AD^2}{2AB\,.\,BD}=\frac{a^2+g^2-d^2}{2ag}$$

$$Cos.\,CBD=\frac{BC^2+BD^2-CD^2}{2BC\,.\,BD}=\frac{b^2+g^2-c^2}{2bg}$$

$$Cos.\,ACB = \frac{BC^2 + AC^2 - AB^2}{2BC\,.\,AC} = \frac{b^2 + f^2 - a^2}{2bf}$$

$$Cos.\,ACD = \frac{CD^2 + AC^2 - AD^2}{2CD\,.\,AC} = \frac{c^2 + f^2 - d^2}{2cf}.$$

Further, the three angles AEB, BEC, CED, give the four following equations:

$$\begin{aligned} AE^2 &= AB^2 + BE^2 - 2AB\,.\,BE\ Cos.\,ABD \\ CE^2 &= BC^2 + BE^2 - 2BC\,.\,BE\ Cos.\,CBD \\ BE^2 &= BC^2 + CE^2 - 2BC\,.\,CE\ Cos.\,ACB \\ DE^2 &= CD^2 + CE^2 - 2CD\,.\,CE\ Cos.\,ACD \end{aligned}$$

or, when for the lines and cosines their values are substituted, the four following:

$$\begin{aligned} x^2 &= a^2 + y^2 - \frac{a^2 + g^2 - d^2}{g}y \\ (f - x)^2 &= b^2 + y^2 - \frac{b^2 + g^2 - c^2}{g}y \\ y^2 &= b^2 + (f - x)^2 - \frac{b^2 + f^2 - a^2}{f}(f - x) \\ (g - y)^2 &= c^2 + (f - x)^2 - \frac{c^2 + f^2 - d^2}{f}(f - x). \end{aligned}$$

2. If the first equation be subtracted from the second, and the third from the fourth, we then obtain the two following equations:

$$f^2 - 2fx = b^2 - a^2 + \frac{a^2 - b^2 + c^2 - d^2}{g}y$$

$$g^2 - 2gy = c^2 - b^2 - \frac{a^2 - b^2 + c^2 - d^2}{f}(f - x);$$

and hence again, by eliminating y,

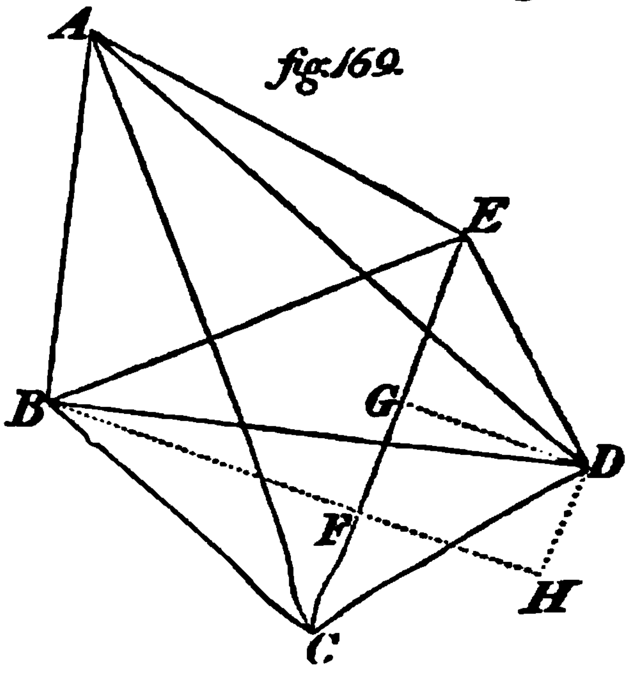
$$x = \frac{f(g^2 + a^2 - d^2)(a^2 - b^2 + c^2 - d^2) - 2fg^2(f^2 + a^2 - b^2)}{(a^2 - b^2 + c^2 - d^2)^2 - 4f^2g^2}.$$

Similar expressions are also obtained for the segments CE, BE, DE.

SECTION CXLIX.

Prob. *From the three angles of a given triangle, a tower is seen, whose base is in the same plane with the triangle; the angles at which it is seen from these, are given: find the distance of the tower from each of these three points.*

Solut. Let AB (*fig.* 169) be the tower, CDE the given triangle; the angles at which the tower is seen from the given points C, D, E, are ACB, ADB, AEB, and let $ACB = \alpha$, $ADB = \beta$, $AEB = \gamma$: find the distances BC, BD, BE.



1. From B, D draw BF, DG perpendicular to CE, and from D draw DH parallel to CE, which meets BF produced in H. Since the triangle CDE is given; consequently the lines CE, CG, DG, are known. Let $\therefore$ $CE = a$, $CG = b$, $DG = FH = c$. If only the lines CF, BF are also determined; we then have likewise the distances sought. Put $\therefore$ $CF = x$, $BF = y$. This gives $BH = y + c$, $DH = GF = b - x$, $EF = a - x$. We consequently have, $BC^2 = BF^2 + CF^2 = y^2 + x^2$, $BD^2 = BH^2 + DH^2 = y^2 + 2cy + c^2 + b^2 - 2bx + x^2$, $BE^2 = BF^2 + EF^2 = y^2 + a^2 - 2ax + x^2$.

2. Since $AB = BC\ Tan.\ \alpha = BD\ Tan.\ \beta = BE\ Tan.\ \gamma$: then also $BC^2\ Tan.\ {}^2\alpha = BD^2\ Tan.\ {}^2\beta = BE^2\ Tan.\ {}^2\gamma$. Substitute now for BC^2, BD^2, BE^2, their values found in 1; hence arise the following equations;

$$Tan.\ {}^2\alpha\ (y^2 + x^2)$$
$$= Tan.\ {}^2\beta\ (y^2 + x^2 + 2cy - 2bx + b^2 + c^2)$$
$$Tan.\ {}^2\alpha\ (y^2 + x^2) = Tan.\ {}^2\gamma\ (y^2 + x^2 - 2ax + a^2),$$

or

$$\frac{Tan.\,^2\alpha - Tan.\,^2\beta}{Tan.\,^2\beta}(y^2 + x^2) = 2cy - 2bx + b^2 + c^2$$

$$\frac{Tan.\,^2\alpha - Tan.\,^2\gamma}{Tan.\,^2\gamma}(y^2 + x^2) = -2ax + a^2.$$

3. By trigonometry

$$Tan.\,^2\alpha - Tan.\,^2\beta = \frac{Sin.\,(\alpha + \beta)\ Sin.\,(\alpha - \beta)}{Cos.\,^2\alpha\ Cos.\,^2\beta}$$

$$Tan.\,^2\alpha - Tan.\,^2\gamma = \frac{Sin.\,(\alpha + \gamma)\ Sin.\,(\alpha - \gamma)}{Cos.\,^2\alpha\ Cos.\,^2\gamma};$$

further,

$$Tan.\,^2\beta = \frac{Sin.\,^2\beta}{Cos.\,^2\beta},\quad Tan.\,^2\gamma = \frac{Sin.\,^2\gamma}{Cos.\,^2\gamma}.$$

4. The substitution of these values in 2, gives the two following equations:

$$\frac{Sin.\,(\alpha+\beta)\ Sin.\,(\alpha-\beta)}{Cos.\,^2\alpha\ Sin.\,^2\beta}(y^2 + x^2) = 2cy - 2bx + b^2 + c^2$$

$$\frac{Sin.\,(\alpha+\gamma)\ Sin.\,(\alpha-\gamma)}{Cos.\,^2\alpha\ Sin.\,^2\gamma}(y^2 + x^2) = -2ax + a^2.$$

5. We eliminate $y^2 + x^2$, by multiplying the first equation by $\frac{Sin.\,(\alpha + \gamma)\ Sin.\,(\alpha - \gamma)}{Sin.\,^2\gamma}$, and the second by $\frac{Sin.\,(\alpha + \beta)\ Sin.\,(\alpha - \beta)}{Sin.\,^2\beta}$, and then subtract them from one another: this gives the following equation:

$$(2cy - 2bx + b^2 + c^2)\frac{Sin.\,(\alpha + \gamma)\ Sin.\,(\alpha - \gamma)}{Sin.\,^2\gamma} -$$

$$(a^2 - 2\,ax)\frac{Sin.\,(\alpha + \beta)\ Sin.\,(\alpha - \beta)}{Sin.\,^2\beta} = 0.$$

Hence we obtain

$$y = \left[\frac{b}{c} - \frac{a\,Sin.\,^2\gamma\;Sin.\,(\alpha+\beta)\;Sin.\,(\alpha-\beta)}{c\,Sin.\,^2\beta\;Sin.\,(\alpha+\gamma)\;Sin.\,(\alpha-\gamma)}\right] x +$$

$$\frac{a^2\,Sin.\,^2\gamma\;Sin.\,(\alpha+\beta)\;Sin.\,(\alpha-\beta)}{2\,c\,Sin.\,^2\beta\;Sin.\,(\alpha+\gamma)\;Sin.\,(\alpha-\gamma)} - \frac{b^2+c^2}{2c}.$$

If we substitute this value in the second equation in 4, we then obtain a quadratic equation for x, whence x, and consequently also y, may be determined.

6. For the sake of brevity, put

$$\frac{a\,Sin.\,^2\gamma\;Sin.\,(\alpha+\beta)\;Sin.\,(\alpha-\beta)}{c\,Sin.\,^2\beta\;Sin.\,(\alpha+\gamma)\;Sin.\,(\alpha-\gamma)} = A,$$

$$\frac{Sin.\,(\alpha+\gamma)\;Sin.\,(\alpha-\gamma)}{Cos.\,^2\alpha\;Sin.\,^2\gamma} = B,$$

we consequently have,

$$y = \left[\frac{b}{c} - A\right] x + \tfrac{1}{2}\,a\,A - \frac{b^2+c^2}{2c}$$

$$B\,(y^2 + x^2) = -\,2\,a\,x + a^2.$$

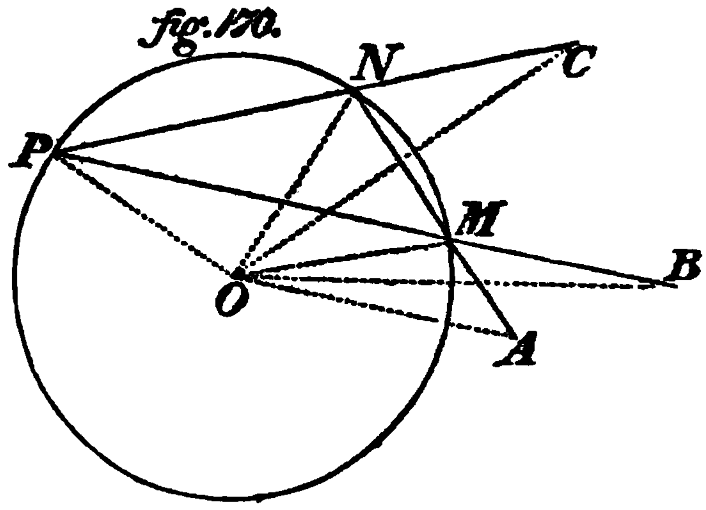
The expressions A, B may be very easily calculated by means of logarithms; the further *calculation will be best* managed by the given numbers themselves.

SECTION CL.

PROB. *A circle is given in magnitude and position: inscribe a triangle within it, whose sides, or their parts produced, pass through three given points.*

SOLUT. Let O (*fig.* 170) be the centre of the given circle, MNP the required triangle, whose sides pass through the three given points A, B, C.

fig. 170.



1. Draw the lines OA, OB, OC: then these, because the points A, B, C, O, are given, are in like manner given in magnitude

and position. Therefore $OA = a$, $OB = b$, $OC = c$, $AOB = \alpha$, $AOC = \beta$.

2. Draw the radii OM, ON, OP; then these lines are radii of the given circle, and consequently given in magnitude, but the position is unknown: $\therefore$ these must be found. Therefore put $AOM = \phi$, $AON = \psi$, $AOP = \zeta$; further, let the radius of the circle $= r$.

3. In the isosceles triangle MON, the angle $MON = \psi - \phi$; consequently $ONM = \dfrac{180^\circ - MON}{2} = 90^\circ - \frac{1}{2} (\psi - \phi)$. Further, in the triangle AON, the angle $OAN = 180^\circ - (ONM + AON) = 90^\circ - \frac{1}{2}(\psi + \phi)$.

4. The triangle AON gives the proportion,

$$AO : ON = Sin.\ ONM : Sin.\ OAN,$$

or

$$a : r = Cos.\ \tfrac{1}{2}(\psi - \phi) : Cos.\ \tfrac{1}{2}(\psi + \phi).$$

Hence we obtain the equation

$$a\ Cos.\ \tfrac{1}{2}(\psi + \phi) = r\ Cos.\ \tfrac{1}{2}(\psi - \phi),$$

or, since

$$Cos.\ \tfrac{1}{2}(\phi + \psi) = Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi - Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi,$$
$$Cos.\ \tfrac{1}{2}(\psi - \phi) = Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi + Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi,$$

the following equation

$$(a - r)\ Cos.\ \tfrac{1}{2}\psi\ Cos.\ \tfrac{1}{2}\phi = (a + r)\ Sin.\ \tfrac{1}{2}\psi\ Sin.\ \tfrac{1}{2}\phi.$$

Divide by $(a+r)\ Cos.\ \frac{1}{2}\psi\ Cos.\ \frac{1}{2}\phi$, and substitute $Tan.\ \frac{1}{2}\psi$ for $\dfrac{Sin.\ \frac{1}{2}\psi}{Cos.\ \frac{1}{2}\psi}$, $Tan.\ \frac{1}{2}\phi$ for $\dfrac{Sin.\ \frac{1}{2}\phi}{Cos.\ \frac{1}{2}\phi}$; this gives

$$Tan.\ \tfrac{1}{2}\psi\ Tan.\ \tfrac{1}{2}\phi = \frac{a - r}{a + r}.$$

5. A similar equation is found, when the triangles MOP, BOP are treated in the same way as the triangles MON, AON were in 3, 4. To obtain this, it is only necessary to substitute the line OB for the line OA, the line OP for the line ON, and the angles BOP, BOM, for the angles AON,

AOM, or b for a, $\zeta - \alpha$ for ψ, and $\phi - \alpha$ for ϕ. We then obtain from the equation found in 4, the following one:

$$Tan.\ \tfrac{1}{2}(\zeta - \alpha)\ Tan.\ \tfrac{1}{2}(\phi - \alpha) = \frac{b-r}{b+r}.$$

6. In like manner we find a third equation, when the triangles NOP, COP are substituted for the triangles MON, AON, and for this purpose, $OC = c$ is put for $OA = a$, $COP = \zeta - \beta$ for $AON = \psi$, and $CON = \psi - \beta$ for $AOM = \phi$. We consequently have

$$Tan.\ \tfrac{1}{2}(\zeta - \beta)\ Tan.\ \tfrac{1}{2}(\psi - \beta) = \frac{c-r}{c+r}.$$

7. For the sake of abbreviation, put $Tan.\ \tfrac{1}{2}\alpha = m$, $Tan.\ \tfrac{1}{2}\beta = n$, $Tan.\ \tfrac{1}{2}\phi = x$, $Tan.\ \tfrac{1}{2}\psi = y$, $Tan.\ \tfrac{1}{2}\zeta = z$; further $\dfrac{a-r}{a+r} = A$, $\dfrac{b-r}{b+r} = B$, $\dfrac{c-r}{c+r} = C$: then,

$Tan.\ \tfrac{1}{2}(\phi - \alpha) = \dfrac{x-m}{1+mx}$, $Tan.\ \tfrac{1}{2}(\zeta - \alpha) = \dfrac{z-m}{1+mz}$,

$Tan.\ \tfrac{1}{2}(\psi - \beta) = \dfrac{y-n}{1+ny}$, $Tan.\ \tfrac{1}{2}(\zeta - \beta) = \dfrac{z-n}{1+nz}$. The three equations in 4, 5, 6 are consequently transformed into the following ones:

$$xy = A$$

$$\frac{x-m}{1+mx} \times \frac{z-m}{1+mz} = B$$

$$\frac{y-n}{1+ny} \times \frac{z-n}{1+nz} = C.$$

8. From the first of these equations we obtain $y = \dfrac{A}{x}$, and from the second $z = \dfrac{B - m^2 + (1+B)mx}{(1-Bm^2)x - (1+B)m}$. If we substitute these values in the third equation, we then obtain, $C =$

$$\frac{A - nx}{nA + x}\left[\frac{B - m^2 + (1+B)mn + [(1+B)m - (1-Bm^2)n]x}{-(1+B)m + (B-m^2)n + [(1+B)mn + 1 - Bm^2]x}\right].$$

9. For the sake of brevity, put

$$B - m^2 + (1 + B)\,mn = M$$
$$(1+B)m - (1 - Bm^2)n = N$$
$$-(1+B)m + (B - m^2)\,n = P$$
$$(1+B)mn + (1 - Bm^2) = Q.$$

By these means we obtain the equation

$$\frac{A - nx}{nA + x} \times \frac{M + Nx}{P + Qx} = C,$$

or

$$(CQ + nN)\,x^2 + (nACQ + CP + nM - NA)\,x = AM - nACP;$$

whence x, and consequently also y and z may be determined. Hence further, the angles ϕ, ψ, ζ may be found.

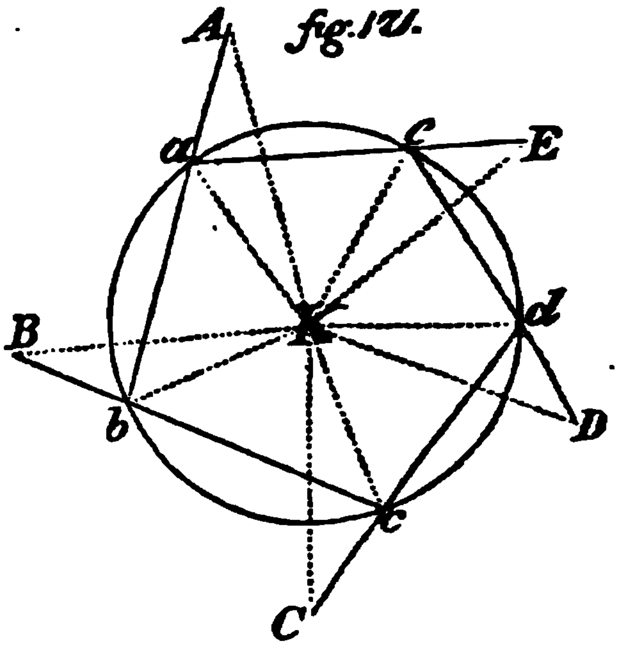
Remark. This apparently easy problem has long engaged the attention of the greatest geometricians. Cramer first proposed it to Castillon, and this last gave an elegant synthetic proof of it in the Berlin Memoirs of 1776. In the same volume there also appeared another, an Analytical Solution by Lagrange, which I have followed, with the exception of a few alterations in the notation. At Euler's request, Lexell, in the 4th vol. of the Petersburgh Memoirs, gave a geometrical construction of Lagrange's Formula. Altajano, at the age of sixteen, then gave a synthetic solution of this problem, in the 4th vol. of the Memoire della Societa Italiana, and likewise showed, how to describe generally in a given circle a polygon, whose sides pass through any number of given points. In the same volume, there are two more synthetic solutions, one by Malfatti, the other by Giordano. Romano also has a geometrical construction of Lagrange's formula (Nuovo Metodo di applicare alla Sintesi la Soluzione Analytica di qualunque Problema Geometrico. Venezia, 1793), which work I beg to recommend to all those who wish for practice in the geometrical construction of analytical formulæ. Carnot, in his Geometrie de Position, has, as well as Altajano, solved the general problem for the polygon in the following ingenious way.

SECTION CLI.

Prob. *A certain number of points and a circle are given: required to inscribe in this circle a polygon, consisting of as many sides as there are points, so that each side may respectively pass through one of these points.*

Solut. Let K (*fig.* 171) be the centre of the given circle,

abcde the required polygon, whose sides pass through the given points A, B, C, D, E.



1. Since the points A, B, C, D, E are given; consequently the lines KA, KB, KC, KD, KE, and the angles AKB, BKC, CKD, DKE, EKA are also given. Let $\therefore$ $KA = a$, $KB = b$, $KC = c$, $KD = d$, $KE = e$, $AKB = \alpha$, $BKC = \beta$, $CKD = \gamma$, $DKE = \delta$, $EKA = \epsilon$; further, let the radius of the circle $= r$. Now if the angles AKa, BKb, CKc, DKd, EKe are known; then the points a, b, c, d, e, are determined, and consequently the problem is solved. Put $\therefore$ $AKa = \tau$, $BKb = \upsilon$, $CKc = \phi$, $DKd = \chi$, $EKe = \psi$.

2. Then in the triangle AKb, we have

$$AK + bK : AB - bK =$$
$$Tan. \tfrac{1}{2}(AbK + bAK) : Tan. \tfrac{1}{2}(AbK - bAK).$$

Now $AbK + bAK = 180^\circ - AKb = 180^\circ - (\alpha + \upsilon)$, $AbK - bAK = baK - bAK = AKa = \tau$; consequently $Tan. \tfrac{1}{2}(AbK + bAK) = Tan. [90^\circ - \tfrac{1}{2}(\alpha + \upsilon)] = Cot. \tfrac{1}{2}(\alpha + \upsilon) = \dfrac{1}{Tan. \tfrac{1}{2}(\alpha + \upsilon)}$, $Tan. \tfrac{1}{2}(AbK - bAK) = Tan. \tfrac{1}{2}\tau$. We $\therefore$ have

$$a + r : a - r = \frac{1}{Tan. \tfrac{1}{2}(\alpha + \upsilon)} : Tan. \tfrac{1}{2}\tau,$$

or,

$$\frac{a - r}{a + r} = Tan. \tfrac{1}{2}\tau \; Tan. \tfrac{1}{2}(\alpha + \upsilon),$$

or,

$$\frac{a - r}{a + r} = Tan. \tfrac{1}{2}\tau \frac{Tan. \tfrac{1}{2}\alpha + Tan. \tfrac{1}{2}\upsilon}{1 - Tan. \tfrac{1}{2}\alpha \; Tan. \tfrac{1}{2}\upsilon};$$

whence we obtain,

$$Tan. \tfrac{1}{2}\tau = \frac{\dfrac{a - r}{a + r} - \dfrac{a - r}{a + r} Tan. \tfrac{1}{2}\alpha \; Tan. \tfrac{1}{2}\upsilon}{Tan. \tfrac{1}{2}\alpha + Tan. \tfrac{1}{2}\upsilon}$$

3. A similar equation is found for each side of the polygon. Thus we obtain as many equations as the polygon has sides, and these equations are:

$$Tan.\ \tfrac{1}{2}\tau = \frac{\dfrac{a-r}{a+r} - \dfrac{a-r}{a+r}\,Tan.\ \tfrac{1}{2}\alpha\ Tan.\ \tfrac{1}{2}\upsilon}{Tan.\ \tfrac{1}{2}\alpha + Tan.\ \tfrac{1}{2}\upsilon}$$

$$Tan.\ \tfrac{1}{2}\upsilon = \frac{\dfrac{b-r}{b+r} - \dfrac{b-r}{b+r}\,Tan.\ \tfrac{1}{2}\beta\ Tan.\ \tfrac{1}{2}\phi}{Tan.\ \tfrac{1}{2}\beta + Tan.\ \tfrac{1}{2}\phi}$$

$$Tan.\ \tfrac{1}{2}\phi = \frac{\dfrac{c-r}{c+r} - \dfrac{c-r}{c+r}\,Tan.\ \tfrac{1}{2}\gamma\ Tan.\ \tfrac{1}{2}\chi}{Tan.\ \tfrac{1}{2}\gamma + Tan.\ \tfrac{1}{2}\chi}$$

$$Tan.\ \tfrac{1}{2}\chi = \frac{\dfrac{d-r}{d+r} - \dfrac{d-r}{d+r}\,Tan.\ \tfrac{1}{2}\delta\ Tan.\ \tfrac{1}{2}\psi}{Tan.\ \tfrac{1}{2}\delta + Tan.\ \tfrac{1}{2}\psi}$$

$$Tan.\ \tfrac{1}{2}\psi = \frac{\dfrac{e-r}{e+r} - \dfrac{e-r}{e+r}\,Tan.\ \tfrac{1}{2}\varepsilon\ Tan.\ \tfrac{1}{2}\tau}{Tan.\ \tfrac{1}{2}\varepsilon + Tan.\ \tfrac{1}{2}\tau}$$

4. By means of these equations, the number of which is always equal to the number of the unknown magnitudes, it is easy to determine these last. If, for instance, we wish to find ϕ: it is merely necessary to substitute in the equation for χ its value taken from the fourth, by which we obtain an equation between ϕ and ψ. From this again we then obtain an equation between ϕ and τ, by substituting for ψ its value taken from the fourth. From this again we get an equation between ϕ and υ, by substituting for τ its value in terms of υ taken from the first equation, and lastly, we obtain an equation, which only contains ϕ, by substituting for υ its value in terms of ϕ taken from the second equation. The solution of this last equation then gives the value of ϕ.

5. For the sake of brevity, put $Tan.\ \tfrac{1}{2}\tau = \tau'$, $Tan.\ \tfrac{1}{2}\upsilon = \upsilon'$, $Tan.\ \tfrac{1}{2}\phi = \phi'$, $Tan.\ \tfrac{1}{2}\chi = \chi'$, $Tan.\ \tfrac{1}{2}\psi = \psi'$; by these means the equations already found take the following form:

$$\tau' = \frac{A \;\; + B\upsilon'}{C \;\; + D\upsilon'}$$

$$\upsilon' = \frac{A' \;\; + B'\phi'}{C' \;\; + D'\phi'}$$

$$\phi = \frac{A'' \;\; + B''\chi'}{C'' \;\; + D''\chi'}$$

$$\chi' = \frac{A''' + B'''\psi'}{C''' + D'''\psi'}$$

$$\psi' = \frac{A^{IV} + B^{IV}\tau'}{C^{IV} + D^{IV}\tau'}$$

If in the first of these equations we substitute for υ' its value in the second, we then obtain an equation between τ' and ϕ', of the form $\tau' = \frac{A^{V} + B^{V}\phi'}{C^{V} + D^{V}\phi'}$. If in this again we substitute for ϕ' its value in the third equation; we then obtain an equation between τ' and χ' of the same form, and by continuing these substitutions, we at last arrive at an equation of a similar form, which contains τ' only, viz. $\tau' = \frac{M + N\tau'}{P + Q\tau'}$, or the equation of the *second degree* $Q\tau'^2 + (P - N)\tau' - M = 0$. Hence follows the very important result, that, however great the number of the given points is, the final equation, for each of the magnitudes, never exceeds the second degree.

TABLES OF TRIGONOMETRICAL FORMULÆ.

1. $Sin.^2\phi + Cos.^2\phi = 1$
2. $Tan.\,\phi = \frac{Sin.\,\phi}{Cos.\,\phi}$
3. $Cot.\,\phi = \frac{Cos.\,\phi}{Sin.\,\phi}$
4. $Sec.\,\phi = \frac{1}{Cos.\,\phi}$
5. $Cosec.\,\phi = \frac{1}{Sin.\,\phi}$
6. $Tan.\,\phi = \frac{1}{Cot.\,\phi}$
7. $Cot.\,\phi = \frac{1}{Tan.\,\phi}$
8. $Sin.^2\,\tfrac{1}{2}\,\phi = \tfrac{1}{2}\,(1 - Cos.\,\phi)$
9. $Cos.^2\,\tfrac{1}{2}\,\phi = \tfrac{1}{2}\,(1 + Cos.\,\phi)$

10. $\mathit{Sin.}\ (\phi \pm \psi) = \mathit{Sin.}\ \phi\ \mathit{Cos.}\ \psi \pm \mathit{Cos.}\ \phi\ \mathit{Sin.}\ \psi$

11. $\mathit{Cos.}\ (\phi \pm \psi) = \mathit{Cos.}\ \phi\ \mathit{Cos.}\ \psi \mp \mathit{Sin.}\ \phi\ \mathit{Sin.}\ \psi$

12. $\mathit{Sin.}\ \phi\ \mathit{Cos.}\ \psi = \frac{1}{2}\ [\mathit{Sin.}\ (\phi + \psi) + \mathit{Sin.}\ (\phi - \psi)]$

13. $\mathit{Cos.}\ \phi\ \mathit{Sin.}\ \psi = \frac{1}{2}\ [\mathit{Sin.}\ (\phi + \psi) - \mathit{Sin.}\ (\phi - \psi)]$

14. $\mathit{Cos.}\ \phi\ \mathit{Cos.}\ \psi = \frac{1}{2}\ \mathit{Cos.}\ (\phi - \psi) + \mathit{Cos.}\ (\phi + \psi)]$

15. $\mathit{Sin.}\ \phi\ \mathit{Sin.}\ \psi = \frac{1}{2}\ [\mathit{Cos.}\ (\phi - \psi) - \mathit{Cos.}\ (\phi + \psi)]$

16. $\mathit{Sin.}\ \phi + \mathit{Sin.}\ \psi = 2\ \mathit{Sin.}\ \frac{1}{2}\ (\phi + \psi)\ \mathit{Cos.}\ \frac{1}{2}\ (\phi - \psi)$

17. $\mathit{Sin.}\ \phi - \mathit{Sin.}\ \psi = 2\ \mathit{Cos.}\ \frac{1}{2}\ (\phi + \psi)\ \mathit{Sin.}\ \frac{1}{2}\ (\phi - \psi)$

18. $\mathit{Cos.}\ \phi + \mathit{Cos.}\ \psi = 2\ \mathit{Cos.}\ \frac{1}{2}\ (\phi + \psi)\ \mathit{Sin.}\ \frac{1}{2}\ (\phi - \psi)$

19. $\mathit{Cos.}\ \phi - \mathit{Cos.}\ \psi = -\ 2\ \mathit{Sin.}\ \frac{1}{2}\ (\phi + \psi)\ \mathit{Sin.}\ \frac{1}{2}\ (\phi - \psi)$

20. $\mathit{Tan.}\ (\phi \pm \psi) = \dfrac{\mathit{Tan.}\ \phi \pm \mathit{Tan.}\ \psi}{1 \mp \mathit{Tan.}\ \phi\ \mathit{Tan.}\ \psi}$

21. $\mathit{Cot.}\ (\phi \pm \psi) = \dfrac{\mathit{Cot}\ \phi\ \mathit{Cot.}\ \psi \pm 1}{\mathit{Cot.}\ \psi \mp \mathit{Cot.}\ \phi}$

22. $\mathit{Tan.}\ \phi + \mathit{Tan.}\ \psi = \dfrac{\mathit{Sin.}\ (\phi + \psi)}{\mathit{Cos.}\ \phi\ \mathit{Cos.}\ \psi}$

23. $\mathit{Tan.}\ \phi - \mathit{Tan.}\ \psi = \dfrac{\mathit{Sin.}\ (\phi - \psi)}{\mathit{Cos.}\ \phi\ \mathit{Cos.}\ \psi}$

24. $\mathit{Cot.}\ \phi + \mathit{Cot.}\ \psi = \dfrac{\mathit{Sin.}\ (\phi + \psi)}{\mathit{Sin.}\ \phi\ \mathit{Sin.}\ \psi}$

25. $\mathit{Cot.}\ \phi - \mathit{Cot.}\ \psi = \dfrac{\mathit{Sin.}\ (\psi - \phi)}{\mathit{Sin.}\ \phi\ \mathit{Sin.}\ \psi}$

26. $\mathit{Sin.}\ 2\,\phi = 2\ \mathit{Sin.}\ \phi\ \mathit{Cos.}\ \phi$

27. $\mathit{Cos.}\ 2\,\phi = \mathit{Cos.}^2\ \phi - \mathit{Sin.}^2\ \phi$

28. $\mathit{Sin.}^2\ \phi = \frac{1}{2}\ (1 - \mathit{Cos.}\ 2\,\phi)$

29. $\mathit{Cos.}^2\ \phi = \frac{1}{2}\ (1 + \mathit{Cos.}\ 2\,\phi)$

30. $\mathit{Sec.}^2\ \phi = 1 + \mathit{Tan.}^2\ \phi$

31. $\mathit{Cosec.}^2\ \phi = 1 + \mathit{Cot.}^2\ \phi$

32. $\mathit{Sin.}^2\ \phi - \mathit{Sin.}^2\ \psi = \mathit{Sin.}\ (\phi + \psi)\ \mathit{Sin.}\ (\phi - \psi)$

33. $\mathit{Cos.}^2\ \phi - \mathit{Cos.}^2\ \psi = -\ \mathit{Sin.}\ (\phi - \psi)\ \mathit{Sin.}\ (\phi - \psi)$

34. $\mathit{Cos.}^2\ \phi - \mathit{Sin.}^2\ \psi = \mathit{Cos.}\ (\phi + \psi)\ \mathit{Cos.}\ (\phi - \psi)$

35. $Tan.^2\phi - Tan.^2\psi = \dfrac{Sin.(\phi+\psi)\,Sin.(\phi-\psi)}{Cos.^2\phi\,Cos.^2\psi}$

36. $Cot.^2\phi - Cot.^2\psi = \dfrac{Sin.(\phi+\psi)\,Sin.(\psi-\phi)}{Sin.^2\phi\,Sin.^2\psi}$

37. $\dfrac{Sin.\,\phi + Sin.\,\psi}{Sin.\,\phi - Sin.\,\psi} = \dfrac{Tan.\,\frac{1}{2}(\phi+\psi)}{Tan.\,\frac{1}{2}(\phi-\psi)}$

38. $\dfrac{Sin.\,\phi + Sin.\,\psi}{Cos.\,\phi + Cos.\,\psi} = Tan.\,\tfrac{1}{2}(\phi+\psi)$

39. $\dfrac{Sin.\,\phi + Sin.\,\psi}{Cos.\,\phi - Cos.\,\psi} = Cot.\,\tfrac{1}{2}(\psi-\phi)$

40. $\dfrac{Sin.\,\phi - Sin.\,\psi}{Cos.\,\phi + Cos.\,\psi} = Tan.\,\tfrac{1}{2}(\phi-\psi)$

41. $\dfrac{Sin.\,\phi - Sin.\,\psi}{Cos.\,\phi - Cos.\,\psi} = -\,Cot.\,\tfrac{1}{2}(\phi+\psi)$

42. $\dfrac{Cos.\,\phi + Cos.\,\psi}{Cos.\,\phi - Cos.\,\psi} = \dfrac{Cot.\,\frac{1}{2}(\phi+\psi)}{Tan.\,\frac{1}{2}(\psi-\phi)}$

43. $\dfrac{Tan.\,\phi + Tan.\,\psi}{Tan.\,\phi - Tan.\,\psi} = \dfrac{Sin.(\phi+\psi)}{Sin.(\phi-\psi)}$

44. $\dfrac{Cot.\,\phi + Cot.\,\psi}{Cot.\,\phi - Cot.\,\psi} = \dfrac{Sin.(\phi+\psi)}{Sin.(\psi-\phi)}$

45. $\dfrac{Sin.\,\phi}{1 + Cos.\,\phi} = Tan.\,\tfrac{1}{2}\phi$

46. $\dfrac{Sin.\,\phi}{1 - Cos.\,\phi} = Cot.\,\tfrac{1}{2}\phi$

47. $\dfrac{1 - Cos.\,\phi}{1 + Cos.\,\phi} = Tan.^2\,\tfrac{1}{2}\phi$

FINIS.

T. C. HANSARD, Paternoster Row Press.

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